PROBLEM D1

1 point

$$\left(\begin{array}{cccc|ccc} 5 & 0 & 2 & -7 & 1 & 7 & -2 \\ -7 & 0 & 2 & 5 & 1 & -5 & -2 \\ 5 & 2 & 0 & -7 & 1 & 2 & 10 \\ 0 & 2 & 5 & -7 & 1 & 2 & 5 \end{array}\right) \sim \left(\begin{array}{cccc|ccc} 5 & 0 & 2 & -7 & 1 & 7 & -2 \\ -2 & 0 & 4 & -2 & 2 & 2 & -4 \\ 0 & 2 & -2 & 0 & 0 & -5 & 12 \\ 0 & 2 & 5 & -7 & 1 & 2 & 5 \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|ccc} 5 & 0 & 2 & -7 & 1 & 7 & -2 \\ -1 & 0 & 2 & -1 & 1 & 1 & -2 \\ 0 & 2 & -2 & 0 & 0 & -5 & 12 \\ 0 & 0 & 7 & -7 & 1 & 7 & -7 \end{array}\right) \sim \left(\begin{array}{cccc|ccc} 0 & 0 & 12 & -12 & 6 & 12 & -12 \\ -1 & 0 & 2 & -1 & 1 & 1 & -2 \\ 0 & 2 & -2 & 0 & 0 & -5 & 12 \\ 0 & 0 & 7 & -7 & 1 & 7 & -7 \end{array}\right) \sim$$

+ reorder

$$\sim \left(\begin{array}{cccc|ccc} -1 & 0 & 2 & -1 & 1 & 1 & -2 \\ 0 & 2 & -2 & 0 & 0 & -5 & 12 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & 1 & -1 \\ 0 & 0 & 1 & -1 & \frac{1}{7} & 1 & -1 \end{array}\right) \sim \left(\begin{array}{cccc|ccc} -1 & 0 & 2 & -1 & 1 & 1 & -2 \\ 0 & 2 & -2 & 0 & 0 & -5 & 12 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & 1 & -1 \\ 0 & 0 & 0 & 0 & \frac{1}{7}-\frac{1}{2} & 0 & 0 \end{array}\right)$$

4 pts

For b_1: $0 = \frac{1}{7} - \frac{1}{2}$, no solution 1pt

For b_2: $x_3 - x_4 = 1$

$x_4 = t,\ x_3 = 1 + t$

$2x_2 - 2x_3 = -5$

$x_2 = x_3 - \frac{5}{2} = 1 + t - \frac{5}{2} = t - \frac{3}{2}$

$-x_1 + 2x_3 - x_4 = 1$

$x_1 = 2x_3 - x_4 - 1 = 2 + 2t - t - 1$

$= t + 1$

2pts

$[t+1,\ t-\frac{3}{2},\ t+1,\ t]$

$t \in \mathbb{R}$

For b_3:

$x_3 - x_4 = -1$

$x_4 = t,\ x_3 = t - 1$

$2x_2 - 2x_3 = 12$

$x_2 = x_3 + 6 = t + 5$

$-x_1 + 2x_3 - x_4 = -2$

$x_1 = 2x_3 - x_4 + 2 = 2t - 2 - t + 2 = t$

$[t,\ t+5,\ t-1,\ t],\ t \in \mathbb{R}$

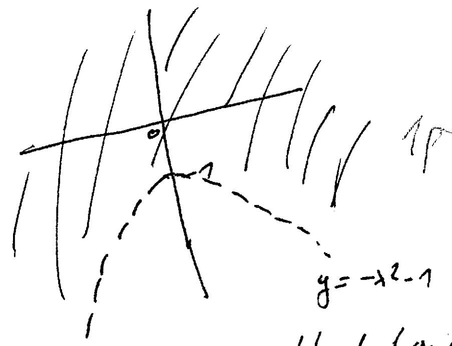
2pts

PROBLEM 2 $f(x,y) = (x^2+y+1)^{|x+y|} = \exp(|x+y| \cdot \log(x^2+y+1))$

D_f: $x^2+y+1 > 0$, $y > -x^2-1$

$D_f = \{[x,y] \in \mathbb{R}^2 : y > -x^2-1\}$ picture:

2pts



1pt

(without boundary)

2pts $$\frac{\partial f}{\partial x} = \exp(|x+y| \cdot \log(x^2+y+1)) \cdot \left(\operatorname{sgn}(x+y) \log(x^2+y+1) + |x+y| \cdot \frac{1}{x^2+y+1} \cdot 2x \right)$$

$$\frac{\partial f}{\partial y} = \exp(|x+y| \cdot \log(x^2+y+1)) \left(\operatorname{sgn}(x+y) \log(x^2+y+1) + |x+y| \cdot \frac{1}{x^2+y+1} \cdot 1 \right)$$

both for $[x,y] \in D_f$, $y+x \neq 0$ 1pt

At the remaining points, i.e. at $[x,-x]$, $x \in \mathbb{R}$

- all these points are in D_f, since $x^2-x+1 > 0$ for $x \in \mathbb{R}$

$$\frac{\partial f}{\partial x}(x,-x) = \lim_{h\to 0} \frac{f(x+h,-x) - f(x,-x)}{h} = \lim_{h\to 0} \frac{\exp(|h| \cdot \log((x+h)^2-x+1)) - 1}{h} =$$

$$= \lim_{h\to 0} \underbrace{\frac{\exp(|h| \lg((x+h)^2-x+1)) - 1}{|h| \lg((x+h)^2-x+1)}}_{\to 1} \cdot \operatorname{sgn} h \cdot \lg((x+h)^2-x+1) = \begin{cases} \xrightarrow{h\to 0+} \lg(x^2-x+1) \\ \xrightarrow{h\to 0-} -\lg(x^2-x+1) \end{cases}$$

2pts

Hence $\frac{\partial f}{\partial x}(x,-x)$ exists iff $\log(x^2-x+1) = 0$, i.e. $x^2-x+1 = 1$
i.e. $x=0$ or $x=1$

So $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial x}(1,-1) = 0$, $\frac{\partial f}{\partial x}(x,-x)$ for $x \neq 0,1$ does not exist

$$\frac{\partial f}{\partial y}(x,-x) = \lim_{h\to 0} \frac{f(x,-x+h) - f(x,-x)}{h} = \lim_{h\to 0} \frac{\exp(|h| \lg(x^2-x+h+1)) - 1}{h}$$

$$= \lim_{h\to 0} \frac{\exp(|h| \lg(x^2-x+h+1)) - 1}{|h| \lg(x^2-x+h+1)} \cdot \operatorname{sgn} h \cdot \lg(x^2-x+h+1) = \begin{cases} \xrightarrow{h\to 0+} \lg(x^2-x+1) \\ \xrightarrow{h\to 0-} -\lg(x^2-x+1) \end{cases}$$

2pts

So, the same result as above:

$\frac{\partial f}{\partial y}(0,0) = \frac{\partial f}{\partial y}(1,-1) = 0$, $\frac{\partial f}{\partial y}(x,-x)$ does not exist for $x \neq 0,1$

IFT3 ~~14:35 – 14:44:30~~ PROBLEM D3

$$\underbrace{\log(x+2y^2)+e^{x+y}-1}_{F(x,y)}=0 \qquad [-1,1]$$

3pts

① $F\in C^\infty(\{[x,y]\in\mathbb{R}^2:\ \underbrace{x+2y^2>0}_{[-1,1]\in ?}\})$

② $F(-1,1)=\log(-1+2)+e^0-1=0$

③ $\frac{\partial F}{\partial y}(-1,1)=\left(\frac{1}{x+2y^2}\cdot 4y+e^{x+y}\cdot 1\right)_{\substack{x=-1\\y=1}}=\frac{1}{-1+2}\cdot 4+e^0=5\neq 0$

$\Rightarrow$ there exists a C^∞ function with the required properties,

in part. $\log(x+2f(x)^2)+e^{x+f(x)}-1=0$ on a nbhood of -1

1pt

differentiate: $\frac{1}{x+2f(x)^2}(1+4f(x)f'(x))+e^{x+f(x)}(1+f'(x))=0$

1pt

$x=-1$, $f(-1)=1$: $\frac{1}{-1+2}(1+4f'(-1))+e^0(1+f'(-1))=0$

$2+5f'(-1)=0 \Rightarrow f'(-1)=-\frac{2}{5}$ 1pt

tangent line: $y=1-\frac{2}{5}(x+1)$ 1pt

Second deriv:

2pts

$$-\frac{1}{(x+2f(x)^2)^2}(1+4f(x)f'(x))^2+\frac{1}{x+2f(x)^2}(4f'(x)\cdot f'(x)+4f(x)f''(x))$$
$$+e^{x+f(x)}(1+f'(x))^2+e^{x+f(x)}\cdot f''(x)=0$$

$x=-1$, $f(-1)=1$, $f'(-1)=-\frac{2}{5}$

$$-\frac{1}{(-1+2)^2}\left(1+4\cdot 1\cdot\left(-\frac{2}{5}\right)\right)^2+\frac{1}{-1+2}\left(4\cdot\frac{4}{25}+4\cdot f''(-1)\right)$$
$$+e^0\left(1-\frac{2}{5}\right)^2+e^0\cdot f''(-1)=0$$

$$-\frac{9}{25}+\frac{16}{25}+4f''(-1)+\frac{9}{25}+f''(-1)=0$$

$$f''(-1)=-\frac{16}{125}$$ 1pt

PROBLEM D4 | $f(x,y,z) = x+y+z$

$M = \{[x,y,z] \in \mathbb{R}^3 : x^2+y^2+z^2 = 12,\ xy \geq 5\}$

① Existence of extrema: f is cts on $\mathbb{R}^3$

M closed ("=" and "≥") and bounded – contained in the closed ball with center 0 and radius $\sqrt{12}$ } 2pts

$\Rightarrow M$ is compact

Thus extrema do exist provided $M \neq \emptyset$

② $M = M_1 \cup M_2$ 1pt

$M_1 = \{[x,y,z] \in \mathbb{R}^3 : x^2+y^2+z^2 = 12,\ xy > 5\}$

$M_2 = \{[x,y,z] \in \mathbb{R}^3 : x^2+y^2+z^2 = 12,\ xy = 5\}$

$g_1(x,y,z) = x^2+y^2+z^2-12, \quad g_2(x,y,z) = xy-5$

For future use: $Df = [1,1,1],\ Dg_1 = [2x,2y,2z],\ Dg_2 = [y,x,0]$

③ Extrema on M_1:

1pt Either $Dg_1 = 0$ – only at $[0,0,0]$, but $[0,0,0] \notin M_1$ ... impossible

or $\exists \lambda : Df + \lambda Dg_1 = 0$

$$\left.\begin{aligned} 1+\lambda\cdot 2x &= 0 \\ 1+\lambda\cdot 2y &= 0 \\ 1+\lambda\cdot 2z &= 0 \end{aligned}\right\} \quad y-x=0,\ z-y=0 \;\Rightarrow x=y=z \qquad \therefore 3x^2=12,\ x^2=4,\ x=\pm 2$$

1pt

$[2,2,2]$, $[-2,-2,-2]$ But $2.2 = 4 < 5 \Rightarrow$ these points do not belong to M_1

2pts

So, no points in M_1

④ On M_2: Case 1: Dg_1 and Dg_2 linearly dependent

$Dg_1 = \alpha Dg_2$

$$\left.\begin{aligned} 2x &= \alpha y \\ 2y &= \alpha x \end{aligned}\right\} \Rightarrow 2x^2 - 2y^2 = 0, \quad y^2 = x^2, \quad y = \pm x$$

$2z = \alpha\cdot 0 \Rightarrow z = 0$

2pts

$x^2+y^2 = 12 \quad x^2 = 6,\ y^2 = 6 \Rightarrow x\cdot y = \pm 6 \neq 5$ ... not in M_2

$Dg_2 = \alpha\, Dg_1$

$\alpha \neq 0 \Rightarrow Dg_1 = \frac{1}{\alpha} Dg_2$ ... impossible by the previous case

$\alpha = 0 \Rightarrow y = x = 0 \Rightarrow xy = 0 \neq 5$ not in M_2

Caso 2 $\quad \exists\, \lambda, \mu : \nabla f + \lambda \nabla g_1 + \mu \nabla g_2 = 0$

1pt

$$\left.\begin{array}{l} 1 + \lambda \cdot 2x + \mu \cdot y = 0 \\ 1 + \lambda \cdot 2y + \mu \cdot x = 0 \end{array}\right\} x - y + 2\lambda (x^2 - y^2) = 0$$

$$1 + \lambda \cdot 2z = 0$$

$$z(x-y) - (x^2 - y^2) = 0$$

$$(x-y)(z-x-y) = 0$$

$y = x$

$x^2 = 5, \; y^2 = 5$

$5 + 5 + z^2 = 12$

$z^2 = 2$

$[\sqrt{5}, \sqrt{5}, \sqrt{2}] \overset{f}{\longmapsto} 2\sqrt{5} + \sqrt{2}$

$[\sqrt{5}, \sqrt{5}, -\sqrt{2}] \longmapsto 2\sqrt{5} - \sqrt{2}$

$[-\sqrt{5}, -\sqrt{5}, \sqrt{2}] \longmapsto -2\sqrt{5} + \sqrt{2}$

$[-\sqrt{5}, -\sqrt{5}, -\sqrt{2}] \longmapsto -2\sqrt{5} - \sqrt{2}$

3pts

$z = x + y$

$x^2 + y^2 + (x+y)^2 = 12$

$\cancel{2x}\, x^2 + y^2 + x^2 + 2xy + y^2 = 12$

$x^2 + xy + y^2 = 6$

$y = \frac{5}{x}$

$x^2 + 5 + \frac{25}{x^2} = 6$

$x^4 - x^2 + 25 = 0$

$D = 1 - 4 \cdot 25 < 0$

$\Rightarrow$ no solution

Conclusion: Maximum $2\sqrt{5} + \sqrt{2}$ at $[\sqrt{5}, \sqrt{5}, \sqrt{2}]$ 2pts

Minimum $-2\sqrt{5} - \sqrt{2}$ at $[-\sqrt{5}, -\sqrt{5}, -\sqrt{2}]$

INT2 ~~10:40+34~~ ~~10:35:34~~ 14 (4) PROBCOOIDS

$$\int \frac{4x^4}{(x+1)^2(x^2+2x+3)}\,dx$$

① division: $(x+1)^2(x^2+2x+3) = (x^2+2x+1)(x^2+2x+3) = x^4+2x^3+3x^2+2x^3+4x^2+6x$
$+x^2+2x+3 = x^4+4x^3+8x^2+8x+3$

$4x^4 : (x^4+4x^3+8x^2+8x+3) = 4$ 1pt

$-16x^3-32x^2-32x-12$

$$\frac{4x^4}{(x+1)^2(x^2+2x+3)} = 4 - \frac{16x^3+32x^2+32x+12}{(x+1)^2(x^2+2x+3)}$$

② partial fractions:

$$-\frac{16x^3+32x^2+32x+12}{(x+1)^2(x^2+2x+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+2x+3}$$ 1pt

$-16x^3-32x^2-32x-12 = A(x+1)(x^2+2x+3) + B(x^2+2x+3) + (Cx+D)\underbrace{(x+1)^2}_{=(x^2+2x+1)}$

$x=-1$: $16-32+32-12 = B\cdot(1-2+3) = 2B$

$B = 2$

al x^3: $-16 = A + C$
al x^2: $-32 = 2A+A+B+2C+D = 3A+2+2C+D$
al x^1: $-32 = 3A+2A+2B+C+2D = 5A+4+C+2D$
al x^0: $-12 = 3A+3B+D = 3A+6+D$

$C = -16-A$
$D = -18-3A$

$-32 = \cancel{3A}+2\cancel{6}-3\cancel{2}-2A-18-3\cancel{A}$

$2A = -16 \Rightarrow A = -8$

$C = -8,\quad D = -18+24 = 6$ 3pts

③ $\int \frac{-8x+6}{x^2+2x+3} = \int \left(-4 \cdot \frac{2x+2}{x^2+2x+3} + \frac{14}{x^2+2x+3}\right) dx = -4 \lg (x^2+2x+3)$ 1pt 1pt

$+ \underbrace{14 \int \frac{1}{x^2+2x+3}\, dx}$

$\|$

$14 \cdot \int \frac{1}{(x+1)^2+2}\, dx = 14 \cdot \int \frac{1}{2\left(\left(\frac{x+1}{\sqrt{2}}\right)^2+1\right)}\, dx = 7\sqrt{2} \int \frac{\frac{1}{\sqrt{2}}}{\left(\frac{x+1}{\sqrt{2}}\right)^2+1}\, dx$ 3pts

$\stackrel{c}{=} 7\sqrt{2} \operatorname{arctg} \frac{x+1}{\sqrt{2}} + c$

Thorosu H:

2pts

1pt $\left\{ \int \frac{4x+3}{(x+1)^2(x^2+2x+3)}\, dx \stackrel{c}{=} 4x - 8 \lg |x+1| - \frac{2}{x+1} - 4 \lg (x^2+2x+3) + 7\sqrt{2} \operatorname{arctg} \frac{x+1}{\sqrt{2}} \right.$

on $(-\infty, -1)$ and on $(-1, +\infty)$ 2pts