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Fractional Estimates for Non-Differentiable Elliptic Systems with general Growth

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Abstract In this paper we study the regularity of weak solutions of the elliptic system $-\operatorname{div}(\mathbf{A}(x,\nabla\mathbf{u})) = \mathbf{b}(x,\nabla\mathbf{u})$ with non-standard φ-growth condition. Here φ is a given Orlicz function. We are interested in the case where $\mathbf{A}$ and $\mathbf{b}$ are not differentiable with respect to x but only Hölder continuous with exponent α. We show that the natural quantity $\mathbf{V}(\nabla\mathbf{u})$ is locally in the Nikolskiĭ space $\mathscr{N}^{\alpha,2}$. From this it follows that the set of singularities of $\mathbf{V}(\nabla\mathbf{u})$ has Hausdorff dimension less or equal $n-2\alpha$, where n is the dimension of the domain Ω. One of the main features of our technique is that it handles the case of the p-Laplacian for $1 < p < \infty$ in a unified way. There is no need to use different approaches for the cases $p \leq 2$ and $p \geq 2$.

Keywords Elliptic Systems; Singular set; Hausdorff dimension; Orlicz Function; non-differentiable

Mathematics Subject Classification (2000) 35J60; 35D10

1 Introduction

In this paper we are concerned with fractional estimates for weak solutions of the system

$$-\operatorname{div}(\mathbf{A}(x,\nabla\mathbf{u})) = \mathbf{b}(x,\nabla\mathbf{u}) \quad \text{in } \Omega \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^n$ is a bounded, open domain. We assume that the elliptic operator satisfies non-standard φ-growth and φ-monotonicity conditions, i. e.

$$\begin{aligned}\big(\mathbf{A}(x,\mathbf{P})-\mathbf{A}(x,\mathbf{Q})\big)\cdot\big(\mathbf{P}-\mathbf{Q}\big) &\geq c\,\varphi''(|\mathbf{P}|+|\mathbf{Q}|)\,|\mathbf{P}-\mathbf{Q}|^2,\\ |\mathbf{A}(x,\mathbf{P})-\mathbf{A}(x,\mathbf{Q})| &\leq c\,\varphi''(|\mathbf{P}|+|\mathbf{Q}|)\,|\mathbf{P}-\mathbf{Q}|,\\ \mathbf{A}(x,\mathbf{0}) &= 0.\end{aligned}$$

where φ is a given Orlicz function. Moreover, we assume that the vector fields $\mathbf{A}:\Omega\times\mathbb{R}^{N\times n}\to\mathbb{R}^{N\times n}$ and $\mathbf{b}:\Omega\to\mathbb{R}^{N\times n}\to\mathbb{R}^{N\times n}$ satisfy the following continuity and growth assumptions with respect to x:

$$\begin{aligned}|\mathbf{A}(x,\mathbf{Q})-\mathbf{A}(x_0,\mathbf{Q})| &\leq c\,|x-x_0|^{\alpha_1}\,\varphi'(|\mathbf{Q}|),\\ |\mathbf{b}(x,\mathbf{Q})| &\leq c\,\big(\varphi'(|\mathbf{Q}|)+g_1(x)\big),\\ |\mathbf{b}(x,\mathbf{Q})-\mathbf{b}(x_0,\mathbf{Q})| &\leq c\,|x-x_0|^{\alpha_2}\,\big(\varphi'(|\mathbf{Q}|)+g_2(x)+g_2(x_0)\big),\\ |\mathbf{b}(x,\mathbf{P})-\mathbf{b}(x,\mathbf{Q})| &\leq c\,\varphi'(|\mathbf{P}|+|\mathbf{Q}|)\left(\frac{|\mathbf{P}-\mathbf{Q}|}{|\mathbf{P}|+|\mathbf{Q}|}\right)^{\alpha_3}\end{aligned}$$

with $\alpha_1,\alpha_2,\alpha_3\in(0,1]$ and suitable $g_1,g_2:\Omega\to[0,\infty)$.

The standard examples for the Orlicz function φ are

$$\varphi_1(t)=\int_0^t(\mu+s^2)^{\frac{p-2}{2}}\,s\,ds,\qquad\qquad \varphi_2(t)=\int_0^t(\mu+s)^{p-2}\,s\,ds,$$

where $\mu\geq 0$. The p-Laplacian corresponds to the choice $\mu=0$. Systems which such a type of growth conditions have been studied by many authors for special situations.

The first partial regularity results for non-linear elliptic systems were achieved by Morrey [21], followed by Giusti and Miranda [15] and Giusti [13]. The work has been continued for example by Evans [9], Giaquinta [10], Carozza, Fusco, Mingione [6], and by Duzaar and Grotowski [8].

Suppose that $\mathbf{u}$ is a weak solution to (1.1) and let Σ denote the set of singularities of $\nabla\mathbf{u}$, see Section 5 for precise definition. In this situation we try to show that Σ is reasonable small, i. e. that the Hausdorff dimension of Σ is small. If $\alpha_1=1$ then we speak of a differentiable elliptic system. For differentiable systems and minimizers with $p=2$ it is shown, e. g. in [10] and [14] that the Hausdorff dimension is strictly less than $n-2$.

The nonlinear, differentiable case with p-growth (the case φ_1) has for example been considered by Acerbi and Fusco [1]. They show that the Hausdorff dimension of Σ is strictly less or equal to $n-p$ for $1<p<2$. For $p\geq 2$ it can seen by [9], [11], and [5] that the Hausdorff dimension is less or equal to $n-2$. Let us point up here that the cases $1<p\leq 2$ and $p\geq 2$ required different techniques in the mentioned papers. It is one of the main advantages of our approach that such distinction is not necessary anymore.

For non-differentiable systems for a long time it has only been know that Σ has Lebesgue measure zero. So the question arose if it is possible gain more control of Σ for non-differentiable systems. In particular, Giaquinta and Modica asked in their paper [12] and also in the book [10], pg. 191, wether the Hausdorff dimension of the singular set could be estimated. In his two articles [19] and [20]

Mingione gave the answer to this question in the case $p \geq 2$, i. e. that the dimension is always less than $n-2\alpha$ if u is Hölder continuous and A is allowed to depend also on u. Further he could show that this result is still true in lower dimension ($n \leq 4$) if one drops the a priori Hölder continuity assumption. For higher dimensions he showed that the dimension is always less than n which was not known even for the Lipschitz case $\alpha_1 = 1$. Our main motivation was to transfer these results to the case of arbitrary Orlicz function, thus including the full case $1 < p < \infty$. Hereby, it was of great importance to us that the used technique will not distinguish the cases $1 < p \leq 2$ and $p \geq 2$. By the difference quotient method Mingione shows that $\nabla \mathbf{u}$ is in the Sobolev-Slobodeckiĭ $W^{\frac{2\beta}{p},p}$ for any $\beta < \alpha$. In Mingione's papers the estimates are actually carried out in Nikolskiĭ spaces and then at the end translated to fractional Sobolev spaces. Rather than estimating the $\nabla \mathbf{u}$ we prefer to estimate the natural quantity $\mathbf{V}(\nabla \mathbf{u})$, which in the case of φ_1 is given by $\mathbf{V}(\nabla \mathbf{u}) = (\mu + |\nabla \mathbf{u}|^2)^{\frac{p-2}{4}} \nabla \mathbf{u}$. Additionally, we choose Σ to be the set of singularities of $\mathbf{V}(\nabla \mathbf{u})$ instead of $\nabla \mathbf{u}$. We will see that this is much more natural for the non-linear system (1.1). We show that $\mathbf{V}(\nabla \mathbf{u})$ is locally in the Nikolskiĭ space $\mathcal{N}^{\alpha,2}$. This will be proven by the difference quotient method. The estimate for the Hausdorff measure of $\Sigma(\mathbf{V}(\nabla \mathbf{u}))$ is then a consequence of this regularity information. We will show that the Hausdorff dimension of Σ is less or equal to $n-2\alpha$.

Since our approach works for arbitrary Orlicz functions, it especially works for the full range $1 < p < \infty$. Therefore, our technique is new even in the case of differentiable $\mathbf{A}$ with no x-dependence of $\mathbf{A}$.

Additionally, we derive estimates of Cacciopoli and Gehring type for $\mathbf{V}(\nabla \mathbf{u})$. The result is based on a new, generalized Poincaré inequality for arbitrary Orlicz functions. This inequality might be of independent interest.

Under similar assumption partial regularity can be proved, i. e. $\mathbf{V}(\nabla u)$ is Hölder continuous on the complement of the singular set. This will be the content of a forthcoming paper.

2 Notation and Basic Properties

Let $\Omega \subset \mathbb{R}^n$ be an bounded, open domain. By Q we will always denote a cube in $\mathbb{R}^n$ with sides parallel to the axis. We write $Q \Subset \Omega$ if the closure of Q is contained in Ω. Let $|Q|$ denote the volume and length(Q) the side length of Q. For $f \in L^1(Q)$ we define

$$\fint_Q f(x)\,dx := \frac{1}{|Q|}\int_Q f(x)\,dx.$$

By kQ, with $k > 0$, we denote the cube with the same center and k times the side length. For functions f,g on Ω we define $\langle f,g\rangle := \int_\Omega f(x)g(x)\,dx$. For $a,b \in \mathbb{R}^n$ we denote by $[a,b]$ the straight line segment from a to b. If $a \neq b$ we define $\fint_a^b \cdots ds$ to be the mean average integral over the line $[a,b]$. For $U,W \subset \mathbb{R}^n$ we define $U+W := \{u+w : u \in U, w \in W\}$. We write $f \sim g$ iff there exist constants $c_0, c_1 > 0$, such that

$$c_0 f \leq g \leq c_1 f\,,$$

where we always indicate on what the constants may depend. Furthermore, we use c (no index) as a generic constant, i. e. its value my change from line to line but does not depend on the important variables.

The following definitions and results are standard in the context of N–function (see e. g. [22]). A real function $\varphi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is said to be an N-function if it satisfies the following conditions: There exists the derivative φ' of φ. This derivative is right continuous, non-decreasing and satisfies $\varphi'(0) = 0$ and $\varphi'(t) > 0$ for $t > 0$. Especially, φ is convex.

We say that φ satisfies the Δ_2–condition, if there exists $c_1 > 0$ such that for all $t \geq 0$ holds $\varphi(2t) \leq c_1\, \varphi(t)$. By $\Delta_2(\varphi)$ we denote the smallest constant c_1. Since $\varphi(t) \leq \varphi(2t)$ the Δ_2 condition is equivalent to $\varphi(2t) \sim \varphi(t)$. For a family φ_λ of N-functions we define $\Delta_2(\{\varphi_\lambda\}) := \sup_\lambda \Delta_2(\varphi_\lambda)$.

By L^φ and $W^{1,\varphi}$ we denote the classical Orlicz and Sobolev-Orlicz spaces, i. e. $f \in L^\varphi$ iff $\int \varphi(|f|)\, dx < \infty$ and $f \in W^{1,\varphi}$ iff $f, \nabla f \in L^\varphi$.

By $(\varphi')^{-1} : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ we denote the function

$$(\varphi')^{-1}(t) := \sup\{u \in \mathbb{R}^{\geq 0} : \varphi'(u) \leq t\}.$$

If φ' is strictly increasing then $(\varphi')^{-1}$ is the inverse function of φ'. Then $\varphi^* : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ with

$$\varphi^*(t) := \int_0^t (\varphi')^{-1}(s)\, ds$$

is again an N–function and $(\varphi^*)'(t) = (\varphi')^{-1}(t)$ for $t > 0$. It is the complementary function of φ. Note that $(\varphi^*)^* = \varphi$. For all $\delta > 0$ there exists c_δ (only depending on $\Delta_2(\{\varphi, \varphi^*\})$ such that for all $t, u \geq 0$ holds

$$t\, u \leq \delta\, \varphi(t) + c_\delta\, \varphi^*(u). \tag{2.1}$$

This inequality is called Young's inequality. For all $t \geq 0$

$$\begin{aligned} \frac{t}{2}\varphi'\Big(\frac{t}{2}\Big) &\leq \varphi(t) \leq t\, \varphi'(t), \\ \varphi\bigg(\frac{\varphi^*(t)}{t}\bigg) &\leq \varphi^*(t) \leq \varphi\bigg(\frac{2\,\varphi^*(t)}{t}\bigg). \end{aligned} \tag{2.2}$$

Therefore, uniformly in $t \geq 0$

$$\varphi(t) \sim \varphi'(t)\, t, \qquad \varphi^*\big(\varphi'(t)\big) \sim \varphi(t), \tag{2.3}$$

where the constants only depend on $\Delta_2(\{\varphi, \varphi^*\})$. If $\rho(t) = a\, \varphi(b\, t)$ for some $a, b > 0$ and all $t \geq 0$, then

$$\rho^*(t) = a\, \varphi^*\Big(\frac{t}{a\, b}\Big). \tag{2.4}$$

If φ and ρ are N–functions with $\varphi(t) \leq \rho(t)$ for all $t \geq 0$, then

$$\rho^*(t) \leq \varphi^*(t) \tag{2.5}$$

for all $t \geq 0$.

In most parts of the paper we will assume that φ satisfies the following assumptions:

Assumption 1 *Let φ be an N-function such that $\Delta_2(\{\varphi,\varphi^*\}) < \infty$. Further assume that φ is C^2 on $(0,\infty)$ and uniformly in $t \geq 0$*

$$\varphi'(t) \sim t\,\varphi''(t), \tag{2.6}$$

As already mentioned in the introduction we will assume that the vector field $\mathbf{A} : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ satisfies the non-standard φ-growth condition, i. e.

$$\begin{aligned} \big(\mathbf{A}(x,\mathbf{P}) - \mathbf{A}(x,\mathbf{Q})\big)\cdot\big(\mathbf{P}-\mathbf{Q}\big) &\geq c\,\varphi''(|\mathbf{P}|+|\mathbf{Q}|)\,|\mathbf{P}-\mathbf{Q}|^2, \\ |\mathbf{A}(x,\mathbf{P}) - \mathbf{A}(x,\mathbf{Q})| &\leq c\,\varphi''(|\mathbf{P}|+|\mathbf{Q}|)\,|\mathbf{P}-\mathbf{Q}| \end{aligned} \tag{2.7}$$

and the continuity and growth condition

$$|\mathbf{A}(x,\mathbf{Q}) - \mathbf{A}(x_0,\mathbf{Q})| \leq c\,|x-x_0|^{\alpha_1}\varphi'(|\mathbf{Q}|), \tag{2.8}$$

where $0 < \alpha_1 \leq 1$ and φ satisfies Assumption 1.

It is interesting to know that for every φ as in Assumption 1 there exists $\mathbf{A} : \Omega \to \mathbb{R}^{N\times n}$ that satisfies (2.7). The construction of such $\mathbf{A}$ can be found in Lemma 21 in the appendix. It is possible to multiply $\mathbf{A}$ by some function $\mu : \Omega \to (0,\infty)$ which is uniformly α_1-Hölder continuous and is bounded from above and below. Then (2.7) and (2.8) still hold.

Remark 1 Our standard examples for $\mathbf{A}$ and φ are

$$\mathbf{A}(x,\mathbf{Q}) := \mu(x)\,|\mathbf{Q}|^{p-2}\mathbf{Q}, \qquad \varphi'(t) := t^{p-1}$$

and

$$\mathbf{A}(x,\mathbf{Q}) := \mu(x)\,(1+|\mathbf{Q}|)^{p-2}\mathbf{Q}, \qquad \varphi'(t) := (1+t)^{p-2}t,$$

where $1 < p < \infty$, $0 < \alpha_1 \leq 1$, and $\mu : \Omega \to (0,\infty)$ is α_1-Hölder continuous and bounded from above and below.

For given φ we define the N-function ψ by

$$\frac{\psi'(t)}{t} := \left(\frac{\varphi'(t)}{t}\right)^{\frac{1}{2}}. \tag{2.9}$$

It is shown in Lemma 25 that ψ also satisfies Assumption 1 and uniformly in $t > 0$ holds $\psi''(t) \sim \sqrt{\varphi''(t)}$. As in Lemma 21 we define $\Psi : \mathbb{R}^{N\times n} \to \mathbb{R}^{\geq 0}$ by $\Psi(\mathbf{Q}) := \psi(|\mathbf{Q}|)$ and let $\mathbf{V}(\mathbf{Q}) := (\nabla_{N\times n}\Psi)(\mathbf{Q}) = \psi'(|\mathbf{Q}|)\frac{\mathbf{Q}}{|\mathbf{Q}|}$. From the same lemma it follows that (2.7) holds with $\mathbf{A},\varphi$ replaced by $\mathbf{V},\psi$.

Remark 2 The examples given in Remark 1 correspond to

$$\mathbf{V}(\mathbf{Q}) := |\mathbf{Q}|^{\frac{p-2}{2}}\mathbf{Q}, \qquad \psi'(t) := t^{\frac{p}{2}}$$

and

$$\mathbf{V}(\mathbf{Q}) := (1+|\mathbf{Q}|)^{\frac{p-2}{2}}\mathbf{Q}, \qquad \psi'(t) := (1+t)^{\frac{p-2}{2}}\,t,$$

where $1 < p < \infty$.

We introduce a family of N-function $\{\varphi_a\}_{a\geq 0}$ by $\varphi_a'(t)/t := \varphi'(a+t)/(a+t)$ which basically states $\varphi_a''(t) \sim \varphi''(a+t)$ uniformly in $a,t \geq 0$. The basic properties of φ_a are given in the appendix, see Definition 22 and thereafter. The connection between $\mathbf{A}$, $\mathbf{V}$, and $\{\varphi_a\}_{a\geq 0}$ is best reflected in the following lemma.

Lemma 3 *Let $\mathbf{A}, \varphi$ satisfy Assumption 1 and (2.7). Let $\psi, \mathbf{V}$ be defined as in (2.9). Then*

$$\big(\mathbf{A}(x,\mathbf{P}) - \mathbf{A}(x,\mathbf{Q})\big)\cdot\big(\mathbf{P}-\mathbf{Q}\big) \sim \big|\mathbf{V}(\mathbf{P}) - \mathbf{V}(\mathbf{Q})\big|^2 \tag{2.10a}$$

$$\sim \varphi_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|), \tag{2.10b}$$

$$\sim |\mathbf{P}-\mathbf{Q}|^2\,\varphi''\big(|\mathbf{P}|+|\mathbf{Q}|\big), \tag{2.10c}$$

uniformly in $\mathbf{P},\mathbf{Q} \in \mathbb{R}^{N\times n}$ and $x \in \Omega$. Moreover,

$$\mathbf{A}(x,\mathbf{Q})\cdot\mathbf{Q} \sim |\mathbf{V}(\mathbf{Q})|^2 \sim \varphi(|\mathbf{Q}|) \tag{2.10d}$$

uniformly in $\mathbf{Q} \in \mathbb{R}^{N\times n}$ and $x \in \Omega$.

Note that if $\varphi''(0)$ does not exists, the expression in (2.10c) is continuously extended by zero for $|\mathbf{P}| = |\mathbf{Q}| = 0$.

The lemma will be proven in the appendix. The different representations of (2.10) will be useful at different stages of our proofs. The one with $\mathbf{A}$ appears when we test the differential operator $-\operatorname{div}(\mathbf{A}(\nabla\mathbf{u}))$ by a suitable test function. The one with $\mathbf{V}$ is useful to write down information, since most of the information on $\mathbf{u}$ will be expressed in information on $\mathbf{V}(\nabla\mathbf{u})$. The representation with φ_a simplifies the proofs. The function $\mathbf{V}$ also appears in the study of minimizers of the form $\int \varphi(|\nabla\mathbf{u}|)\,dx$.

For the right hand side of the system (1.1) we assume that the vector field $\mathbf{b} : \Omega \to \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ satisfies the following continuity and growth assumptions with respect to x:

$$|\mathbf{b}(x,\mathbf{Q})| \leq c\,\big(\varphi'(|\mathbf{Q}|) + g_1(x)\big), \tag{2.11a}$$

$$|\mathbf{b}(x,\mathbf{Q}) - \mathbf{b}(x_0,\mathbf{Q})| \leq c\,|x-x_0|^{\alpha_2}\,\big(\varphi'(|\mathbf{Q}|) + g_2(x) + g_2(x_0)\big), \tag{2.11b}$$

$$|\mathbf{b}(x,\mathbf{P}) - \mathbf{b}(x,\mathbf{Q})| \leq c\,\varphi'(|\mathbf{P}|+|\mathbf{Q}|)\left(\frac{|\mathbf{P}-\mathbf{Q}|}{|\mathbf{P}|+|\mathbf{Q}|}\right)^{\alpha_3} \tag{2.11c}$$

with $\alpha_2, \alpha_3 \in (0,1]$, $g_1, g_2 : \Omega \to \mathbb{R}^{\geq 0}$, and $\varphi^*(|g_1|), \varphi^*(|g_2|) \in L^q$ for some $q > 1$. Again, it will be useful to introduce a suitable family of $N-$functions $\{\varphi_{a,\omega}\}_{a\geq 0}$ to clarify the natural choice of the growth condition (2.11c). Especially, let $\omega_3(t) := 1/(\alpha_3+1)\,t^{\alpha_3+1}$, i. e. $\omega_3'(t) = t^{\alpha_3}$ then (2.11c) can be rewritten as

$$|\mathbf{b}(x,\mathbf{P}) - \mathbf{b}(x,\mathbf{Q})| \leq c\,\varphi'_{|\mathbf{P}|,\omega_3}(|\mathbf{P}-\mathbf{Q}|), \tag{2.12}$$

where $\varphi_{a,\omega}$ is given in Definition 22. We will see later in the proof of Theorem 11 and Lemma 12 that this form of continuity condition is the natural one.

3 Cacciopoli Estimates and a Gehring Type Result

In the following assume that $\mathbf{u}$ is a weak solution of system (1.1) in the sense that $\mathbf{u}$ satisfies (1.1) in the distributional sense and that $\nabla\mathbf{u} \in L^{\varphi}(\Omega)$. In view of (2.10d) this is equivalent to $\mathbf{V}(\nabla\mathbf{u}) \in L^2(\Omega)$. We start with a lemma of Cacciopoli type. At this point we would like to mention that in order to keep to notations short we will often skip the explicit dependence on x. For example we will rather write $\mathbf{A}(\nabla\mathbf{u})$ instead of $\mathbf{A}(x,(\nabla\mathbf{u})(x))$. Nevertheless, $\mathbf{A}$ will still depend on x.

Theorem 4 *Let $\mathbf{u}$ be a weak solution of system* (1.1)*. Then there exists $c > 1$ such that for all cubes Q with $2Q \Subset \Omega$ holds*

$$\fint_{Q} \varphi(|\nabla\mathbf{u}|)\,dx \le c \fint_{2Q} \varphi\left(\frac{|\mathbf{u}-\langle\mathbf{u}\rangle_Q|}{R}\right)dx + c \fint_{2Q} \varphi^*(|g_1|)\,dx, \tag{3.1}$$

where R is the side length of the cube Q. The constant c only depends on $\Delta_2(\{\phi,\phi^*\})$ and the constants in (2.6), (2.7), (2.8), *and* (2.11).

Proof For fixed Q and $R := \text{length}(Q)$ let $\eta \in C_0^\infty(2Q)$ be a cut-off function with $\chi_Q \le \eta \le \chi_{2Q}$ and $|\nabla\eta| \le c/R$. We pick the test function $\xi := \eta^q(\mathbf{u}-\langle\mathbf{u}\rangle_Q)$ and obtain

$$\begin{aligned}\langle\mathbf{A}(\nabla\mathbf{u}),\eta^q\nabla\mathbf{u}\rangle &= -\langle\mathbf{A}(\nabla\mathbf{u}), q\,\eta^{q-1}(\mathbf{u}-\langle\mathbf{u}\rangle_Q)\otimes(\nabla\eta)\rangle \\ &\quad + \langle\mathbf{b}(\nabla\mathbf{u}),\eta^q(\mathbf{u}-\langle\mathbf{u}\rangle_Q)\rangle.\end{aligned}$$

The exponent q will be chosen as follows. By Lemma 31 there exist $\varepsilon, c_2 > 0$ with $\varphi(\lambda t) \le c_2\,\lambda^{1+\varepsilon}\,\varphi(t)$ uniformly in $t \ge 0$ and $\lambda \in [0,1]$. We fix $q > 1$ large enough such that $(1+\varepsilon)(q-1) \ge q$. In particular, uniformly in $t \ge 0$

$$\varphi(\eta^{q-1}t) \le c_2\,\eta^q\,\varphi(t). \tag{3.2}$$

The monotonicity and growth conditions on $\mathbf{A}$ and $\mathbf{b}$ imply

$$\begin{aligned}\int_{2Q} \eta^q\varphi(|\nabla\mathbf{u}|)\,dx &\le c\int_{2Q} \eta^{q-1}\,\varphi'(|\nabla\mathbf{u}|)\frac{|\mathbf{u}-\langle\mathbf{u}\rangle_Q|}{R}\,dx \\ &\quad + c\int_{2Q} \eta^q\,\big(\varphi'(|\nabla\mathbf{u}|)+g_1\big)|\mathbf{u}-\langle\mathbf{u}\rangle_Q|\,dx.\end{aligned}$$

According to Young's inequality (6.26) we derive for $\varepsilon > 0$

$$\begin{aligned}\int_{2Q} \eta^q\,\varphi(|\nabla\mathbf{u}|)\,dx &\le \varepsilon\int_{2Q} \varphi^*\big(\eta^{q-1}\varphi'(|\nabla\mathbf{u}|)\big)\,dx + \varepsilon\int_{2Q} \eta^q\varphi(|\nabla\mathbf{u}|)\,dx \\ &\quad + c_\varepsilon\int_{2Q} \varphi\left(\frac{|\mathbf{u}-\langle\mathbf{u}\rangle_Q|}{R}\right)dx + c_\varepsilon\int_{2Q} \varphi^*(|g_1|)\,dx,\end{aligned}$$

where we have used that $R \le c(\Omega)$, since Ω is bounded. (This is the only place in the paper where we use the boundedness of Ω.) Note that by (3.2) and (2.3)

$$\varphi^*\big(\eta^{q-1}\varphi'(|\nabla\mathbf{u}|)\big) \le c\,\eta^q\,\varphi^*\big(\varphi'(|\nabla\mathbf{u}|)\big) \sim \eta^q\,\varphi(|\nabla\mathbf{u}|).$$

Thus for small $\varepsilon > 0$ we deduce

$$\int_Q \varphi(|\nabla \mathbf{u}|)\,dx \leq c \int_{2Q} \varphi\left(\frac{|\mathbf{u} - \langle \mathbf{u}\rangle_Q|}{R}\right) dx + c \int_{2Q} \varphi^*(|g_1|)\,dx,$$

where we have used $\eta^q \geq \chi_Q$. This the Theorem.

Remark 5 It is easy to see that Theorem 4 and the results below remain valid if we use balls instead of cubes.

From Theorem 4 we want to derive an estimate of Gehring type, i.e. some reverse Hölder estimate. It is standard to use the ingenious lemma of Giaquinta and Modica:

Proposition 6 (Giaquinta-Modica) *Let $Q_0 \subset \mathbb{R}^n$ be a cube, $G \in L^1(Q_0)$, and $H \in L^{q_0}(Q_0)$ for some $q_0 > 1$. Suppose that for some $\theta \in (0,1)$, $c_1 > 0$, and all cubes Q with $2Q \subset Q_0$*

$$\fint_Q |G|\,dx \leq c_1 \left(\fint_{2Q} |G|^\theta\,dx\right)^{\frac{1}{\theta}} + \fint_{2Q} |H|\,dx.$$

Then there exist $q_1 > 1$ and $c_2 > 1$ such that $G \in L^{q_1}_{\mathrm{loc}}(Q)$ and for all $q_2 \in [1, q_1]$

$$\left(\fint_Q |G|^{q_2}\,dx\right)^{\frac{1}{q_2}} \leq c_2 \fint_{2Q} |G|\,dx + c_2 \left(\fint_{2Q} |H|^{q_2}\,dx\right)^{\frac{1}{q_2}}.$$

Another important tool in our proof will be the following generalization of the Poincaré's inequality.

Theorem 7 (Poincaré type) *Let φ be an N–function with $\Delta_2(\{\varphi, \varphi^*\}) < \infty$. Further, let $Q \subset \mathbb{R}^n$ be some cube with side length R and let $\omega \in L^\infty(Q)$ with $\omega \geq 0$ and $\int_Q \omega(x)\,dx = 1$. Then there exists $0 < \theta < 1$, which only depends on $\Delta_2(\{\varphi, \varphi^*\})$, and there exists $K > 0$, which only depends on $\Delta_2(\{\varphi, \varphi^*\})$ and $R^n \|\omega\|_\infty$, such that for all $\mathbf{v} \in W^{1,\varphi}(Q)$ holds*

$$\fint_Q \varphi\left(\frac{|\mathbf{v} - \langle \mathbf{v}\rangle_\omega|}{R}\right) dx \leq K \left(\fint_Q \big(\varphi(|\nabla \mathbf{v}|)\big)^\theta\,dx\right)^{\frac{1}{\theta}}, \tag{3.3a}$$

where $\langle \mathbf{v}\rangle_\omega := \int_Q \mathbf{v}(x)\,\omega(x)\,dx$.
Note that for the special choice $\omega := |Q|^{-1}\chi_Q$ we have $\langle \mathbf{v}\rangle_\omega = \langle \mathbf{v}\rangle_Q$.

Proof Since $\Delta_2(\varphi^*) < \infty$ it follows from [17] (Lemma 1.2.2+1.2.3) that φ^θ is quasiconvex for some $1 - \frac{1}{n} < \theta < 1$, i.e. there exists an N–function ρ with $\varphi^\theta \sim \rho$ and $\Delta_2(\{\rho, \rho^*\}) < \infty$. It is important to remark that θ and $\Delta_2(\{\rho, \rho^*\})$ only depend on $\Delta_2(\{\varphi, \varphi^*\})$. We deduce that $\varphi(\rho^{-1}(t)) \sim t^{\frac{1}{\theta}}$. Let $L := \fint_Q \rho(\nabla \mathbf{v})\,dx$.

If $L = 0$ then $\mathbf{v}$ is constant on Q and there is nothing to show. So we assume that $L > 0$. From [18] (Lemma 1.50) we know that for almost all $x \in Q$ holds

$$|\mathbf{v}(x) - \langle \mathbf{v}\rangle_\omega| \le c \int_Q \frac{|\nabla \mathbf{v}(y)|}{|x-y|^{n-1}}\, dy, \tag{3.4}$$

where the constant only depends on $R^n \|\omega\|_\infty$.

With (3.4) and $\Delta_2(\varphi) < \infty$ we estimate

$$(I) := \fint_Q \varphi\left(\left|\frac{\mathbf{v} - \langle \mathbf{v}\rangle_\omega}{R}\right|\right) dx \le c \fint_Q \varphi\left(\int_Q \frac{|\nabla \mathbf{v}(\xi)|}{R\,|x-\xi|^{n-1}}\, d\xi\right) dx.$$

Since $\int_Q R^{-1}\,|x-\xi|^{1-n}\, dx \le c$ independent of Q and $x \in Q$, we can apply Jensen's inequality to the convex function ρ and the measure $R^{-1}\,|x-\xi|^{1-d}\, d\xi$. This implies

$$\begin{aligned}
(I) &\le c \fint_Q \varphi \circ \rho^{-1} \circ \left(\int_Q \rho\big(|\nabla \mathbf{v}(\xi)|\big)\, R^{-1}\,|x-\xi|^{1-n}\, d\xi\right) dx,\\
&\le c \fint_Q \left(\int_Q \rho\big(|\nabla \mathbf{v}(\xi)|\big)\, R^{-1}\,|x-\xi|^{1-n}\, d\xi\right)^{\frac{1}{\theta}} dx\\
&\le c\, R^{-1/\theta} \fint_Q L^{1/\theta} R^{n/\theta} \left(\fint_Q L^{-1} \rho\big(|\nabla \mathbf{v}(\xi)|\big)\, |x-\xi|^{1-n}\, d\xi\right)^{\frac{1}{\theta}} dx,
\end{aligned}$$

where we have used $\Delta_2(\{\rho, \rho^*\}) < \infty$. Now Jensen's inequality applied to the convex function $t \mapsto t^{1/\theta}$ and the measure $L^{-1}\rho(|\nabla \mathbf{v}(\xi)|)\, d\xi$ gives

$$\begin{aligned}
(I) &\le c\, R^{(n-1)/\theta} \fint_Q L^{1/\theta} \fint_Q L^{-1} \rho\big(|\nabla \mathbf{v}(\xi)|\big) \big(|x-\xi|^{1-n}\big)^{\frac{1}{\theta}}\, d\xi\, dx\\
&\le c\, R^{(n-1)/\theta} L^{1/\theta - 1} \fint_Q \rho\big(|\nabla \mathbf{v}(\xi)|\big)\, d\xi\, R^{(1-n)/\theta},
\end{aligned}$$

which is possible since $\frac{1-n}{\theta} > -n$. By definition of L

$$(I) \le c \left(\fint_Q \rho\big(|\nabla \mathbf{v}(\xi)|\big)\, d\xi\right)^{\frac{1}{\theta}} \le c \left(\fint_Q \varphi^\theta\big(|\nabla \mathbf{v}(\xi)|\big)\, d\xi\right)^{\frac{1}{\theta}}.$$

This proves the theorem.

Remark 8 Theorem 7 is probably well-known among experts, but we could not find a reference. A proof of the simplified case $\theta = 1$ and $\omega = |Q|^{-1}\chi_Q$ can be found in [4]. Nevertheless, we need the sharper version with $\theta < 1$ in Theorem 9 in order to apply Proposition 6. Moreover, we will need the version with general ω in a forthcoming article.

We are now able to prove the reverse Hölder estimate.

Theorem 9 *Let* $\mathbf{u}$ *be a weak solution of system* (1.1)*. Then there exists* $q_2 > 1$ *and* $c > 1$ *such that for all cubes* Q *with* $2Q \Subset \Omega$ *and all* $q \in [1, q_1]$ *holds*

$$\left(\fint_Q |\mathbf{V}(\nabla \mathbf{u})|^{2q}\,dx\right)^{\frac{1}{q}} \leq c \fint_{2Q} |\mathbf{V}(\nabla \mathbf{u})|^2\,dx + c\left(\fint_{2Q} (\varphi^*(g_1))^q\,dx\right)^{\frac{1}{q}}, \tag{3.5a}$$

Especially, by (2.10d) *we have* $\varphi(|\nabla \mathbf{u}|) \in L^{q_1}_{\mathrm{loc}}(\Omega)$*. The constants* c *and* q_1 *only depend on* $\Delta_2(\{\phi, \phi^*\})$ *and the constants in* (2.6)*,* (2.7)*,* (2.8)*, and* (2.11)*.*

Proof Due to Theorem 4 we have

$$\fint_Q \varphi(|\nabla \mathbf{u}|)\,dx \leq c \fint_{2Q} \varphi\left(\frac{|\mathbf{u} - \langle \mathbf{u}\rangle_Q|}{R}\right) dx + c \fint_{2Q} \varphi^*(g_1)\,dx.$$

By Theorem 7 there exists $\theta \in (0,1)$ only depending on $\Delta_2(\{\varphi, \varphi^*\})$ such that

$$\fint_Q \varphi(|\nabla \mathbf{u}|)\,dx \leq c\left(\fint_{2Q} (\varphi(|\nabla \mathbf{u}|))^\theta\,dx\right)^{\frac{1}{\theta}} + c \fint_{2Q} \varphi^*(g_1)\,dx.$$

From Proposition 6 we deduce that there exists $q_1 > 1$ such that for all $q \in [1, q_1]$

$$\left(\fint_Q (\varphi(|\nabla \mathbf{u}|))^q\,dx\right)^{\frac{1}{q}} \leq c \fint_{2Q} \varphi(|\nabla \mathbf{u}|)\,dx + c\left(\fint_{2Q} (\varphi^*(g_1))^q\,dx\right)^{\frac{1}{q}}.$$

This and (2.10d) proves the theorem.

Remark 10 Note that similar results regarding higher integrability have been proved in [7] by A. Chianchi and N. Fusco.

4 Modified Difference Quotient Method

In this section we derive higher regularity of the solution $\mathbf{u}$, especially we show that $\mathbf{V}(\nabla \mathbf{u})$ is locally in the Nikolskiĭ space $\mathcal{N}^{\alpha,2}$. To prove this we use a modified version of the difference quotient method. Instead of plain differences we will at a certain stage consider averages of differences. Let us introduce the notations: For $x, s \in \mathbb{R}^n$ we define

$$T_s(x) := x + s, \qquad\qquad (\tau_s f)(x) := f(x+s) - f(x).$$

The main theorem is the following.

Theorem 11 *Let $\mathbf{u}$ be a weak solution of system (1.1). Then there exists $c_3 > 0$ such that the following holds: If $Q \subset \Omega$ is a cube with $20Q \Subset \Omega$ and if $h \in \mathbb{R}^n \setminus \{0\}$ with $|h| \le R$ then*

$$\fint_Q |\tau_h \mathbf{V}(\nabla \mathbf{u})|^2\,dx \le c_3\,\beta(R,|h|)\left(\fint_{20Q} |\mathbf{V}(\nabla \mathbf{u})|^2\,dx + \fint_{20Q} \varphi^*(g_2)\,dx\right), \tag{4.1}$$

where $\beta(R,|h|) := \frac{|h|^2}{R^2} + \frac{|h|^{1+\alpha_1}}{R} + |h|^{\frac{2}{2-\alpha_3}} + |h|^{2\alpha_1} + |h|^{1+\alpha_2}$. The constant c_3 only depend on $\Delta_2(\{\phi,\phi^*\})$ and the constants in (2.6), (2.7), (2.8), and (2.11).

We split the proof in two parts and begin with the following lemma.

Lemma 12 *Let $\mathbf{u}$ be a weak solution of system (1.1). Then for every $\delta > 0$ there exists $c_\delta > 0$ such that the following holds: If $Q \subset \Omega$ is a cube with $4Q \Subset \Omega$ and if $h, s \in \mathbb{R}^n \setminus \{0\}$ with $|s| \le |h| \le R$ then*

$$\begin{aligned}\fint_Q |\tau_s \mathbf{V}(\nabla \mathbf{u})|^2\,dx &\le \delta\,\frac{|s|}{|h|}\fint_0^s \fint_{2Q} |\tau_\lambda \mathbf{V}(\nabla \mathbf{u})|^2\,dx\,d\lambda \\ &\quad + c_\delta\,\beta(R,|h|)\fint_{4Q} |\mathbf{V}(\nabla \mathbf{u})|^2\,dx + c\,|s|^{1+\alpha_2}\fint_{4Q} \varphi^*(g_2)\,dx\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\fint_0^h \fint_Q |\tau_\lambda \mathbf{V}(\nabla \mathbf{u})|^2\,dx\,d\lambda &\le \delta \fint_0^h \fint_{2Q} |\tau_\lambda \mathbf{V}(\nabla \mathbf{u})|^2\,dx\,d\lambda \\ &\quad + c_\delta\,\beta(R,|h|)\fint_{4Q} |\mathbf{V}(\nabla \mathbf{u})|^2\,dx + c\,|h|^{1+\alpha_2}\fint_{4Q} \varphi^*(g_2)\,dx,\end{aligned} \tag{4.3}$$

where $\beta(R,|h|)$ is defined as in Theorem 11.

Proof As in the classical approach we first apply τ_s to our system (1.1). Let Q, R, and s be as required and $\xi \in C_0^\infty(Q)$. Then

$$\langle \tau_s(\mathbf{A}(\nabla \mathbf{u})), \nabla \xi\rangle = \langle \tau_s(\mathbf{b}(\nabla \mathbf{u})), \xi\rangle.$$

(Recall that $\langle f, g\rangle := \int_\Omega f(x) g(x)\,dx$.) As before we skipped the explicit dependence on x in order to keep the notations short, but $\mathbf{A}$ and $\mathbf{b}$ nevertheless depend on x. As in the proof of Theorem 9 we can choose with the help of (6.25) some $q > 1$ and $c_2 > 0$ such that

$$\varphi_a(\eta^{q-1} t) \le c_2\,\eta^q\,\varphi_a(t). \tag{4.4}$$

uniformly in $t, a \ge 0$. Let $\eta \in C_0^\infty$ be a cut off function with $\chi_Q \le \eta \le \chi_{2Q}$ and $|\nabla \eta| \le c/R$. Then we use the test function $\xi := \eta^q\,\tau_s \mathbf{u}$. We get

$$\big\langle \tau_s(\mathbf{A}(\nabla \mathbf{u})), \nabla(\eta^q \tau_s \mathbf{u})\big\rangle = \big\langle \tau_s(\mathbf{b}(\nabla \mathbf{u})), \eta^q \tau_s \mathbf{u}\big\rangle. \tag{4.5}$$

Everything will be derived from this equation. We define

$$\begin{aligned}
\mathscr{A}_s(x) &:= \mathbf{A}\big(x, \nabla\mathbf{u}(x+s)\big) - \mathbf{A}\big(x, \nabla\mathbf{u}(x)\big),\\
\mathscr{B}_s(x) &:= \mathbf{A}\big(x+s, \nabla\mathbf{u}(x+s)\big) - \mathbf{A}\big(x, \nabla\mathbf{u}(x+s)\big),\\
\mathscr{C}_s(x) &:= \mathbf{b}\big(x, \nabla\mathbf{u}(x+s)\big) - \mathbf{b}\big(x, \nabla\mathbf{u}(x)\big),\\
\mathscr{D}_s(x) &:= \mathbf{b}\big(x+s, \nabla\mathbf{u}(x+s)\big) - \mathbf{b}\big(x, \nabla\mathbf{u}(x+s)\big),
\end{aligned}$$

then

$$\begin{aligned}
\big(\tau_s(\mathbf{A}(\nabla\mathbf{u}))\big)(x) &= \mathscr{A}_s(x) + \mathscr{B}_s(x),\\
\big(\tau_s(\mathbf{b}(\nabla\mathbf{u}))\big)(x) &= \mathscr{C}_s(x) + \mathscr{D}_s(x).
\end{aligned}$$

Now (4.5) reads

$$\langle \mathscr{A}_s + \mathscr{B}_s, \nabla(\eta^q \tau_s \mathbf{u})\rangle = \langle \mathscr{C}_s + \mathscr{D}_s, \eta^q \tau_s \mathbf{u}\rangle. \tag{4.6}$$

Let (I), (II), (III), and (IV) be the four summands in (4.6). Let us collect the fundamental estimates for $\mathscr{A}_s$, $\mathscr{B}_s$, $\mathscr{C}_s$, and $\mathscr{D}_s$:

$$\mathscr{A}_s \cdot \tau_s\nabla\mathbf{u} \sim \varphi_{|\nabla\mathbf{u}|}(|\tau_s\nabla\mathbf{u}|) \sim |\tau_s\mathbf{V}(\nabla\mathbf{u})|^2, \tag{4.7a}$$

$$|\mathscr{A}_s| \leq c\,\varphi'_{|\nabla\mathbf{u}|}(|\tau_s\nabla\mathbf{u}|), \tag{4.7b}$$

$$|\mathscr{B}_s| \leq c\,|s|^{\alpha_1}\varphi'(|\nabla\mathbf{u}\circ T_s|), \tag{4.7c}$$

$$|\mathscr{C}_s| \leq c\,\varphi'_{|\nabla\mathbf{u}|,\omega_3}(|\tau_s\nabla\mathbf{u}|), \tag{4.7d}$$

$$|\mathscr{D}_s| \leq c\,|s|^{\alpha_2}\big(\varphi'(|\nabla\mathbf{u}\circ T_s|) + g_2 + g_2\circ T_s\big), \tag{4.7e}$$

where ω_3 is defined by $\omega_3'(t) := t^{\alpha_3}$. These inequalities follow directly from the assumptions (2.7) and (2.8) on $\mathbf{A}$, the assumptions (2.11) and (2.12) on $\mathbf{b}$, and the fundamental lemma 3 which provides the connection of $\mathbf{A}$, $\mathbf{V}$, φ, and $\{\varphi_a\}_{a\geq 0}$. As an example we will derive (4.7a) and (4.7b) in detail:

$$\begin{aligned}
&\mathscr{A}_s(x)\cdot(\tau_s\nabla\mathbf{u})(x)\\
&\quad= \Big(\mathbf{A}\big(x,\nabla\mathbf{u}(x+s)\big) - \mathbf{A}\big(x,\nabla\mathbf{u})(x)\big)\Big)\cdot(\tau_s\nabla\mathbf{u})(x)\\
&\quad\sim \varphi''(|\nabla\mathbf{u}(x+s)| + |\nabla\mathbf{u}(x)|)\,|(\tau_s\nabla\mathbf{u})(x)|^2 \qquad \text{by (2.7)}\\
&\quad\sim \varphi_{|\nabla\mathbf{u}(x)|}(|(\tau_s\nabla\mathbf{u})(x)|) \qquad \text{by Lemma 24}\\
&\quad\sim \big|\big(\tau_s\mathbf{V}(\nabla\mathbf{u})\big)(x)\big|^2 \qquad \text{by (2.10)},
\end{aligned}$$

$$\begin{aligned}
|\mathscr{A}_s(x)| &= \big|\mathbf{A}\big(x,(\nabla\mathbf{u})(x+s)\big) - \mathbf{A}\big(x,(\nabla\mathbf{u})(x)\big)\big|\\
&\leq c\,\varphi''(|\nabla\mathbf{u}(x+s)| + |\nabla\mathbf{u}(x)|)\,|(\tau_s\nabla\mathbf{u})(x)| \qquad \text{by (2.7)}\\
&\sim c\,\varphi'_{|\nabla\mathbf{u}(x)|}(|(\tau_s\nabla\mathbf{u})(x)|) \qquad \text{by Lemma 24}
\end{aligned}$$

We split $\nabla\xi$ into $\nabla\xi = \eta^q\tau_s\nabla\mathbf{u} + q\,\eta^{q-1}(\tau_s\mathbf{u})\otimes\nabla\eta$. Then

$$(I) = \langle\mathscr{A}_s, \eta^q\tau_s\nabla\mathbf{u}\rangle + \langle\mathscr{A}_s, q\,\eta^{q-1}(\tau_s\mathbf{u})\otimes\nabla\eta\rangle =: (I_1) + (I_2).$$

Analogously, we split (II) into $(II_1)+(II_2)$. By (4.7a)

$$(I_1) = \langle \mathscr{A}_s, \eta^q \tau_s \nabla \mathbf{u} \rangle \sim \int_{2Q} \eta^q |\tau_s \mathbf{V}(\nabla \mathbf{u})|^2 \, dx. \tag{4.8}$$

This term is the good term while all other terms have to be controlled. Using (4.7b) and the estimates on $\nabla\eta$ we estimate (I_2).

$$(I_2) \leq c \int_{2Q} \eta^{q-1} \varphi'_{|\nabla \mathbf{u}|}(|\tau_s \nabla \mathbf{u}|) \frac{|\tau_s \mathbf{u}|}{R} \, dx. \tag{4.9}$$

Since our good term (I_1) only carries information on $\nabla \mathbf{u}$ we have to find a way to estimate $\tau_s \mathbf{u}$ in terms of $\nabla \mathbf{u}$: The representation

$$(\tau_s \mathbf{u})(x) = \int_0^s \sum_i (\partial_i \mathbf{u})(x+\lambda) \frac{s_i}{|s|} d\lambda$$

provides the estimates

$$|(\tau_s \mathbf{u})(x)| \leq |s| \fint_0^s |(\nabla \mathbf{u} \circ T_\lambda)(x)| \, d\lambda. \tag{4.10}$$

From (4.9) and (4.10) we get

$$(I_2) \leq c \int_{2Q} \eta^{q-1} \varphi'_{|\nabla \mathbf{u}|}(|\tau_s \nabla \mathbf{u}|) \fint_0^s \frac{|s|}{R} |\nabla \mathbf{u} \circ T_\lambda| \, d\lambda \, dx. \tag{4.11}$$

Let us define

$$(J) := \eta^{q-1} \varphi'_{|\nabla \mathbf{u}|}(|\tau_s \nabla \mathbf{u}|) \frac{|h|}{R} |\nabla \mathbf{u} \circ T_\lambda|$$

Please notice the $|h|/R$ instead of $|s|/R$. We will need the remaining factor $|s|/|h|$ later. From Lemma 29 (with $a = \nabla \mathbf{u}$, $b = \nabla \mathbf{u} \circ T_s$, $\omega(t) = \frac{1}{2}t^2$, and $e = \nabla \mathbf{u} \circ T_\lambda$) we deduce

$$(J) \leq c\, \eta^{q-1} \Big(\varphi'_{|\nabla \mathbf{u} \circ T_\lambda|}(|\tau_{s-\lambda} \nabla \mathbf{u} \circ T_\lambda|) + \varphi'_{|\nabla \mathbf{u} \circ T_\lambda|}(|\tau_\lambda \nabla \mathbf{u}|) \Big) \frac{|h|}{R} |\nabla \mathbf{u} \circ T_\lambda|.$$

Now, Young's inequality (6.27), (4.4), (6.22), and (2.10d) imply

$$\begin{aligned}
(J) &\leq \delta \, (\varphi_{|\nabla \mathbf{u} \circ T_\lambda|})^* \Big(\eta^{q-1} \varphi'_{|\nabla \mathbf{u} \circ T_\lambda|}(|\tau_{s-\lambda} \nabla \mathbf{u} \circ T_\lambda|) \Big) \\
&\quad + \delta \, (\varphi_{|\nabla \mathbf{u} \circ T_\lambda|})^* \Big(\eta^{q-1} \varphi'_{|\nabla \mathbf{u} \circ T_\lambda|}(|\tau_\lambda \nabla \mathbf{u}|) \Big) \\
&\quad + c_\delta \, \varphi_{|\nabla \mathbf{u} \circ T_\lambda|} \bigg(\frac{|h|}{R} |\nabla \mathbf{u} \circ T_\lambda| \bigg) \\
&\leq c_2 \, \delta \, \eta^q \, \varphi_{|\nabla \mathbf{u} \circ T_\lambda|}(|\tau_{s-\lambda} \nabla \mathbf{u} \circ T_\lambda|) + c_2 \, \delta \, \eta^q \, \varphi_{|\nabla \mathbf{u} \circ T_\lambda|}(|\tau_\lambda \nabla \mathbf{u}|) \\
&\quad + c_\delta \frac{|h|^2}{R^2} \varphi(|\nabla \mathbf{u} \circ T_\lambda|) \\
&\sim \delta \, \eta^q \, |\tau_{s-\lambda} \mathbf{V}(\nabla \mathbf{u}) \circ T_\lambda|^2 + \delta \, \eta^q \, |\tau_\lambda \mathbf{V}(\nabla \mathbf{u})|^2 + c_\delta \frac{|h|^2}{R^2} |\mathbf{V}(\nabla \mathbf{u} \circ T_\lambda)|^2.
\end{aligned}$$

Let us combine this with (4.11) then

$$
\begin{aligned}
(I_2) \leq \delta \int_{2Q} \eta^q \frac{|s|}{|h|} \fint_0^s |\tau_{s-\lambda} \mathbf{V}(\nabla \mathbf{u}) \circ T_\lambda|^2 + |\tau_\lambda \mathbf{V}(\nabla \mathbf{u})|^2 \, d\lambda \, dx \\
+ c_\delta \frac{|h|^2}{R^2} \int_{2Q} |\mathbf{V}(\nabla \mathbf{u} \circ T_\lambda)|^2 \, dx,
\end{aligned} \tag{4.12}
$$

Note that in general for $|s| \leq |h| \leq R$

$$
\int_{2Q} \fint_0^s |(f \circ T_\lambda)(x)| \, d\lambda \, dx \leq c \int_{4Q} |f(x)| \, dx, \tag{4.13}
$$

$$
\int_{2Q} \fint_0^s |(\tau_{s-\lambda} f \circ T_\lambda)(x)| \, d\lambda \, dx \leq c \int_{4Q} \fint_0^s |\tau_\lambda f(x)| \, d\lambda \, dx. \tag{4.14}
$$

This, (4.12), and $|s| \leq |h|$ imply

$$
(I_2) \leq \delta \int_{4Q} \eta^q \frac{|s|}{|h|} \fint_0^s |\tau_\lambda \mathbf{V}(\nabla \mathbf{u})|^2 \, d\lambda \, dx + c_\delta \frac{|h|^2}{R^2} \int_{4Q} |\mathbf{V}(\nabla \mathbf{u})|^2 \, dx, \tag{4.15}
$$

We estimate (II_1) with (4.7c) and $\chi_Q \leq \eta \leq \chi_{2Q}$

$$
(II_1) = \langle \mathscr{B}_s, \eta^q \tau_s \nabla \mathbf{u} \rangle \leq c \int_{2Q} \eta^q |s|^{\alpha_1} \varphi'(|\nabla \mathbf{u} \circ T_s|) |\tau_s \nabla \mathbf{u}| \, dx.
$$

Note that by Young's inequality (6.27)

$$
\begin{aligned}
&|s|^{\alpha_1} \varphi'(|\nabla \mathbf{u} \circ T_s|) |\tau_s \nabla \mathbf{u}| \\
&\quad \leq \delta \, \varphi_{|\nabla \mathbf{u} \circ T_s|}(|\tau_s \nabla \mathbf{u}|) + c_\delta \, (\varphi_{|\nabla \mathbf{u} \circ T_s|})^* \big(|s|^{\alpha_1} \varphi'(|\nabla \mathbf{u} \circ T_s|)\big) \\
&\quad \sim \delta \, |\tau_s \mathbf{V}(\nabla \mathbf{u})|^2 + c_\delta \, |s|^{2\alpha_1} \varphi(|\nabla \mathbf{u} \circ T_s|) \qquad \text{by (2.10), (6.23).}
\end{aligned}
$$

In particular, with (2.10d)

$$
(II_1) \leq \delta \int_{2Q} \eta^q |\tau_s \mathbf{V}(\nabla \mathbf{u})|^2 \, dx + c_\delta \, |h|^{2\alpha_1} \int_{3Q} |\mathbf{V}(\nabla \mathbf{u})|^2 \, dx. \tag{4.16}
$$

We estimate (II_2) with (4.7c) and (4.10)

$$
\begin{aligned}
(II_2) &= \langle \mathscr{B}_s, q \, \eta^{q-1} (\tau_s \mathbf{u}) \otimes \nabla \eta \rangle \\
&\leq c \int_{2Q} |s|^{\alpha_1} \varphi'(|\nabla \mathbf{u} \circ T_s|) \frac{|s|}{R} \fint_0^s |\nabla \mathbf{u} \circ T_\lambda| \, d\lambda \, dx.
\end{aligned}
$$

By Young's inequality (6.26b), (4.13), and (2.10d)

$$(II_2) \le c \int_{2Q} \frac{|s|^{1+\alpha_1}}{R} \left(\varphi(|\nabla \mathbf{u} \circ T_s|) + \fint_0^s \varphi(|\nabla \mathbf{u} \circ T_\lambda|)\, d\lambda \right) dx$$
$$\le c \frac{|s|^{1+\alpha_1}}{R} \int_{4Q} |\mathbf{V}(\nabla \mathbf{u})|^2\, dx. \tag{4.17}$$

We now come to (III). By (4.7d) and (4.10)

$$(III) = \langle \mathscr{C}_s, \eta^q \tau_s \mathbf{u} \rangle$$
$$\le c \int_{2Q} \eta^q \varphi'_{|\nabla \mathbf{u}|,\omega_3}(|\tau_s \nabla \mathbf{u}|)\, |s| \fint_0^s |\nabla \mathbf{u} \circ T_\lambda|\, d\lambda\, dx.$$

Analogously to the term (J) above we estimate with (6.19)

$$(J_2) := \varphi'_{|\nabla \mathbf{u}|,\omega_3}(|\tau_s \nabla \mathbf{u}|)\, |h|\, |\nabla \mathbf{u} \circ T_\lambda|$$
$$\le \Big(\varphi'_{|\nabla \mathbf{u} \circ T_\lambda|,\omega_3}(|\tau_{s-\lambda} \nabla \mathbf{u} \circ T_\lambda|) + \varphi'_{|\nabla \mathbf{u} \circ T_\lambda|,\omega_3}(|\tau_\lambda \nabla \mathbf{u}|) \Big)\, |h|\, |\nabla \mathbf{u} \circ T_\lambda|.$$

Define the N-functions σ and κ by $\sigma'(t) := t^{\frac{\alpha_3}{2-\alpha_3}}$, $\kappa'(t) := t$. Then $\kappa'(1) = \sigma'(1) = \omega_3'(1) = 1$, $\sigma(t) \sim t^{\frac{2}{2-\alpha_3}}$, $\sigma^*(t) \sim t^{\frac{2}{\alpha_3}}$, and $\sigma^*(\omega_3'(t)) \sim t^2 \sim \kappa(t)$. Particularly, σ, κ, ω_3 satisfy the assumptions of Lemma 34. By Young's inequality (6.31), $\varphi_{a,\kappa} = \varphi_a$ and (2.10)

$$(J_2) \le \delta\, \varphi_{|\nabla \mathbf{u} \circ T_\lambda|}(|\tau_{s-\lambda} \nabla \mathbf{u} \circ T_\lambda|) + \delta\, \varphi_{|\nabla \mathbf{u} \circ T_\lambda|}(|\tau_\lambda \nabla \mathbf{u}|)$$
$$+ c_\delta\, \varphi_{|\nabla \mathbf{u} \circ T_\lambda|,\sigma}\big(|h|\, |\nabla \mathbf{u} \circ T_\lambda|\big)$$
$$\sim \delta\, |\tau_{s-\lambda} \mathbf{V}(\nabla \mathbf{u}) \circ T_\lambda|^2 + \delta\, |\tau_\lambda \mathbf{V}(\nabla \mathbf{u})|^2 + c_\delta\, \sigma(|h|)\, \varphi\big(|\nabla \mathbf{u} \circ T_\lambda|\big).$$
$$\le c\,\delta\, |\tau_{s-\lambda} \mathbf{V}(\nabla \mathbf{u}) \circ T_\lambda|^2 + \delta\, |\tau_\lambda \mathbf{V}(\nabla \mathbf{u})|^2 + c_\delta\, |h|^{\frac{2}{2-\alpha_3}}\, |\mathbf{V}(\nabla \mathbf{u} \circ T_\lambda)|^2.$$

Therefore

$$(III) \le \delta \int_{2Q} \eta^q \fint_0^s \frac{|s|}{|h|} |\tau_\lambda \mathbf{V}(\nabla \mathbf{u})|^2\, d\lambda\, dx + c_\delta\, |h|^{\frac{2}{2-\alpha_3}} \int_{4Q} |\mathbf{V}(\nabla \mathbf{u})|^2\, dx, \tag{4.18}$$

where we have used (4.13) and (4.14) once more. We finally get to last term (IV). With (4.7e), (4.10), (4.13), and (2.10d)

$$(IV) = \langle \mathscr{D}_s, \eta^q \tau_s \mathbf{u} \rangle$$
$$\le c \int_{2Q} |s|^{1+\alpha_2} \big(\varphi'(|\nabla \mathbf{u} \circ T_s|) + g_2 + g_2 \circ T_s\big) \fint_0^s |\nabla \mathbf{u} \circ T_\lambda|\, d\lambda\, dx$$
$$\le c\, |s|^{1+\alpha_2} \int_{2Q} \fint_0^s \varphi(|\nabla \mathbf{u}| \circ T_s) + \varphi^*(g_2) + \varphi^*(g_2 \circ T_s)\, d\lambda\, dx \tag{4.19}$$
$$\le c\, |s|^{1+\alpha_2} \left(\int_{4Q} |\mathbf{V}(\nabla \mathbf{u})|^2\, dx + \int_{4Q} \varphi^*(g_2)\, dx \right).$$

If we combine all estimates (4.8), (4.15), (4.16), (4.17), (4.18), and (4.19), apply (4.13) to all terms involving T_s, and divide by $|Q|$ we get (4.2). Note that for any integrable function $k : \mathbb{R}^n \to \mathbb{R}$ by Fubini holds

$$\fint_0^h \frac{|s|}{|h|} \fint_0^s |k(\lambda)|\, d\lambda\, ds = \fint_0^h \int_\lambda^h \frac{1}{|h|}\, ds\, |k(\lambda)|\, d\lambda \leq \fint_0^h |k(\lambda)|\, d\lambda. \tag{4.20}$$

Thus (4.3) follow from (4.2) by application of $\fint_0^h ds$. This proves the Lemma.

We are able to get rid of the first term on the right hand side in (4.3) with a Giaquinta-Modica type lemma.

Lemma 13 *Let $\gamma_1, \ldots, \gamma_M : (0,\infty) \times (0,\infty) \to [0,\infty)$ be such that $\gamma_m(R,|h|)$, $m = 1, \ldots, M$, is non-decreasing in R and $|h|$. Let $\mathbf{v} \in L^2_{\mathrm{loc}}(\Omega)$, $w_1, \ldots, w_M \in L^1_{\mathrm{loc}}(\Omega)$ be such that the following holds: For every $\delta > 0$ there exists $c_\delta > 0$ such that for every cube $Q \subset \Omega$ with side length R and $4Q \Subset \Omega$ and every $h \in \mathbb{R}^n \setminus \{0\}$ with $|h| \leq R$ holds*

$$\begin{aligned} \fint_0^h \fint_Q |\tau_s \mathbf{v}|^2\, dx\, ds &\leq \delta \fint_0^h \fint_{2Q} |\tau_s \mathbf{v}|^2\, dx\, ds \\ &\quad + c_\delta \sum_{m=1}^M \gamma_m(R,h) \fint_{4Q} |w_m|\, dx. \end{aligned} \tag{4.21}$$

Then there exists $N_2 = N_2(n)$ and $\tilde{c} > 0$ such that for every cube $Q_0 \subset \Omega$ with side length R_0 and $5Q_0 \Subset \Omega$ and for every $h_0 \in \mathbb{R}^n \setminus \{0\}$ with $|h_0| \leq \frac{R}{10}$ holds

$$\fint_{Q_0} |\tau_{h_0} \mathbf{v}|^2\, dx \leq \tilde{c} \sum_{m=1}^M \gamma_m(N_2 R_0, N_2 |h_0|) \fint_{5Q_0} |w_m|\, dx.$$

Proof Let Q_0, R_0, and h_0 be as specified and let $\Omega_0 := 5\,Q_0$. We construct a family $\{W_j\}_{j\geq 1}$ of cubes in the following way:

(a) Split the set $5Q_0$ into 5^n equivalent cubes. Take these 5^n cubes as our initial family of cubes. In particular, Q_0 is contained in this family.
(b) Replace any cube Q of the family which does not satisfy $4Q \subset \Omega_0$ into 2^n equivalent cubes. Repeat this step recursively.

Then we obtain a family of cubes which we denote by $\{W_j\}_{j\geq 1}$ with the following properties:

(i) $\Omega_0 = \bigcup W_j$ up to a set of measure zero.
(ii) $\Omega_0 = \bigcup 4W_j$.
(iii) The W_j, $j \geq 1$, are pairwise disjoint.
(iv) $Q_0 \in \{W_j\}$.
(v) There exists $N_1 = N_1(n) \in \mathbb{N}$ such that $\mathcal{N}(j) \leq N_1$ for all $j \in \mathbb{N}$, where $\mathcal{N}(j) := \#\{k : 4W_k \cap W_j \neq \emptyset\}$.

(vi) There exists $N_2 = N_2(n) \in \mathbb{N}$ such that $\frac{1}{N_2} R_k \le R_j \le N_2 R_k$ for every $k \in \mathscr{N}(j)$, where R_j is the side length of W_j.

Set $h_j := \frac{R_j}{R_0} h_0$ and $\omega_j := \left(\frac{R_j}{R_0}\right)^2$. Especially, $\frac{h_j}{R_j} = \frac{h_0}{R_0}$. We apply (4.21) for every W_j and h_j, multiply the result by $|W_j|\,\omega_j$ and sum up. We obtain

$$
\begin{aligned}
\sum_j \omega_j \fint_0^{h_j} \int_{W_j} |\tau_s \mathbf{v}|^2 \,dx\,ds \le \delta \sum_j \omega_j \fint_0^{h_j} \int_{2W_j} |\tau_s \mathbf{v}|^2 \,dx\,ds \\
+ c_\delta \sum_{m=1}^M \gamma_m(R_j, |h_j|) \sum_j \omega_j \int_{4W_j} |w_m|\,dx \\
=: (I) + (II).
\end{aligned} \tag{4.22}
$$

Note that by triangle inequality

$$
\begin{aligned}
\fint_0^{h_j} \int_{2W_j} |\tau_s \mathbf{v}|^2 \,dx\,ds &\le \fint_0^{h_j} \int_{2W_j} N_2 \sum_{j=1}^{N_2} |\tau_{s/N_2} \mathbf{v} \circ T_{\frac{j}{N_2}s}|^2 \,dx\,ds \\
&\le N_2^2 \fint_0^{h_j} \int_{4W_j} |\tau_{s/N_2} \mathbf{v}|^2 \,dx\,ds \\
&= N_2^2 \fint_0^{h_j/N_2} \int_{4W_j} |\tau_s \mathbf{v}|^2 \,dx\,ds.
\end{aligned}
$$

This and $4W_j \subset \cup_{k\in\mathscr{N}(j)} W_k$ implies

$$
(I) \le \delta N_2^2 \sum_k \sum_{k\in\mathscr{N}(j)} \omega_j \fint_0^{h_j/N_2} \int_{W_k} |\tau_s \mathbf{v}|^2 \,dx\,ds.
$$

From (vi) we deduce $h_j \le N_2 h_k$ and $\omega_j \le N_2^2 \omega_k$, so

$$
\begin{aligned}
(I) &\le \delta N_2^5 \sum_k \sum_{k\in\mathscr{N}(j)} \omega_k \fint_0^{h_k} \int_{W_k} |\tau_s \mathbf{v}|^2 \,dx\,ds \\
&\le \delta\, c N_1 N_2^5 \sum_k \omega_k \fint_0^{h_k} \int_{W_k} |\tau_s \mathbf{v}|^2 \,dx\,ds.
\end{aligned} \tag{4.23}
$$

Analogously, we have

$$
(II) \le c_\delta\, c N_1 N_2^2 \sum_{m=1}^M \gamma_m(N_2 R_k, N_2 h_k) \sum_k \omega_k \int_{W_k} |w_m|\,dx. \tag{4.24}
$$

If we combine (4.22), (4.23), and (4.24) and absorb (I) for small $\delta > 0$ on the left-hand side then with $\tilde{c} = \tilde{c}(N_1 N_2)$

$$\sum_j \omega_j \fint_0^{h_j} \int_{W_j} |\tau_s \mathbf{v}|^2 \, dx \, ds \leq \tilde{c} \sum_{m=1}^{M} \gamma_m(N_2 R_k, N_2 h_k) \sum_k \omega_k \int_{W_k} |w_m| \, dx$$
$$\leq \tilde{c} \sum_{m=1}^{M} \gamma_m(N_2 R_k, N_2 h_k) \int_{6Q_0} |w_m| \, dx,$$

where we have used $\omega_k \leq 1$ and (ii). Since $Q_0 \in \{W_j\}$ by (iv) and $\omega_0 = 1$ we get

$$\fint_0^{h_0} \int_{Q_0} |\tau_s \mathbf{v}|^2 \, dx \, ds \leq \tilde{c} \sum_{m=1}^{M} \gamma_m(N_2 R_k, N_2 h_k) \int_{6Q_0} |w_m| \, dx.$$

This proves the lemma.

We are now prepared to prove Theorem 11.

Proof (Proof of Theorem 11) Let Q, R, h be as specified. From (4.3) we know that the requirements of Lemma 13 are satisfied with

$$\gamma_1(R, |h|) := \beta(R, |h|), \qquad w_1 := \varphi(|\nabla \mathbf{u}|),$$
$$\gamma_1(R, |h|) := \beta(R, |h|), \qquad w_2 := \varphi^*(g_2).$$

Thus Lemma 13 and $\gamma_j(N_2 R, N_2|h|) \leq c\, \gamma_j(R, |h|)$ implies

$$\begin{aligned} &\fint_0^h \fint_Q |\tau_s \mathbf{V}(\nabla \mathbf{u})|^2 \, dx \, ds \\ &\qquad \leq c\, \beta(R, |h|) \fint_{6Q} |\mathbf{V}(\nabla \mathbf{u})|^2 \, dx + c\, |h|^{1+\alpha_2} \fint_{6Q} \varphi^*(g_2) \, dx, \end{aligned} \tag{4.25}$$

We use (4.25) to estimate the first term on the right-hand side of (4.2). We get

$$\fint_Q |\tau_s \mathbf{V}(\nabla \mathbf{u})|^2 \, dx \leq c\, \beta(R, |h|) \fint_{20Q} |\mathbf{V}(\nabla \mathbf{u})|^2 \, dx + c\, |s|^{1+\alpha_2} \fint_{4Q} \varphi^*(g_2) \, dx.$$

This proves Theorem 11.

5 Dimension of the Singular Set

For a function $\mathbf{f} \in L^1_{\text{loc}}(\Omega)$ with $\Omega \subset \mathbb{R}^n$ open we define the singular sets

$$\Sigma_1(\mathbf{f}) := \left\{ x \in \Omega : \liminf_{\rho \searrow 0} \fint_{B_\rho(x)} |\mathbf{f} - \langle \mathbf{f} \rangle_{B_\rho(x)}| \, dy > 0 \right\},$$
$$\Sigma_2(\mathbf{f}) := \left\{ x \in \Omega : \nexists \lim_{\rho \searrow 0} \langle \mathbf{f} \rangle_{B_\rho(x)} \right\} \cup \left\{ x \in \Omega : \limsup_{\rho \searrow 0} |\langle \mathbf{f} \rangle_{B_\rho(x)}| = \infty \right\},$$

where $B_\rho(x)$ is a ball centered at x with radius ρ. Further, define $\Sigma(\mathbf{f}) := \Sigma_1(\mathbf{f}) \cup \Sigma_2(\mathbf{f})$. By $\mathscr{H}^{(\beta)}$ we denote the β-dimensional Hausdorff measure. To estimate the Hausdorff dimension of $\Sigma(\mathbf{f})$ we will need the following theorem.

Theorem 14 *Let $\Omega \subset \mathbb{R}^n$ be open and let $0 < \alpha$. Assume that $f \in \mathscr{N}^{p,\alpha}(\Omega)$. Especially, $f \in L^p(\Omega)$ and there exists $c > 0$ such that for any $\tilde{\Omega} \Subset \Omega$ and all $0 < h < dist(\tilde{\Omega}, \partial\Omega)$ holds*

$$\|\tau_h f\|_{L^p(\tilde{\Omega})} \leq c\,|h|^\alpha.$$

Then for any $\beta > n - p\,\alpha$ with $\beta \geq 0$ we have $\mathscr{H}^{(\beta)}(\Sigma(f)) = 0$. As a consequence the Hausdorff dimension of $\Sigma(f)$ is less or equal to $n - p\,\alpha$.

Proof It has been shown in Theorem 1 of [16] under the restriction $0 < \alpha < \frac{n}{p}$ that $\mathscr{H}^{(\beta)}(\Sigma_2(f)) = 0$. The restriction $\alpha < \frac{n}{p}$ however was only used to ensure that the case $\beta < 0$ cannot occur and $\mathscr{H}^{(\beta)}$ is well defined. In our formulation this condition is replaced by $\beta \geq 0$. The proof in [16] remains true without any changes. Horihata construct a function ϕ_∞ to which he applies the fundamental lemma of Giusti [14], i. e. $\mathscr{H}^{(\beta)}(E_\beta) = 0$ where

$$E_\beta := \Big\{x \in \Omega\, :\, \limsup_{\rho \searrow 0} \rho^{-\beta} \int_{B_\rho(x)} |\phi_\infty(y)|\, dy > 0\Big\}.$$

For any $x_0 \notin E_\beta$ Horihata shows on p. 202 that for $0 < r < R < \tilde{\delta}/2$ with $\tilde{\delta} := \operatorname{dist}(x, \partial\Omega)$ holds

$$\big|\langle \mathbf{f}\rangle_{B_r(x_0)} - \langle \mathbf{f}\rangle_{B_R(x_0)}\big| \leq c(n,\beta,\varepsilon,p)\, R^{\varepsilon/p}.$$

But considering p. 203, second line of (19), and p. 204, second line of (24), it can easily be seen that as a byproduct he shows

$$\fint_{B_r(x_0)} \big|\mathbf{f} - \langle \mathbf{f}\rangle_{B_r(x_0)}\big|\, dy + \fint_{B_R(x_0)} \big|\mathbf{f} - \langle \mathbf{f}\rangle_{B_r(x_0)}\big|\, dy \leq c(n,\beta,\varepsilon,p)\, R^{\varepsilon/p}.$$

The limit $R \searrow 0$ directly implies that any $x_0 \notin E_\beta$ satisfies $x_0 \notin \Sigma_1(\mathbf{f})$. Therefore, $\mathscr{H}^{(\beta)}(E_\beta) = 0$ gives $\mathscr{H}^{(\beta)}(\Sigma_1(\mathbf{f})) = 0$. This proves the Theorem.

Remark 15 For $\mathbf{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ let us define

$$\Sigma_3(\mathbf{f}) := \left\{x \in \Omega : \limsup_{\rho \searrow 0} \fint_{B_\rho(x)} |\mathbf{f}|\, dy = \infty\right\}.$$

Then from

$$\fint_{B_\rho(x)} |\mathbf{f}|\, dy \leq \fint_{B_\rho(x)} |\mathbf{f} - \langle \mathbf{f}\rangle_{B_\rho(x)}|\, dy + |\langle \mathbf{f}\rangle_{B_\rho(x)}|$$

it follows that $\Sigma_3(\mathbf{f}) \subset \Sigma_1(\mathbf{f}) \cup \Sigma_2(\mathbf{f}) = \Sigma(\mathbf{f})$.

Remark 16 Please note that it would also be possible to prove Theorem 14 by embeddings from $\mathscr{N}^{2,\alpha}$ the to Bessel potential spaces $L^{\alpha-\varepsilon,2}$ with $\varepsilon > 0$ and use the classical capacity estimates for these spaces. The limit $\varepsilon \to 0$ provides an alternative proof of Theorem 14. See [2] for further references.

Remark 17 Note that Theorem 14 and Remark 15 can easily be generalized in the following sense: In the construction of Σ_1 and Σ_2 the balls $B_\rho(x)$ can be replaced by cubes $Q_\rho(x)$ (sides parallel to the axis). It is even possible to use balls B_ρ or cubes Q_ρ (with sides parallel to the axis) which are not centered at x but only contain x. This follows easily from the fact that for any B with $x \in B$ the expressions

$$\fint_B |f - \langle f\rangle_B|\,dy \quad \text{and} \quad \fint_B |f|\,dx.$$

with $B \ni x$ can be estimated from above by the same expressions with B replaced by some larger ball $B_\rho(x)$ centered at x.

We will now estimate the singularities of $\mathbf{V}(\nabla \mathbf{u})$.

Theorem 18 *Let* $\mathbf{u}$ *be a weak solution of system* (1.1)*. Define*

$$\alpha := \min\left\{\tfrac{1}{2-\alpha_3}, \alpha_1, \frac{1+\alpha_2}{2}\right\} \leq 1.$$

Then for any $\beta > n - 2\alpha$ *with* $\beta \geq 0$ *holds*

$$\mathscr{H}^{(\beta)}\big(\Sigma(\mathbf{V}(\nabla \mathbf{u}))\big) = 0.$$

Especially, the singular set $\Sigma(\mathbf{V}(\nabla \mathbf{u}))$ *has Hausdorff dimension less or equal to* $n - 2\alpha$*.*

Proof Let Q_j be a countable sequence of cubes with $\Omega \subset \bigcup_j Q_j$ and $20Q_j \Subset \Omega$. Then from Theorem 11 we know that $\mathbf{V}(\nabla \mathbf{u}) \in \mathscr{N}^{2,\alpha}(Q_j)$. Hence, it follows from Theorem 14 that

$$\mathscr{H}^{(\beta)}(\Sigma(\mathbf{V}) \cap Q_j) = 0.$$

This immediately implies $\mathscr{H}^{(\beta)}(\Sigma(\mathbf{V})) = 0$ which proves the Theorem.

6 Appendix

For $\mathbf{P}_0, \mathbf{P}_1 \in \mathbb{R}^{N\times n}$, $\theta \in [0,1]$ we define $\mathbf{P}_\theta := (1-\theta)\mathbf{P}_0 + \theta\mathbf{P}_1$. The following fact is standard and can e. g. be found in [1].

Lemma 19 *Let* $\alpha > -1$ *then uniformly in* $\mathbf{P}_0, \mathbf{P}_1 \in \mathbb{R}^{N\times n}$ *with* $|\mathbf{P}_0| + |\mathbf{P}_1| > 0$ *holds*

$$\big(|\mathbf{P}_0| + |\mathbf{P}_1|\big)^\alpha \sim \int_0^1 |\mathbf{P}_\theta|^\alpha\, d\theta. \tag{6.1}$$

Lemma 20 *Let φ be an N-function with $\Delta_2(\{\varphi,\varphi^*\}) < \infty$. Then uniformly for all $\mathbf{P}_0, \mathbf{P}_1 \in \mathbb{R}^{N\times n}$ with $|\mathbf{P}_0| + |\mathbf{P}_1| > 0$ holds*

$$\int_0^1 \frac{\varphi'(|\mathbf{P}_\theta|)}{|\mathbf{P}_\theta|}\, d\theta \sim \frac{\varphi'(|\mathbf{P}_0| + |\mathbf{P}_1|)}{|\mathbf{P}_0| + |\mathbf{P}_1|}, \tag{6.2}$$

where the constants only depend on $\Delta_2(\{\varphi,\varphi^*\})$.

Proof From $\varphi(t) \sim t\,\varphi'(t)$ and the convexity of φ we derive

$$\int_0^1 \frac{\varphi'(|\mathbf{P}_\theta|)}{|\mathbf{P}_\theta|}\, d\theta \geq c \int_0^1 \frac{\varphi(|\mathbf{P}_\theta|)}{(|\mathbf{P}_0| + |\mathbf{P}_1|)^2}\, d\theta \geq c\,\frac{\varphi(\int_0^1 |\mathbf{P}_\theta|\, d\theta)}{(|\mathbf{P}_0| + |\mathbf{P}_1|)^2}$$

Since by Lemma 19 $\int_0^1 |\mathbf{P}_\theta|\, d\theta \sim |\mathbf{P}_0| + |\mathbf{P}_1|$ there follows

$$\int_0^1 \frac{\varphi'(|\mathbf{P}_\theta|)}{|\mathbf{P}_\theta|}\, d\theta \geq c\,\frac{\varphi(|\mathbf{P}_0| + |\mathbf{P}_1|)}{(|\mathbf{P}_0| + |\mathbf{P}_1|)^2} \geq c\,\frac{\varphi'(|\mathbf{P}_0| + |\mathbf{P}_1|)}{|\mathbf{P}_0| + |\mathbf{P}_1|}.$$

This proves the first part. Since $\Delta_2(\varphi^*) < \infty$, there exists (as in the proof of Theorem 7) some $\theta \in (0,1)$ and an N–function ρ with $\varphi^\theta \sim \rho$ and $\Delta_2(\{\rho,\rho^*\}) < \infty$. Note that θ and $\Delta_2(\{\rho,\rho^*\})$ depend only on $\Delta_2(\{\varphi,\varphi^*\})$. From $\varphi(t) \sim t\,\varphi'(t)$, $\varphi(t) \sim (\rho(t))^{\frac{1}{\theta}}$, and $\rho(t) \sim t\rho'(t)$ we deduce

$$\int_0^1 \frac{\varphi'(|\mathbf{P}_\theta|)}{|\mathbf{P}_\theta|}\, d\theta \sim \int_0^1 \big(\rho'(|\mathbf{P}_\theta|)\big)^{\frac{1}{\theta}}\, |\mathbf{P}_\theta|^{\frac{1}{\theta}-2}\, d\theta.$$

Using the monotonicity of ρ' and Lemma 19 with $\alpha := 1/\theta - 2$ we get

$$\begin{aligned}\int_0^1 \frac{\varphi'(|\mathbf{P}_\theta|)}{|\mathbf{P}_\theta|}\, d\theta &\leq c \int_0^1 \big(\rho'(|\mathbf{P}_0| + |\mathbf{P}_1|)\big)^{\frac{1}{\theta}}\, |\mathbf{P}_\theta|^{\frac{1}{\theta}-2}\, d\theta\\ &\leq c\,\big(\rho'(|\mathbf{P}_0| + |\mathbf{P}_1|)\big)^{\frac{1}{\theta}}\, \big(|\mathbf{P}_0| + |\mathbf{P}_1|\big)^{\frac{1}{\theta}-2}\\ &\sim \frac{\varphi'(|\mathbf{P}_0| + |\mathbf{P}_1|)}{|\mathbf{P}_0| + |\mathbf{P}_1|}.\end{aligned}$$

This proves the lemma.

Lemma 21 *Let φ be as in Assumption 1. Let $\Phi : \mathbb{R}^{N\times n} \to \mathbb{R}^{\geq 0}$ be given by $\Phi(\mathbf{Q}) := \varphi(|\mathbf{Q}|)$ and let $\mathbf{A}(\mathbf{Q}) := (\nabla_{N\times n}\Phi)(\mathbf{Q})$. Then $\mathbf{A}(\mathbf{Q}) = \varphi'(|\mathbf{Q}|)\frac{\mathbf{Q}}{|\mathbf{Q}|}$ for $\mathbf{Q} \neq \mathbf{0}$, $\mathbf{A}(\mathbf{0}) = \mathbf{0}$, and $\mathbf{A}$ satisfies (2.7).*

Proof Note that $\varphi'(0) = 0$, since φ is an $N-$function. Observe that for all $\mathbf{Q} \in \mathbb{R}^{N\times n} \setminus \{\mathbf{0}\}$

$$(\partial_{jk}\partial_{lm}\Phi)(\mathbf{Q}) = \varphi'(|\mathbf{Q}|)\left(\frac{\delta_{jk,lm}}{|\mathbf{Q}|} - \frac{Q_{jk}Q_{lm}}{|\mathbf{Q}|^3}\right) + \varphi''(|\mathbf{Q}|)\frac{Q_{jk}}{|\mathbf{Q}|}\frac{Q_{lm}}{|\mathbf{Q}|}.$$

Especially, with (2.6)

$$|(\partial_{jk}\partial_{lm}\Phi)(\mathbf{Q})| \leq c\,\frac{\varphi'(|\mathbf{Q}|)}{|\mathbf{Q}|} + c\,\varphi''(|\mathbf{Q}|) \leq c\,\frac{\varphi'(|\mathbf{Q}|)}{|\mathbf{Q}|}. \tag{6.3}$$

Moreover,

$$\begin{aligned} A_{jk}(\mathbf{P}) - A_{jk}(\mathbf{Q}) &= (\partial_{jk}\Phi)(\mathbf{P}) - (\partial_{jk}\Phi)(\mathbf{Q}) \\ &= \sum_{lm}\int_0^1 (\partial_{jk}\partial_{lm}\Phi)([\mathbf{Q},\mathbf{P}]_s)(P_{lm}-Q_{lm})\,ds, \end{aligned} \tag{6.4}$$

where $[\mathbf{Q},\mathbf{P}]_s := (1-s)\,\mathbf{Q} + s\,\mathbf{P}$. So by (6.3), Lemma 20, and (2.6)

$$\begin{aligned} |\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})| &\le c\int_0^1 \frac{\varphi'(|[\mathbf{Q},\mathbf{P}]_s|)}{|[\mathbf{Q},\mathbf{P}]_s|}\,ds\,|\mathbf{P}-\mathbf{Q}| \\ &\le c\,\frac{\varphi'(|\mathbf{P}|+|\mathbf{Q}|)}{|\mathbf{P}|+|\mathbf{Q}|}\,|\mathbf{P}-\mathbf{Q}| \le c\,\varphi''(|\mathbf{P}|+|\mathbf{Q}|)\,|\mathbf{P}-\mathbf{Q}|. \end{aligned}$$

On the other hand due to (2.6) there exists $\varepsilon > 0$ with $\varphi'(t)/t > \varepsilon\,\varphi''(t)$. So by (6.4) for $\mathbf{G},\mathbf{B} \in \mathbb{R}^{N\times n}$ with $\mathbf{G} \neq \mathbf{0}$ holds

$$\begin{aligned} \sum_{lm} B_{jk}(\partial_{jk}\partial_{lm}\Phi)(\mathbf{G})B_{lm} &= \frac{\varphi'(|\mathbf{G}|)}{|\mathbf{G}|}\left(|\mathbf{B}|^2 - \frac{|\mathbf{B}\mathbf{G}|^2}{|\mathbf{G}|^2}\right) + \varphi''(|\mathbf{G}|)\frac{|\mathbf{B}\mathbf{G}|^2}{|\mathbf{G}|^2} \\ &\ge \varepsilon\,\varphi''(|\mathbf{G}|)\left(|\mathbf{B}|^2 - \frac{|\mathbf{B}\mathbf{G}|^2}{|\mathbf{G}|^2}\right) + \varphi''(|\mathbf{G}|)\frac{|\mathbf{B}\mathbf{G}|^2}{|\mathbf{G}|^2} \\ &\ge \varepsilon\,\varphi''(|\mathbf{G}|)|\mathbf{B}|^2. \end{aligned}$$

This, (6.4), and Lemma 20 imply

$$\begin{aligned} \langle \mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q}), \mathbf{P}-\mathbf{Q}\rangle &\ge \varepsilon\int_0^1 \varphi''(([\mathbf{Q},\mathbf{P}]_s)|\mathbf{P}-\mathbf{Q}|^2\,ds \\ &\ge \varepsilon\,c\,\varphi''(|\mathbf{P}|+|\mathbf{Q}|)\,|\mathbf{P}-\mathbf{Q}|^2. \end{aligned}$$

This proves the lemma.

We will now introduce some auxiliary N-functions and prove some of their fundamental properties.

Definition 22 Let φ, ω be N-functions with $\Delta_2(\{\varphi,\varphi^*,\omega,\omega^*\}) < \infty$. Further assume that $\omega'(1) = 1$. Then for $a \ge 0$ we define $\varphi'_{a,\omega}(t) : \mathbb{R}^{\ge 0} \to \mathbb{R}^{\ge 0}$ by

$$\varphi'_{a,\omega}(t) := \varphi'(a+t)\,\omega'\left(\frac{t}{a+t}\right). \tag{6.5}$$

Further we define $\varphi_{a,\omega} : \mathbb{R}^{\ge 0} \to \mathbb{R}^{\ge 0}$ by $\varphi_{a,\omega}(t) := \int_0^t \varphi'_{a,\omega}(s)\,ds$.

By $\varphi_a(t)$ we denote the function φ_{a,ω_0} with $\omega'_0(t) = t$, i. e.

$$\varphi'_a(t) := \varphi'(a+t)\,\frac{t}{a+t}. \tag{6.6}$$

We remark that the requirement $\omega'(1) = 1$ is symmetric with respect to $\omega \leftrightarrow \omega^*$, since $\omega'(1) = 1$ implies $(\omega^*)'(1) = (\omega')^{-1}(1) = 1$. Thus φ, ω satisfy the be requirements of Definition 22 if and only if φ^*, ω^* satisfy the requirements.

Lemma 23 *Let φ, ω be as in Definition 22. Then for all $a \geq 0$ the function $\varphi_{a,\omega}$ is an N-function and $\Delta_2(\{\varphi_{a,\omega}\}_{a\geq 0}) < \infty$, i. e. the family $\varphi_{a,\omega}$ satisfies the Δ_2-condition uniformly in $a \geq 0$.*

Proof The assertion is obvious for $a = 0$, since $\varphi'_{0,\omega} = \varphi'$. If $a > 0$ then $\varphi'(a + t)$ and $\omega'(t/(t+a))$ are strictly in increasing, so $\varphi'_{a,\omega}(t)$ is strictly increasing. Moreover, $\varphi'_{a,\omega}(0) = \varphi'(a)\,\omega'(0) = 0$. Thus $\varphi_{a,\omega}$ is an N-function.

Due to (2.3) and $\Delta_2(\{\varphi, \omega\}) < \infty$ there holds $\varphi'(t) \sim \varphi'(2t)$ and $\omega'(t) \sim \omega'(2t)$ uniformly in $t \geq 0$. Moreover, for all $a, t \geq 0$ holds $a + 2t \sim a + t$ and $2t/(a+2t) \sim t/(a+t)$. Thus

$$\varphi'_{a,\omega}(2t) = \varphi'(a+2t)\,\omega'\left(\frac{2t}{a+2t}\right) \sim \varphi'(a+t)\,\omega'\left(\frac{t}{a+t}\right) = \varphi'_{a,\omega}(t)$$

uniformly in $a, t \geq 0$. Again (2.3) implies that $\varphi_{a,\omega}(2t) \sim \varphi_{a,\omega}(t)$ uniformly in $a, t \geq 0$. This proves the assertion.

Lemma 24 *Let φ satisfy Assumption 1. Then uniformly in $s, t \in \mathbb{R}^n$, $|s| + |t| > 0$*

$$\begin{aligned} \varphi''(|s|+|t|)\,|s-t| &\sim \varphi'_{|s|}(|s-t|), \\ \varphi''(|s|+|t|)\,|s-t|^2 &\sim \varphi_{|s|}(|s-t|). \end{aligned} \tag{6.7}$$

Proof Due to (2.3) and $\Delta_2(\varphi) < \infty$ there holds $\varphi'(r) \sim \varphi'(2r)$ uniformly in $r \geq 0$. Moreover, $|s| + |t| \sim |s| + |s-t|$ uniformly in $s, t \in \mathbb{R}^n$. Thus

$$\varphi''(|s|+|t|) \sim \frac{\varphi'(|s|+|t|)}{|s|+|t|} \sim \frac{\varphi'(|s|+|s-t|)}{|s|+|s-t|} = \frac{\varphi'_{|s|}(|s-t|)}{|s-t|}.$$

This proves the first inequality in (6.7). The second follows from (2.3).

Lemma 25 *Let φ be as in Assumption 1. Then also φ^* satisfies the Assumption 1. If we define the N-function ψ for $t > 0$ by*

$$\psi'(t) := \sqrt{\phi'(t)\,t}$$

then ψ and ψ^* satisfy the Assumption 1. Moreover, $\psi''(t) \sim \sqrt{\varphi''(t)}$ uniformly in $t > 0$.

Proof From $(\varphi^*)'(t) = (\varphi')^{-1}(t)$, (2.6), and (2.3) (with φ replaced by φ^*) we deduce for $t > 0$

$$(\varphi^*)''(t) = \frac{1}{\varphi''((\varphi^*)'(t))} \sim \frac{((\varphi^*)'(t))^2}{\varphi((\varphi^*)'(t))} \sim \frac{((\varphi^*)'(t))^2}{(\varphi^*)(t)} \sim \frac{\varphi^*(t)}{t^2}.$$

This proves that φ^* satisfies Assumption 1. From $\Delta_2(\varphi) < \infty$ we deduce $\varphi'(2t) \sim \varphi'(t)$, $\psi'(2t) \sim \psi'(t)$, and $\psi(2t) \sim \psi(t)$. Especially, $\Delta_2(\psi') < \infty$. Let $K \geq 64$ then with repetitive use of (2.2) and the monotonicity of φ' we estimate for all $t \geq 0$

$$\begin{aligned} K\,\psi\left(\frac{2t}{K}\right) &\leq 2t\,\psi'\left(\frac{2t}{K}\right) = 2t\sqrt{\varphi'\left(\frac{2t}{K}\right)\cdot\frac{2t}{K}} \leq 2t\sqrt{\varphi'\left(\frac{t}{2}\right)\cdot\frac{2t}{K}} \\ &= \frac{4}{\sqrt{K}}\,t\,\psi'\left(\frac{t}{2}\right) \leq \frac{8}{\sqrt{K}}\,\psi(t) \leq \psi(t). \end{aligned}$$

Due to (2.4) and (2.5) this is equivalent to $K\,\psi^*(t/2) \geq \psi^*(t)$ for all $t \geq 0$. This proves $\Delta_2(\psi^*) < \infty$. Moreover, for $t > 0$ we deduce from (2.6)

$$\psi''(t) = \tfrac{1}{2}\big(t\,\varphi'(t)\big)^{-1/2}\big(t\,\varphi''(t) + \varphi'(t)\big) \sim \sqrt{\varphi''(t)}.$$

Overall, we have shown that ψ satisfies Assumption 1. Thus by the first part of the Lemma ψ^* also satisfies Assumption 1.

We are now able to prove Lemma 3:

Proof **(Proof of Lemma 3)** Let $\mathbf{A}, \varphi, \psi, \mathbf{V}$ be as in Lemma 3. Due to (2.7) holds uniformly in $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N\times n}$ and $x \in \Omega$

$$\big(\mathbf{A}(x,\mathbf{P}) - \mathbf{A}(x,\mathbf{Q})\big)\cdot\big(\mathbf{P}-\mathbf{Q}\big) \sim |\mathbf{P}-\mathbf{Q}|^2\,\varphi''\big(|\mathbf{P}|+|\mathbf{Q}|\big). \tag{6.8}$$

On the other hand by Lemma 24 holds uniformly in $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N\times n}$

$$|\mathbf{P}-\mathbf{Q}|^2\,\varphi''\big(|\mathbf{P}|+|\mathbf{Q}|\big) \sim \varphi_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|). \tag{6.9}$$

Moreover, by Lemma 25 and Lemma 21 the estimates (2.7) holds with $\mathbf{A}$ and φ replaced by $\mathbf{V}$ and ψ. This and Lemma 25 implies

$$\big|\mathbf{V}(\mathbf{P}) - \mathbf{V}(\mathbf{Q})\big|^2 \sim \big(|\mathbf{P}-\mathbf{Q}|\,\psi''(|\mathbf{P}|+|\mathbf{Q}|)\big)^2 \tag{6.10}$$

$$\sim |\mathbf{P}-\mathbf{Q}|^2\,\varphi''\big(|\mathbf{P}|+|\mathbf{Q}|\big) \tag{6.11}$$

uniformly in $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N\times n}$. The combination of (6.8), (6.9), and (6.10) prove (2.10a), (2.10b), and (2.10c), whereas (2.10d) is just the special case $\mathbf{P} = \mathbf{0}$ using $\mathbf{A}(\mathbf{0}) = \mathbf{V}(\mathbf{0}) = \mathbf{0}$. This proves the Lemma.

Lemma 26 *Let φ, ω be as in Definition 22. Then*

$$(\varphi_{a,\omega})^*(t) \sim (\varphi^*)_{\varphi'(a),\omega^*}(t) \tag{6.12}$$

uniformly in $a, t \geq 0$. Especially, we have uniformly in $t \geq 0$

$$(\varphi_a)^*(t) \sim (\varphi^*)_{\varphi'(a)}(t). \tag{6.13}$$

Proof Due to (2.3) and $\Delta_2(\{\varphi,\omega\}) < \infty$ there holds $\varphi'(t) \sim \varphi'(2t)$ and $\omega'(t) \sim \omega'(2t)$ uniformly in $t \geq 0$. If $0 \leq t \leq a$ then $a+t \sim a$ and if $a \leq t < \infty$ then $a+t \sim t$. Therefore

$$\varphi'_{a,\omega}(t) = \varphi'(a+t)\,\omega'\bigg(\frac{t}{a+t}\bigg) \sim \begin{cases} \varphi'(a)\,\omega'\big(\frac{t}{a}\big) & \text{for } 0 \leq t \leq a, \\ \varphi'(t)\,\omega'(1) & \text{for } t \geq a \end{cases}$$

uniformly in $a, t \geq 0$. Thus by $\omega'(1) = 1$

$$\varphi'_{a,\omega}(t) \sim \begin{cases} \varphi'(a)\,\omega'\big(\frac{t}{a}\big) & \text{for } 0 \leq t \leq a, \\ \varphi'(t) & \text{for } t \geq a. \end{cases} \tag{6.14}$$

Let $u := \varphi'_{a,\omega}(t)$ then (6.14) implies

$$\big((\varphi_{a,\omega})^*\big)'(u) = t \sim \begin{cases} a\,(\omega')^{-1}\big(\frac{u}{\varphi'(a)}\big) & \text{for } 0 \leq u \leq \varphi'_{a,\omega}(a), \\ (\varphi')^{-1}(u) & \text{for } u \geq \varphi'_{a,\omega}(a) \end{cases}$$

uniformly in $a,t \geq 0$. Therefore

$$\big((\varphi_{a,\omega})^*\big)'(u) \sim \begin{cases} a\,(\omega^*)'\big(\frac{u}{\varphi'(a)}\big) & \text{for } 0 \leq u \leq \varphi'_{a,\omega}(a), \\ (\varphi^*)'(u) & \text{for } u \geq \varphi'_{a,\omega}(a). \end{cases} \tag{6.15}$$

Because of $\varphi'_{a,\omega}(a) = \varphi'(2a)\,\omega'(\frac{1}{2}) \sim \varphi'(a)$ it is possible in (6.15) to shift the border for t from $\varphi'_{a,\omega}(a)$ to $\varphi'(a)$. Especially,

$$\big((\varphi_{a,\omega})^*\big)'(u) \sim \begin{cases} a\,(\omega^*)'\big(\frac{u}{\varphi'(a)}\big) & \text{for } 0 \leq u \leq \varphi'(a), \\ (\varphi^*)'(u) & \text{for } u \geq \varphi'(a). \end{cases} \tag{6.16}$$

On the other hand we replace in (6.14) φ by φ^*, a by $\varphi'(a)$, ω by ω^*, and t by u then

$$(\varphi^*)'_{\varphi'(a),\omega^*}(u) \sim \begin{cases} (\varphi^*)'(\varphi'(a))\,(\omega^*)'\big(\frac{u}{\varphi'(a)}\big) & \text{for } 0 \leq u \leq \varphi'(a), \\ (\varphi^*)'(u) & \text{for } u \geq \varphi'(a). \end{cases}$$

Note that $(\omega^*)'(1) = (\omega')^{-1}(1) = 1$ and $(\varphi^*)'(\varphi'(a)) = a$, so

$$(\varphi^*)'_{\varphi'(a),\omega^*}(u) \sim \begin{cases} a\,(\omega^*)'\big(\frac{u}{\varphi'(a)}\big) & \text{for } 0 \leq u \leq \varphi'(a), \\ (\varphi^*)'(u) & \text{for } u \geq \varphi'(a) \end{cases} \tag{6.17}$$

uniformly in $a,t \geq 0$. From (6.16) and (6.17) follows

$$\big((\varphi_{a,\omega})^*\big)'(u) \sim (\varphi^*)'_{\varphi'(a),\omega^*}(u)$$

uniformly in $a,u \geq 0$. This and (2.3) prove (6.12). Inequality (6.13) follows from (6.12) with the special choice $\omega'(t) = t$.

Lemma 27 *Let φ, ω be as in Definition 22. Then the families $\varphi_{a,\omega}$ and $(\varphi_{a,\omega})^*$ satisfy the Δ_2-condition uniformly in $a \geq 0$, i. e. it holds $\Delta_2(\{\varphi_{a,\omega}\}_{a\geq 0}) < \infty$ and $\Delta_2(\{(\varphi_{a,\omega})^*\}_{a\geq 0}) < \infty$.*

Proof From Lemma 23 follows $\Delta_2(\{\varphi_{a,\omega}\}_{a\geq 0}) < \infty$. By the same lemma we get $\Delta_2(\{(\varphi^*)_{\varphi'(a),\omega^*}\}_{a\geq 0}) < \infty$. Due to Lemma 26 this implies $\Delta_2(\{(\varphi_{a,\omega})^*\}_{a\geq 0}) < \infty$. This proves the assertion.

Lemma 28 *Let φ, ω be as in Definition 22. The uniformly in $a,b \in \mathbb{R}^n$*

$$\varphi_{|a|,\omega}(|a-b|) \sim \varphi_{|b|,\omega}(|a-b|). \tag{6.18}$$

Proof The proof is obvious for $a=b$, so let us assume that $|a-b|>0$. From (2.3) and $|a|+|a-b| \sim |b|+|a-b|$ we deduce

$$\begin{aligned} \frac{\varphi_{|a|,\omega}(|a-b|)}{|a-b|^2} &\sim \frac{\varphi'_{|a|}(|a-b|)}{|a-b|} = \varphi'(|a|+|a-b|)\,\omega'\bigg(\frac{|a-b|}{|a|+|a-b|}\bigg) \\ &\sim \varphi'(|b|+|a-b|)\,\omega'\bigg(\frac{|a-b|}{|b|+|a-b|}\bigg) = \frac{\varphi'_{|b|,\omega}(|a-b|)}{|a-b|} \\ &\sim \frac{\varphi_{|b|,\omega}(|a-b|)}{|a-b|^2} \end{aligned}$$

This proves the assertion.

Lemma 29 *Let φ, ω be as in Definition 22. Then there exists $c_1 > 0$ such that for all $a,b,e \in \mathbb{R}^n$*

$$\varphi'_{|a|,\omega}(|b-a|) \leq c_1\, \varphi'_{|e|,\omega}(|b-e|) + c_1\, \varphi'_{|e|,\omega}(|a-e|). \tag{6.19}$$

Proof If $|b-e| \leq |a-e|$ then $|a-b| \leq 2\,|a-e|$ and

$$\begin{aligned} \varphi'_{|a|,\omega}(|b-a|) &\leq \varphi'_{|a|,\omega}(2\,|a-e|) \\ &\sim \varphi'_{|a|,\omega}(\,|a-e|) \qquad \text{by Lemma 23} \\ &\sim \varphi'_{|e|,\omega}(\,|a-e|) \qquad \text{by (6.18)} \end{aligned} \tag{6.20}$$

This proves the assertion in the case $|b-e| \leq |a-e|$. Assume now that $|a-e| \leq |b-e|$. From (2.3) and (6.18) we know $\varphi'_{|a|,\omega}(|a-b|) \sim \varphi'_{|b|,\omega}(|a-b|)$. The rest follows from (6.20) with a and b interchanged.

Lemma 30 *Let φ, ω be as in Definition 22. Then uniformly in $\lambda \in [0,1]$ and $a \geq 0$ holds*

$$\varphi_{a,\omega}(\lambda\, a) \sim \omega(\lambda)\, \varphi(a). \tag{6.21}$$

Especially,

$$\varphi_a(\lambda\, a) \sim \lambda^2\, \varphi(a), \tag{6.22}$$

$$(\varphi_a)^*(\lambda\, \varphi'(a)) \sim \lambda^2\, \varphi(a). \tag{6.23}$$

Proof Because of (2.3) and (6.14) holds

$$\varphi_{a,\omega}(\lambda\, a) \sim \lambda\, a\, \varphi'_{a,\omega}(\lambda\, a) \sim \lambda\, a\, \varphi'(a)\, \omega'(\lambda) \sim \varphi(a)\, \omega(\lambda).$$

This proves (6.21) while (6.22) is a special case of (6.21) with $\omega'(t) = t$. Moreover, (6.13), (6.22), and (2.3) imply

$$(\varphi_a)^*(\lambda\, \varphi'(a)) \sim (\varphi^*)_{\varphi'(a)}(\lambda\, \varphi'(a)) \sim \lambda^2 \varphi^*(\varphi'(a)) \sim \lambda^2\, \varphi(a).$$

So, (6.23) is proven.

Lemma 31 *Let φ be an N-function with $\Delta_2(\{\varphi,\varphi^*\})<\infty$. Then there exist $\varepsilon>0$, $c_2>0$ which only depend on $\Delta_2(\{\varphi,\varphi^*\})$ such that for all $t\geq 0$ and all $\lambda\in[0,1]$*

$$\varphi(\lambda t)\leq c_2\,\lambda^{1+\varepsilon}\,\varphi(t). \tag{6.24}$$

In particular, there exists $\varepsilon>0$ and $c_2>0$ such that

$$\varphi_a(\lambda t)\leq c_2\,\lambda^{1+\varepsilon}\,\varphi_a(t) \tag{6.25}$$

uniformly in $a,t\geq 0$ and $\lambda\in[0,1]$.

Proof Since $\Delta_2(\{\varphi,\varphi^*\})<\infty$ there exists, as in Theorem 7, an N-function ρ and $\theta\in(0,1)$ with $\varphi^\theta\sim\rho$. This implies uniformly in $t\geq 0$ and $\lambda\in[0,1]$

$$\varphi(\lambda t)\sim\big(\rho(\lambda t)\big)^{\frac{1}{\theta}}\leq\big(\lambda\,\rho(t)\big)^{\frac{1}{\theta}}\sim\lambda^{\frac{1}{\theta}}\,\varphi(t),$$

where we have used the convexity of ρ and $\rho(0)=0$. Inequality (6.24) follows with $\varepsilon:=\frac{1}{\theta}-1$. Now, (6.25) follows from Lemma 27.

Lemma 32 (Young type inequality) *Let φ be an N-function which fulfills $\Delta_2(\{\varphi,\varphi^*\})<\infty$. Then for all $\delta>0$ there exists c_δ which only depends on δ and $\Delta_2(\{\varphi,\varphi^*\})$ such that for all $t,u\geq 0$*

$$t\,u\leq\delta\,\varphi(t)+c_\delta\,\varphi^*(u), \tag{6.26a}$$

$$t\,\varphi'(u)+\varphi'(t)\,u\leq\delta\,\varphi(t)+c_\delta\,\varphi(u). \tag{6.26b}$$

Let φ,ω be as in Definition 22. Then for all $\delta>0$ there exists c_δ such that for all $t,u,a\geq 0$

$$t\,u\leq\delta\,\varphi_{a,\omega}(t)+c_\delta\,(\varphi_{a,\omega})^*(u), \tag{6.27}$$

$$t\,\varphi'_{a,\omega}(u)+\varphi'_{a,\omega}(t)\,u\leq\delta\,\varphi_{a,\omega}(t)+c_\delta\,\varphi_{a,\omega}(u). \tag{6.28}$$

Proof Inequality (6.26a) is well known, see (2.1). Now (6.26b) follows from (6.26a) and (2.3). Because of Lemma 27 we can apply (6.26a) and (6.26b) to the family $\{\varphi_{a,\omega}\}_{a\geq 0}$. This proves (6.27) and (6.28).

Remark 33 Note that Lemma 32 together with Lemma 3 generalize many known estimates. One example are the quasi-norms estimates of Barrett and Liu in [3].

Lemma 34 *Let $\varphi,\sigma,\kappa,\omega$ be N-function with $\sigma'(1)=\kappa'(1)=\omega'(1)=1$ and $\Delta_2(\{\varphi,\varphi^*,\sigma,\sigma^*,\kappa,\kappa^*,\omega,\omega^*\})<\infty$. Moreover, let*

$$\kappa(t)\sim\sigma^*\big(\omega'(t)\big) \tag{6.29}$$

uniformly in $t\geq 0$. Then uniformly in $a,t\geq 0$

$$(\varphi_{a,\sigma})^*\big(\varphi'_{a,\omega}(t)\big)\sim\varphi_{a,\kappa}(t). \tag{6.30}$$

Moreover, for every $\delta>0$ there exists $c_\delta>0$ such that uniformly in $a,t,u\geq 0$

$$\varphi'_{a,\omega}(t)\,u\leq\delta\,\varphi_{a,\kappa}(t)+c_\delta\,\varphi_{a,\sigma}(u). \tag{6.31}$$

Proof Let us remark that if $\sigma = \kappa = \omega$ then (6.29) and (6.30) follow immediately from (2.3).

From (6.14) and (6.17) we deduce

$$\varphi'_{a,\omega}(t) \sim \begin{cases} \varphi'(a)\,\omega'\big(\frac{t}{a}\big) & \text{for } 0 \le t \le a, \\ \varphi'(t) & \text{for } t \ge a. \end{cases} \tag{6.32}$$

$$\big((\varphi_{a,\sigma})^*\big)'(u) \sim (\varphi^*)'_{\varphi'(a),\sigma^*}(u) \sim \begin{cases} a\,(\sigma^*)'\big(\frac{u}{\varphi'(a)}\big) & \text{for } 0 \le u \le \varphi'(a), \\ (\varphi^*)'(u) & \text{for } u \ge \varphi'(a) \end{cases} \tag{6.33}$$

Because of $\varphi'_{a,\omega}(a) = \varphi'(2a)\,\sigma\big(\frac{1}{2}\big) \sim \varphi'(a)$ it is possible in (6.15) to shift the border for u from $\varphi'(a)$ to $\varphi'_{a,\omega}(a)$, i. e.

$$\big((\varphi_{a,\sigma})^*\big)'(u) \sim \begin{cases} a\,(\sigma^*)'\big(\frac{u}{\varphi'(a)}\big) & \text{for } 0 \le u \le \varphi'_{a,\omega}(a), \\ (\varphi^*)'(u) & \text{for } u \ge \varphi'_{a,\omega}(a) \end{cases}.$$

Repeatedly use of (2.3) implies

$$\begin{aligned} (\varphi_{a,\sigma})^*(u) &\sim \begin{cases} u\,a\,(\sigma^*)'\big(\frac{u}{\varphi'(a)}\big) & \text{for } 0 \le u \le \varphi'_{a,\omega}(a), \\ \varphi^*(u) & \text{for } u \ge \varphi'_{a,\omega}(a) \end{cases} \\ &\sim \begin{cases} \varphi(a)\,\sigma^*\big(\frac{u}{\varphi'(a)}\big) & \text{for } 0 \le u \le \varphi'_{a,\omega}(a), \\ \varphi^*(u) & \text{for } u \ge \varphi'_{a,\omega}(a) \end{cases}. \end{aligned} \tag{6.34}$$

Now the composition of (6.32) and (6.34) gives

$$\begin{aligned} (\varphi_{a,\sigma})^*\Big(\varphi'_{a,\omega}(t)\Big) &\sim \begin{cases} \varphi(a)\,\sigma^*\Big(\omega'\big(\frac{t}{a}\big)\Big) & \text{for } 0 \le t \le a, \\ \varphi^*\big(\varphi'(t)\big) & \text{for } t \ge a \end{cases} \\ &\sim \begin{cases} \varphi(a)\,\kappa\big(\frac{t}{a}\big) & \text{for } 0 \le t \le a, \\ \varphi(t) & \text{for } t \ge a \end{cases} \\ &\sim \varphi_{a,\kappa}(t) \qquad \text{by (6.14) and (2.3).} \end{aligned}$$

Now (6.14) concludes

$$(\varphi_{a,\sigma})^*\big(\varphi'_{a,\omega}(t)\big) \sim \varphi_{a,\kappa}(t)$$

This proves (6.30). Now (6.31) is a direct implication of (6.30) and Young's inequality (6.28).

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