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BMO estimates for the p-Laplacian

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ABSTRACT

We prove BMO estimates of the inhomogeneous p-Laplace system given by $-\text{div}\,(|\nabla u|^{p-2}\nabla u) = \text{div}\, f$. We show that $f \in \text{BMO}$ implies $|\nabla u|^{p-2}\nabla u \in \text{BMO}$, which is the limiting case of the nonlinear Calderón–Zygmund theory. This extends the work of DiBenedetto and Manfredi (1993) [2], which was restricted to the super-quadratic case $p \geq 2$, to the full case $1 < p < \infty$ and even more general growth. Moreover, we prove that $A(\nabla u)$ inherits the Campanato and VMO regularity of f.

1. Introduction

We study solutions of an inhomogeneous elliptic system

$$-\operatorname{div}(A(\nabla u)) = -\operatorname{div} f \tag{1.1}$$

on a domain $\Omega \subset \mathbb{R}^n$, where $u : \Omega \to \mathbb{R}^N$ and $f : \Omega \to \mathbb{R}^{N\times n}$. We assume that $f \in \text{BMO}$, where BMO is the space of functions with bounded mean oscillation, and A is given by

$$A(\nabla u) = \varphi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}$$

for a suitable N-function φ. Throughout the paper we will assume φ satisfies the following assumption.

Assumption 1.1. Let φ be a convex function on $[0, \infty)$ such that φ is C^1 on $[0, \infty)$ and C^2 on $(0, \infty)$. Moreover, let $\varphi'(0) = 0$, $\lim_{t\to\infty} \varphi'(t) = \infty$ and

$$\varphi'(t) \sim t\varphi''(t) \tag{1.2}$$

uniformly in $t > 0$. The implicit constants in (1.2) are called the *characteristics of* φ.

The assumptions on φ are such that the induced operator $-\operatorname{div}(A(\nabla u))$ is strictly monotone. If we define the energy

$$\mathcal{J}(v) := \int \varphi(|\nabla v|)\,dx - \int f \cdot \nabla v\,dx,$$

then the system (1.1) is its Euler–Lagrange system and solutions of (1.1) are local minimizers of $\mathcal{J}$.

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A significant example of the considered model is the p-Laplacian system, for which $p \in (1, \infty)$, $\varphi(t) = \frac{1}{p}t^p$, $A(\nabla u) = |\nabla u|^{p-2}\nabla u$, and the system (1.1) has the form

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = -\operatorname{div} f.$$

Note that $\varphi(t) = \frac{1}{p}t^p$ satisfies[1] Assumption 1.1. In the rest of this introduction we restrict ourselves to this case. If $f \in L^{p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, then naturally $|\nabla u| \in L^p$ or equivalently $A(\nabla u) \in L^{p'}$. The question of the nonlinear Calderón–Zygmund theory (originated by Iwaniec in [1]) is, whether higher integrability of f transfers to higher integrability of ∇u and $A(\nabla u)$. Iwaniec showed that $f \in L^q(\mathbb{R}^n)$ implies $A(\nabla u) \in L^q(\mathbb{R}^n)$, whenever $q \in [p', \infty)$.

This raises the question of what happens in the limiting case $q = \infty$. We know from the linear theory of the Laplace equation (corresponding to $p = 2$) that $f \in L^\infty$ cannot imply $\nabla u \in L^\infty$. This is related to the fact that the mapping $f \mapsto \nabla u$ is (in the linear case) given by a singular integral operator. It is well known that such operators are in general not bounded from L^∞ to L^∞. However, it is possible to replace L^∞ by the space BMO, since singular integral operators map BMO to BMO. Therefore, the natural question arises if $f \in \mathrm{BMO}$ implies $A(\nabla u) \in \mathrm{BMO}$. The first BMO result was done by DiBenedetto and Manfredi in [2]. Their result, however, only treated the super-quadratic case $p \geq 2$. Our inequalities are more precise and therefore valid for all $p \in (1, \infty)$ and even for more general growth.

Theorem 1.2. *Let $B \subset \mathbb{R}^n$ be a ball. Let u be a solution of (1.1) on $2B$, with φ satisfying Assumption 1.1.*
If $f \in \mathrm{BMO}(2B)$, then $A(\nabla u) \in \mathrm{BMO}(B)$. Moreover,

$$\|A(\nabla u)\|_{\mathrm{BMO}(B)} \leq c \fint_{2B} |(A(\nabla u)) - \langle A(\nabla u)\rangle_{2B}|\,dx + c\|f\|_{\mathrm{BMO}(2B)}.$$

The constant c depends only on the characteristics of φ.

This theorem is a special case of our main result in Theorem 5.3. The technique used in the proof, is in the spirit of the pioneering work of Iwaniec and is based on comparison arguments with p-harmonic functions.

Additionally to Theorem 1.2, we are able to transfer any modulus of continuity of the mean oscillation from f to $A(\nabla u)$. This includes the case of VMO, see Corollary 5.4. Moreover, $f \in C^{0,\beta}(2B)$ implies $A(\nabla u) \in C^{0,\beta}(B)$ with corresponding local estimates, see Corollary 5.5. The β is restricted by the regularity of the p-harmonic functions.

Our results also hold in the context of differential forms on $\Omega \subset \mathbb{R}^n$, where we get the corresponding estimates, see Remark 5.9. By conjugation we can also treat solutions of systems of the form $d^*(A(dv + g)) = 0$.

The special case $f = 0$ in Corollary 5.5 allows us to derive new decay estimates for φ-harmonic functions. On the one hand we get decay estimates for $A(\nabla u)$, see Remark 5.6. On the other hand by conjugation, see Remark 5.9 we also get decay estimates for ∇u, see (5.7).

We study systems, where the right-hand side is given in divergence form, since it simplifies the presentation. The results can also be applied to the situation, where the right-hand side $\operatorname{div} f$ of (1.1) is replaced by a function g. Note that any functional from $(W_0^{1,\varphi}(\Omega))^*$ can be represented in such a divergence form. Whenever, such g can be represented as $g = \operatorname{div} f$ with $f \in \mathrm{BMO}_\omega$ (a refinement of BMO, see Section 5), then our results immediately provide corresponding inequalities. For example we show in Remark 5.7 that $g \in L^n$ implies locally $A(\nabla u) \in \mathrm{VMO}$. This complements the results of [3,4], who proved $A(\nabla u) \in L^\infty$ for $g \in L^{n,1}$ (Lorentz space; subspace of L^n), where the result of [3] is for equations only but up to the boundary.

2. Notation and preliminary results

We use c as a generic constant, which may change from line to line, but does not depend on the crucial quantities. Moreover we write $f \sim g$ if and only if there exist constants $c, C > 0$ such that $cf \leq g \leq Cf$. Note that we do not point out the dependencies of the constants on the fixed dimensions n and N. For $v \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and a ball $B \subset \mathbb{R}^n$ we define

$$\langle v\rangle_B := \fint_B v(x)dx := \frac{1}{|B|}\int_B v(x)dx, \tag{2.1}$$

where $|B|$ is the n-dimensional Lebesgue measure of B. For $\lambda > 0$ we denote by λB the ball with the same center as B but λ times the radius.

A real function $\psi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is said to be an N-function if it satisfies the following conditions: $\psi(0) = 0$ and there exists the derivative ψ', which is right continuous, non-decreasing and satisfies $\psi'(0) = 0$, $\psi'(t) > 0$ for $t > 0$, and $\lim_{t\to\infty} \psi'(t) = \infty$. Especially, φ is convex. Assumption 1.1 assures that φ is an N-function. The complementary function φ^* is given by

$$\varphi^*(u) := \sup_{t\geq 0}\big(ut - \varphi(t)\big)$$

and satisfies $(\varphi^*)'(t) = (\varphi')^{-1}(t)$. Moreover, for any $t \geq 0$ there holds

$$\varphi(t) \leq \varphi'(t)t \leq \varphi(2t), \qquad \varphi^*(\varphi'(t)) \leq \varphi(2t). \tag{2.2}$$

[1] Also $\varphi(t) = \frac{1}{p}\int_0^t (\mu + s)^{p-2}s\,ds$ and $\varphi(t) = \frac{1}{p}\int_0^t (\mu^2 + s^2)^{\frac{p-2}{2}}s\,ds$ with $\mu \geq 0$ satisfy Assumption 1.1.

It follows from the Assumption 1.1 (see for example [5]) that φ and φ^* satisfy the Δ_2-condition, i.e. $\varphi(2t) \leq c\varphi(t)$ and $\varphi^*(2t) \leq c\varphi^*(t)$ uniformly in $t \geq 0$, where the constants only depend on the characteristics of φ. For further properties of the N-function we refer to [6].

As a further consequence of Assumption 1.1 there exists $1 < p \leq q < \infty$ and $K_1 > 0$ such that

$$\varphi(st) \leq K_1 \max\{s^p, s^q\}\varphi(t) \tag{2.3}$$

for all $s, t \geq 0$. The exponents p and q are called the lower and upper indices of φ, respectively. We say that φ is of type $T(p, q, K_1)$ if it satisfies (2.3), where we allow $1 \leq p \leq q < \infty$ in this definition. Note that (2.3) implies

$$\min\{s^p, s^q\}\varphi(t) \leq K_1\varphi(st) \tag{2.4}$$

for all $a, t \geq 0$. Every $\varphi \in T(p, q, K_1)$ satisfies the Δ_2-condition; indeed $\varphi(2t) \leq K_1 2^q \varphi(t)$.

Lemma 2.1. *Let φ be of type $T(p, q, K_1)$, then $\varphi^* \in T(q', p', K_2)$ for some $K_2 = K_2(p, q, K_1)$.*

This lemma is well known. However, for the sake of completeness, we include the proof in the Appendix. In particular, if $\varphi \in T(p, q, K)$ with $1 < p \leq q < \infty$, then also φ^* satisfies the Δ_2-condition. Under the assumption of Lemma 2.1 we also get the following versions of *Young's inequality*. For all $\delta \in (0, 1]$ and all $t, s \geq 0$ it holds

$$\begin{aligned} ts &\leq K_1 K_2^{q-1} \delta^{1-q} \varphi(t) + \delta\varphi^*(s), \\ ts &\leq \delta\varphi(t) + K_2 K_1^{p'-1} \delta^{1-p'} \varphi^*(s). \end{aligned} \tag{2.5}$$

For an N-function φ we introduce the family of shifted N-functions $\{\varphi_a\}_{a\geq 0}$ by $\varphi_a'(t)/t := \varphi'(a+t)/(a+t)$. If φ satisfies Assumption 1.1 then $\varphi_a''(t) \sim \varphi''(a+t)$ uniformly in $a, t \geq 0$. The following lemmas show important invariants in terms of shifts.

Lemma 2.2 (*Lemma 22, [7]*)**.** *Let φ hold Assumption 1.1. Then $(\varphi_{|P|})^*(t) \sim (\varphi^*)_{|A(P)|}(t)$ holds uniformly in $t \geq 0$ and $P \in \mathbb{R}^{N\times n}$. The implicit constants depend on p, q and K only.*

We define

$$\overline{p} := \min\{p, 2\} \quad \text{and} \quad \overline{q} := \max\{q, 2\}. \tag{2.6}$$

Lemma 2.3. *Let φ be of type $T(p, q, K_1)$ and $P \in \mathbb{R}^{N\times n}$, then $\varphi_{|P|}$ is of type $T(\overline{p}, \overline{q}, \overline{K})$ and $(\varphi_{|P|})^*$ and $(\varphi^*)_{|A(P)|}$ are of type $T(\overline{q}', \overline{p}', K)$.*

The proof of this lemma is postponed to the Appendix.

We define $V : \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ by

$$|V(Q)|^2 = A(Q)\cdot Q \quad \text{and} \quad \frac{V(Q)}{|V(Q)|} = \frac{A(Q)}{|A(Q)|} = \frac{Q}{|Q|},$$

in particular we have

$$V(Q) = \sqrt{\varphi'(|Q|)|Q|}\,\frac{Q}{|Q|}.$$

In the case of the p-Laplacian, we have $\varphi(t) = \frac{1}{p}t^p$, $A(Q) = |Q|^{p-2}Q$ and $V(Q) = |Q|^{\frac{p-2}{2}}Q$.

The connection between A, V, and the shifted N-functions is best reflected in the following lemma, which is a summary of Lemmas 3, 21, and 26 of [8].

Lemma 2.4. *Let φ satisfy Assumption 1.1. Then*

$$\begin{aligned} \big(A(P) - A(Q)\big)\cdot\big(P - Q\big) &\sim |V(P) - V(Q)|^2 && (2.7a)\\ &\sim \varphi_{|Q|}\big(|P - Q|\big) && (2.7b)\\ &\sim \big(\varphi^*\big)_{|A(Q)|}\big(|A(P) - A(Q)|\big) && (2.7c) \end{aligned}$$

uniformly in $P, Q \in \mathbb{R}^{N\times n}$. Moreover,

$$A(Q)\cdot Q = |V(Q)|^2 \sim \varphi(|Q|), \tag{2.7d}$$

and

$$|A(P) - A(Q)| \sim \big(\varphi_{|Q|}\big)'\big(|P - Q|\big), \tag{2.7e}$$

uniformly in $P, Q \in \mathbb{R}^{N\times n}$.

The following lemma is a simple modification of Lemma 35 and Corollary 26 of [7] by use of Young's inequality in the form (2.5) and Lemma 2.2.

Lemma 2.5 (*Shift Change*)**.** *For every $\varepsilon \in (0, 1]$, it holds that*

$$\varphi_{|P|}(t) \le c\varepsilon^{1-\overline{p}'}\varphi_{|Q|}(t) + \varepsilon|V(P) - V(Q)|^2,$$
$$(\varphi_{|P|})^*(t) \le c\varepsilon^{1-\overline{q}}(\varphi_{|Q|})^*(t) + \varepsilon|V(P) - V(Q)|^2,$$
$$(\varphi^*)_{|A(P)|}(t) \le c\varepsilon^{1-\overline{q}}(\varphi^*)_{|A(Q)|}(t) + \varepsilon|V(P) - V(Q)|^2,$$

for all $P, Q \in \mathbb{R}^{N\times n}$ and all $t \ge 0$. The constants only depend on the characteristics of φ.

By L^φ and $W^{1,\varphi}$ we denote the classical Orlicz and Sobolev–Orlicz spaces, i.e. $f \in L^\varphi$ if and only if $\int \varphi(|f|)\,dx < \infty$ and $f \in W^{1,\varphi}$ if and only if $f, \nabla f \in L^\varphi$. By $W_0^{1,\varphi}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,\varphi}(\Omega)$.

We define for B a ball and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$

$$M_B^\sharp f = \fint_B |f - \langle f\rangle_B|dx,$$
$$(M^\sharp f)(x) = \sup_{B\ni x} M_B^\sharp f.$$

The space BMO of bounded mean oscillations is defined via the following semi-norm (for Ω open)

$$\|v\|_{\text{BMO}(\Omega)} := \sup_{B\subset\Omega} \fint_B |f - \langle f\rangle_B|\,dx = \sup_{B\subset\Omega} M_B^\sharp f;$$

saying that $v \in \text{BMO}(B)$, whenever its semi-norm is bounded. Therefore $f \in \text{BMO}(\mathbb{R}^n)$ if and only if $M^\sharp f \in L^\infty(\mathbb{R}^n)$.

We need also the following refinements of BMO, see [9]. For a non-decreasing function $\omega : (0, \infty) \to (0, \infty)$ we define

$$M_{\omega,B}^\sharp f = \frac{1}{\omega(R_B)} \fint_B |f - \langle f\rangle_B|dx,$$

where R_B is the radius of B. We define the semi-norm

$$\|v\|_{\text{BMO}_\omega(\Omega)} := \sup_{B\subset\Omega} M_{\omega,B}^\sharp f.$$

The choice $\omega(r) = 1$ gives the usual BMO semi-norm, while $\omega(r) = r^\alpha$ with $0 < \alpha \le 1$ induces the Campanato space (which are equivalent to the Hölder spaces $C^{0,\alpha}$).

3. Reverse Hölder estimate

In this section we refine the reverse Hölder estimate of Lemma 3.4 [10], where the case $f = 0$ was considered. For this we need the following version of Sobolev–Poincaré from [8, Lemma 7].

Theorem 3.1 (*Sobolev–Poincaré*)**.** *Let φ be an N-function such that φ and φ^* satisfies the Δ_2-condition. Then there exists $0 < \theta_0 < 1$ and $c > 0$ such that the following holds. If $B \subset \mathbb{R}^n$ is some ball with radius R and $v \in W^{1,\varphi}(B, \mathbb{R}^N)$, then*

$$\fint_B \varphi\left(\frac{|v - \langle v\rangle_B|}{R}\right) dx \le c\left(\fint_B \varphi^{\theta_0}(|\nabla v|)\,dx\right)^{\frac{1}{\theta_0}}. \tag{3.1}$$

For gradients of solutions of (1.1) we can deduce the following reverse Hölder inequality.

Lemma 3.2. *Let u be a solution of (1.1). There exists $\theta \in (0, 1)$ such that for all $P, f_0 \in \mathbb{R}^{N\times n}$ and all balls B satisfying $2B \subset \Omega$*

$$\fint_B |V(\nabla u) - V(P)|^2\,dx \le c\left(\fint_{2B} |V(\nabla u) - V(P)|^{2\theta}\,dx\right)^{\frac{1}{\theta}} + c\fint_{2B} (\varphi^*)_{|A(P)|}(|f - f_0|)\,dx$$

holds. The constants c and θ only depend on the characteristics of φ.

Proof. Let $\eta \in C_0^\infty(2B)$ with $\chi_B \le \eta \le \chi_{2B}$ and $|\nabla\eta| \le c/R$, where R is the radius of B. Let $\alpha \ge \overline{q}$, then $(\alpha-1)\overline{p}' \ge \alpha$. We define $\xi := \eta^\alpha(u-z)$, where z is a linear function such that $\langle u-z\rangle_{2B} = 0$ and $\nabla z = P$. Using ξ as a test function in the weak formulation of (1.1) we get for all $f_0 \in \mathbb{R}^{N\times n}$

$$\begin{aligned}
(\mathrm{I}) &:= |B|^{-1}\langle A(\nabla u) - A(P), \eta^\alpha(\nabla u - P)\rangle \\
&= |B|^{-1}\langle f - f_0, \eta^\alpha(\nabla u - P)\rangle + |B|^{-1}\langle f - f_0, \alpha\eta^{\alpha-1}(u-z)\otimes\nabla\eta\rangle \\
&\quad - |B|^{-1}\langle A(\nabla u) - A(P), \alpha\eta^{\alpha-1}(u-z)\otimes\nabla\eta\rangle \\
&=: (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}).
\end{aligned}$$

With the help of Lemma 2.4 we get

$$(\mathrm{I}) \sim \fint_B \eta^\alpha |V(\nabla u) - V(P)|^2\,dx.$$

By (2.5) for $\varphi_{|P|}$ and $\delta \in (0,1)$, by $(\varphi_{|P|})^* \sim (\varphi^*)_{|A(P)|}$ due to Lemma 2.2, $(\alpha-1)\overline{p}' \ge \alpha$ and by Lemma 2.4 we estimate

$$\begin{aligned}
(\mathrm{II}) &\le c\delta^{1-\overline{p}'}\fint_{2B} (\varphi^*)_{|A(P)|}(|f-f_0|)\,dx + \delta\fint_{2B}\eta^\alpha\varphi_{|P|}(|\nabla u - P|)\,dx \\
&\le c\delta^{1-\overline{p}'}\fint_{2B} (\varphi^*)_{|A(P)|}(|f-f_0|)\,dx + \delta c\fint_{2B}\eta^\alpha|V(\nabla u) - V(P)|^2\,dx.
\end{aligned}$$

Similarly, we estimate with Lemma 2.4

$$(\mathrm{III}) \le c\fint_{2B}(\varphi^*)_{|A(P)|}(|f-f_0|)\,dx + c\fint_{2B}\varphi_{|P|}\left(\frac{|u-z|}{R}\right)dx.$$

With Lemma 2.4, Young's inequality with $\varphi_{|P|}$, $(\alpha-1)\overline{q} \ge \alpha$ and (2.2) (second part) in combination with Lemma 2.4 we deduce analogously

$$\begin{aligned}
(\mathrm{IV}) &\le c\fint_{2B}\varphi'_{|P|}(|A(\nabla u) - P|)\eta^{\alpha-1}\frac{|u-z|}{R}\,dx \\
&\le \delta\fint_{2B}\eta^\alpha(\varphi_{|P|})^*\big(\varphi'_{|P|}(|\nabla u - P|)\big)\,dx + c\delta^{1-\overline{q}}\fint_{2B}\varphi_{|P|}\left(\frac{|u-z|}{R}\right)dx \\
&\le \delta\fint_{2B}\eta^\alpha|V(\nabla u) - V(P)|^2\,dx + c\delta^{1-\overline{q}}\fint_{2B}\varphi_{|P|}\left(\frac{|u-z|}{R}\right)dx.
\end{aligned}$$

Moreover, it follows from Theorem 3.1 for $\varphi_{|P|}$ for some $\theta \in (0,1)$, Lemma 2.4 and the facts that $\langle u-z\rangle_{2B} = 0$ and $\nabla z = P$ that

$$\begin{aligned}
\fint_{2B}\varphi_{|P|}\left(\frac{|u-z|}{R}\right)dx &\le c\left(\fint_{2B}\big(\varphi_{|P|}(|\nabla u - P|)\big)^\theta\,dx\right)^{\frac{1}{\theta}} \\
&\le c\left(\fint_{2B}|V(\nabla u) - V(P)|^{2\theta}\,dx\right)^{\frac{1}{\theta}}.
\end{aligned}$$

For small δ we can absorb corresponding terms into (I) such that the claim follows. □

Our aim is to give estimates in terms of $A(\nabla u)$. We will give estimates exploiting reverse Hölder inequalities as well as BMO properties. These will enable us to replace the right hand side of Lemma 3.2 with adequate quantities. At first we need the following lemma for improving reverse Hölder estimates. The lemma is a minor modification of the [11, Remark 6.12] and [12, Lemma 3.2].

Lemma 3.3. *Let $B \subset \mathbb{R}^n$ be a ball, let $g, h : \Omega \to \mathbb{R}$ be integrable functions and $\theta \in (0,1)$ such that*

$$\fint_B |g|\,dx \le c_0\left(\fint_{2B}|g|^\theta\,dx\right)^{\frac{1}{\theta}} + \fint_{2B}|h|\,dx$$

for all balls B with $2B \subset \Omega$. Then for every $\gamma \in (0,1)$ there exists $c_1 = c_1(c_0,\gamma)$ such that

$$\fint_B |g|\,dx \le c_1\left(\fint_{2B}|g|^\gamma\,dx\right)^{\frac{1}{\gamma}} + c_1\fint_{2B}|h|\,dx.$$

We will use this result to prove the following inverse Jensen inequality.

Corollary 3.4. *Let $\Omega \subset \mathbb{R}^n$ and ψ be an N-function of type $T(1, q, K)$, $g \in L^{\psi}(\Omega)$ and $h \in L^1_{\mathrm{loc}}(\Omega)$. If there exists $\theta \in (0, 1)$ such that*

$$\fint_B \psi(|g|)\,dx \le c_0 \left(\fint_{2B} \psi(|g|)^{\theta}\,dx \right)^{\frac{1}{\theta}} + \fint_{2B} |h|\,dx,$$

for all balls B with $2B \subset \Omega$, then there exists $c_1 = c_1(c_0, K, q)$ such that

$$\fint_B \psi(|g|)\,dx \le c_1 \psi\left(\fint_{2B} |g|\,dx \right) + c_1 \fint_{2B} |h|\,dx.$$

Proof. By Lemma 3.3 we gain for a fixed $\gamma < \frac{1}{q}$

$$\fint_B \psi(|g|)\,dx \le c_1 \left(\fint_{2B} \psi(|g|)^{\gamma}\,dx \right)^{\frac{1}{\gamma}} + c_1 \fint_{2B} |h|\,dx.$$

Due to Lemma A.3, which can be found in the Appendix, the function $((\psi(t))^{\gamma})^{-1}$ is quasi-convex; i.e. it is uniformly proportional to a convex function. Therefore, the result follows by Jensen's inequality. □

The estimate of Lemma 3.2 can be improved in the following way.

Corollary 3.5. *Let u be a solution of (1.1). For all $P \in \mathbb{R}^{N\times n}$ and all balls B such that $2B \subset \Omega$*

$$\fint_B |V(\nabla u) - V(P)|^2\,dx \le c(\varphi^*)_{|A(P)|}\left(\fint_{2B} |A(\nabla u) - A(P)|\,dx \right) + c(\varphi^*)_{|A(P)|}(\|f\|_{\mathrm{BMO}(2B)})$$

holds. The constants only depend on the characteristics of φ.

Proof. If follows from Lemma 2.4 that

$$|V(\nabla u) - V(P)|^2 \sim (\varphi^*)_{|A(P)|}(|A(\nabla u) - A(P)|).$$

Therefore, we can apply Corollary 3.4 on the inequality proven in Lemma 3.2 to gain

$$\fint_B |V(\nabla u) - V(P)|^2\,dx \le c(\varphi^*)_{|A(P)|}\left(\fint_{2B} |A(\nabla u) - A(P)|\,dx \right) + c \fint_{2B} (\varphi^*)_{|A(P)|}(|f - f_0|)\,dx,$$

for any $f_0 \in \mathbb{R}^{N\times n}$. The result follows by using Lemma A.1 to the last integral

$$\fint_{2B} (\varphi^*)_{|A(Q)|}(|f - f_0|)\,dx \le c(\varphi^*)_{|A(Q)|}(\|f\|_{\mathrm{BMO}(2B)}).$$

This inequality reflects the reverse Jensen property of the BMO norm. □

4. Comparison

The key idea in the proof of our main result is to compare the solution u with a suitable φ-harmonic function h. Later we transfer the good properties of h to u. Regularity of φ-harmonic functions is well known in the case of a p-Laplace system with $\varphi(t) = t^p$ for $p \in (1, \infty)$. Recently, the result was extended in [10, Theorem 6.4] for general φ satisfying Assumption 1.1:

Theorem 4.1 (*Decay Estimate for φ-harmonic Maps*)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set, let φ satisfy Assumption 1.1, and let $h \in W^{1,\varphi}(\Omega, \mathbb{R}^N)$ be φ-harmonic on Ω. Then there exists $\alpha > 0$ and $c > 0$ such that for every ball $B \subset \Omega$ and every $\lambda \in (0, 1)$ there holds*

$$\fint_{\lambda B} |V(\nabla h) - \langle V(\nabla h)\rangle_{\lambda B}|^2\,dx \le c\lambda^{2\alpha} \fint_B |V(\nabla h) - \langle V(\nabla h)\rangle_B|^2\,dx.$$

Note that c and α depend only on the characteristics of φ.

For a given solution u of (1.1) let $h \in W^{1,\varphi}(B)$ be the unique solution of

$$\begin{aligned} -\operatorname{div} A(\nabla h) &= 0 \quad \text{in } B, \\ h &= u \quad \text{on } \partial B. \end{aligned} \tag{4.1}$$

The next lemma estimates the distance of h from u.

Lemma 4.2. *Let u be a solution of (1.1). Further let h solve (4.1). Then for every $\delta > 0$ there exists $c_\delta \geq 1$ such that*

$$\fint_B |V(\nabla u) - V(\nabla h)|^2\,dx \leq \delta (\varphi^*)_{|\langle A(\nabla u)\rangle_{2B}|}\left(\fint_{2B} |A(\nabla u) - \langle A(\nabla u)\rangle_{2B}|\,dx\right) + c\delta^{1-\overline{q}}\,(\varphi^*)_{|\langle A(\nabla u)\rangle_{2B}|}(\|f\|_{\mathrm{BMO}(2B)})$$

holds.

Proof. We have for any $f_0 \in \mathbb{R}^{N\times n}$

$$\text{(I)} := |B|^{-1}\langle A(\nabla u) - A(\nabla h), \nabla u - \nabla h\rangle = |B|^{-1}\langle f - f_0, \nabla u - \nabla h\rangle =: \text{(II)}.$$

Firstly, by Lemma 2.4

$$\text{(I)} \sim \fint_B |V(\nabla u) - V(\nabla h)|^2\,dx.$$

Secondly, by Young's inequality (2.5) with $\varphi_{|\nabla u|}$ and Lemma 2.4 we get

$$\text{(II)} \leq \varepsilon\text{(I)} + c\varepsilon^{1-\overline{p}'}\fint_B (\varphi_{|\nabla u|})^*(|f - f_0|)\,dx.$$

We absorb the first term of the right hand side for some small $\varepsilon > 0$ and apply Lemma 2.2

$$\text{(I)} \leq c\fint_B (\varphi^*)_{|A(\nabla u)|}(|f - f_0|)\,dx.$$

With the shift change of Lemma 2.5 with $A(Q) := \langle A(\nabla u)\rangle_{2B}$ we get for $\gamma > 0$

$$\text{(II)} \leq c\gamma^{1-\overline{q}}\fint_B (\varphi^*)_{|A(Q)|}(|f - f_0|)\,dx + \gamma\fint_B |V(\nabla u) - V(Q)|^2\,dx.$$

We set $f_0 = \langle f\rangle_{2B}$ and estimate the first integral by Lemma A.1. The second integral is estimated by Corollary 3.5 with $P := Q$. The claim follows by choosing $\gamma > 0$ conveniently. □

5. Proof of the main result

We need the following calculation:

$$|\langle g\rangle_{\frac{1}{2}B} - \langle g\rangle_B| \leq \fint_{\frac{1}{2}B} |g - \langle g\rangle_B| dx \leq 2^n M_B^\sharp g.$$

By m iterations of the previous we find

$$|\langle g\rangle_{2^{-m}B} - \langle g\rangle_B| \leq 2^n \sum_{i=0}^{m-1} M^\sharp_{2^{-i}B} g \leq m2^n \max_{0\leq i\leq m-1} M^\sharp_{2^{-i}B} g. \tag{5.1}$$

Proposition 5.1. *Let $B \subset \mathbb{R}^n$ be a ball. Let α be the decay exponent for φ-harmonic functions as in Theorem 4.1. Then for every $m \in \mathbb{N}$ there exists $c_m \geq 1$ such that*

$$\begin{aligned} M^\sharp_{2^{-m}B}(A(\nabla u)) &\leq c2^{-m\frac{2\alpha}{\overline{p}'}} \sum_{0\leq i\leq m} M^\sharp_{2^{1-i}B}(A(\nabla u)) + c_m\|f\|_{\mathrm{BMO}(2B)} \\ &\leq c2^{-m\frac{2\alpha}{\overline{p}'}} m \max_{0\leq i\leq m} M^\sharp_{2^{1-i}B}(A(\nabla u)) + c_m\|f\|_{\mathrm{BMO}(2B)}. \end{aligned}$$

The constant c_m is dependent only on the characteristics of φ and α. The constant c is independent of m and α.

Proof. Define $A(Q) := \langle A(\nabla u)\rangle_{2B}$ and $A(Q_m) := \langle A(\nabla u)\rangle_{2^{-m}B}$. With Lemma 2.3 we find $(\varphi^*)_{|A(P)|}$ is of type $T(\overline{q}', \overline{p}', K)$ for some K independent of P.

Let h be the φ-harmonic function on B with $u = h$ on the boundary ∂B as defined by (4.1). Then $V(\nabla h)$ satisfies the decay estimate of Theorem 4.1 and we can get

$$\begin{aligned}
\text{(I)} &:= \fint_{2^{-m}B} |V(\nabla u) - \langle V(\nabla u)\rangle_{2^{-m}B}|^2\,dx \\
&\le c\fint_{2^{-m}B} |V(\nabla h) - \langle V(\nabla h)\rangle_{2^{-m}B}|^2\,dx + c\fint_{2^{-m}B} |V(\nabla u) - V(\nabla h)|^2\,dx \\
&\le c2^{-m2\alpha}\fint_B |V(\nabla h) - \langle V(\nabla h)\rangle_B|^2\,dx + c2^{mn}\fint_B |V(\nabla u) - V(\nabla h)|^2\,dx \\
&\le c2^{-m2\alpha}\fint_B |V(\nabla u) - \langle V(\nabla u)\rangle_B|^2\,dx + c2^{mn}\fint_B |V(\nabla u) - V(\nabla h)|^2\,dx \\
&\le c2^{-m2\alpha}\fint_B |V(\nabla u) - V(Q)|^2\,dx + c2^{mn}\fint_B |V(\nabla u) - V(\nabla h)|^2\,dx.
\end{aligned}$$

Now using Corollary 3.5 and Lemma 4.2 we get

$$\text{(I)} \le c(2^{-m2\alpha} + \delta 2^{mn})(\varphi^*)_{|A(Q)|}\left(\fint_{2B} |A(\nabla u) - A(Q)|\,dx\right) + c2^{mn}\delta^{1-\overline{q}}(\varphi^*)_{|A(Q)|}(\|f\|_{\mathrm{BMO}(2B)}). \tag{5.2}$$

We use Lemma 2.5 to change the shift $A(Q)$ to $A(Q_m)$ (for the first integral with $\varepsilon = 1$ and for the second integral with $\varepsilon = \gamma$).

$$\begin{aligned}
\text{(I)} &\le c(2^{-m2\alpha} + \delta 2^{mn})(\varphi^*)_{|A(Q_m)|}\big(M^\sharp_{2B}(A(\nabla u))\big) + c2^{mn}\delta^{1-\overline{q}}\gamma^{1-\overline{q}}(\varphi^*)_{|A(Q_m)|}(\|f\|_{\mathrm{BMO}(2B)}) \\
&\quad + c(2^{-m2\alpha} + \delta 2^{mn} + \gamma)\,|V(Q) - V(Q_m)|^2.
\end{aligned}$$

From Lemma 2.4 we know that

$$|V(Q) - V(Q_m)|^2 \le c(\varphi^*)_{|A(Q_m)|}(|A(Q) - A(Q_m)|)$$

and from (5.1) that

$$|A(Q) - A(Q_m)| \le 2^n \sum_{0\le i\le m-1} M^\sharp_{2^{-i}B}(A(\nabla u)).$$

The previous two estimates and $(\varphi^*)_{|A(Q_m)|} \in T(\overline{q}', \overline{p}', K)$ imply

$$|V(Q) - V(Q_m)|^2 \le c(\varphi^*)_{|A(Q_m)|}\left(\sum_{0\le i\le m-1} M^\sharp_{2^{-i}B}(A(\nabla u))\right).$$

Overall, we get

$$\text{(I)} \le c(2^{-m2\alpha} + \delta 2^{mn} + \gamma)(\varphi^*)_{|A(Q_m)|}\left(\sum_{0\le i\le m} M^\sharp_{2^{1-i}B}(A(\nabla u))\right) + c2^{mn}\delta^{1-\overline{q}}\gamma^{1-\overline{q}}(\varphi^*)_{|A(Q_m)|}(\|f\|_{\mathrm{BMO}(2B)}).$$

We fix $\gamma := 2^{-m2\alpha}$ and $\delta := 2^{-m2\alpha-mn}$ to get

$$\text{(I)} \le c2^{-m2\alpha}(\varphi^*)_{|A(Q_m)|}\left(\sum_{0\le i\le m} M^\sharp_{2^{1-i}B}(A(\nabla u))\right) + c2^{mn+(m4\alpha+mn)(\overline{q}-1)}(\varphi^*)_{|A(Q_m)|}(\|f\|_{\mathrm{BMO}(2B)}).$$

Note that for all $b \in [0, 1/K]$ and $t \ge 0$ we have by (2.4)

$$b(\varphi^*)_{|A(Q_m)|}(t) = \frac{1}{K}(bK)(\varphi^*)_{|A(Q_m)|}(t) \le (\varphi^*)_{|A(Q_m)|}\big((bK)^{\frac{1}{\overline{p}'}}t\big).$$

Without loss of generality we can assume in the following that m is so large that $c2^{-m2\alpha} \le 1/K$. Therefore

$$\begin{aligned}
\text{(I)} &\le (\varphi^*)_{|A(Q_m)|}\left(c2^{-m\frac{2\alpha}{\overline{p}'}}\sum_{0\le i\le m} M^\sharp_{2^{1-i}B}(A(\nabla u))\right) + (\varphi^*)_{|A(Q_m)|}(c_m\|f\|_{\mathrm{BMO}(2B)}) \\
&\le (\varphi^*)_{|A(Q_m)|}\left(c2^{-m\frac{2\alpha}{\overline{p}'}}\sum_{0\le i\le m} M^\sharp_{2^{1-i}B}(A(\nabla u)) + c_m\|f\|_{\mathrm{BMO}(2B)}\right).
\end{aligned} \tag{5.3}$$

On the other hand

$$\fint_{2^{-m}B} (\varphi^*)_{|A(Q_m)|}(|A(\nabla u) - A(Q_m)|)\,dx \le c \fint_{2^{-m}B} (A(\nabla u) - A(Q_m)) \cdot (\nabla u - Q_m)\,dx$$
$$\le c \fint_{2^{-m}B} |V(\nabla u) - V(\langle \nabla u\rangle_{2^{-m}B})|^2\,dx$$

by Lemma 2.4 and $\langle A(\nabla u) - A(Q_m)\rangle_{2^{-m}B} = \langle \nabla u - \langle \nabla u\rangle_{2^{-m}B}\rangle_{2^{-m}B} = 0$.

Consequently we get using Lemma 2.1, Jensen's inequality and Lemma A.2

$$(\varphi^*)_{|A(Q_m)|}\big(cM^\sharp_{2^{-m}B}(A(\nabla u))\big) \le c(\varphi^*)_{|A(Q_m)|}\left(\fint_{2^{-m}B} |A(\nabla u) - A(Q_m)|\,dx\right)$$
$$\le c \fint_{2^{-m}B} (\varphi^*)_{|A(Q_m)|}(|A(\nabla u) - A(Q_m)|)\,dx \le (\mathrm{I}). \tag{5.4}$$

If we apply the inverse of $(\varphi^*)_{|A(Q_m)|}$ to the combination of (5.3) and (5.4) we get the claim. □

Remark 5.2. Let u be such that it satisfies (1.1) on $\mathbb{R}^n$ and $M^\sharp(A(\nabla u)) < \infty$ almost everywhere (for example $A(\nabla u) \in L^{p'}(\mathbb{R}^n)$). Then for suitable large m (such that $c2^{-m\frac{2\alpha}{q'}} \le \frac{1}{2}$), we deduce from Proposition 5.1 by taking the supremum over all balls containing x

$$M^\sharp(A(\nabla u))(x) \le c\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

In particular, $\|A(\nabla u)\|_{\mathrm{BMO}(\mathbb{R}^n)} \le c\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}$.

We can now prove our main result that the BMO_ω-regularity of f transfers to $A(\nabla u)$. Note that the case $\omega = 1$ is just Theorem 1.2.

Theorem 5.3. *Let $B \subset \mathbb{R}^n$ be a ball. Let u be a solution of (1.1) on $2B$, with φ satisfying Assumption 1.1. Let $\omega : (0,\infty) \to (0,\infty)$ be non-decreasing such that for some $\beta \in \left(0, \frac{2\alpha}{p'}\right)$ the function $\omega(r)r^{-\beta}$ is almost decreasing in the sense that there is $c_0 > 0$ that $\omega(r)r^{-\beta} \le c_0\,\omega(s)s^{-\beta}$ for all $r > s$. Then*

$$\max_{i\ge 0} M^\sharp_{\omega,2^{-m}B}(A(\nabla u)) \le cM^\sharp_{\omega,2B}(A(\nabla u)) + c\|f\|_{\mathrm{BMO}\omega(2B)}.$$

Moreover,

$$\|A(\nabla u)\|_{\mathrm{BMO}\omega(B)} \le cM^\sharp_{\omega,2B}(A(\nabla u)) + c\|f\|_{\mathrm{BMO}\omega(2B)}.$$

The constants depend on the characteristics of φ, β and c_0.

Proof. Let $\sigma := \frac{2\alpha}{p'}$, then $0 \le \beta < \sigma$. We divide the estimate of Proposition 5.1 by $\omega(2^{-m}R)$, where R is the radius of B.

$$M^\sharp_{\omega,2^{-m}B}(A(\nabla u)) \le c2^{-m\sigma} m \max_{0\le i\le m} \frac{\omega(2^{1-i}R)}{\omega(2^{-m}R)} M^\sharp_{\omega,2^{1-i}B}(A(\nabla u)) + c_m \frac{1}{\omega(2^{-m}R)} \|f\|_{\mathrm{BMO}(2B)}$$
$$\le c2^{-m\sigma} m \max_{0\le i\le m} \frac{(2^{1-i}R)^\beta}{(2^{-m}R)^\beta} M^\sharp_{\omega,2^{1-i}B}(A(\nabla u)) + c_m \frac{\omega(2R)}{\omega(2^{-m}R)} \|f\|_{\mathrm{BMO}\omega(2B)}$$
$$\le c2^{-m(\sigma-\beta)} m \max_{0\le i\le m} M^\sharp_{\omega,2^{1-i}B}(A(\nabla u)) + c_m 2^{(1+m)\beta} \|f\|_{\mathrm{BMO}\omega(2B)}.$$

Since $\sigma > \beta$, we find m_0 such that $c2^{-m(\sigma-\beta)}m \le \frac{1}{2}$ for all $m \ge m_0$. This implies

$$M^\sharp_{\omega,2^{-m}B}(A(\nabla u)) \le \frac{1}{2} \max_{0\le i\le m} M^\sharp_{\omega,2^{1-i}B}(A(\nabla u)) + c_m 2^{(1+m)\beta} \|f\|_{\mathrm{BMO}\omega(2B)}.$$

Applying this to all $m \in [m_0, 2m_0]$ we get

$$\max_{m_0\le m\le 2m_0} M^\sharp_{\omega,2^{-m}B}(A(\nabla u)) \le \frac{1}{2} \max_{0\le i\le 2m_0} M^\sharp_{\omega,2^{1-i}B}(A(\nabla u)) + c_{m_0} \|f\|_{\mathrm{BMO}\omega(2B)}.$$

Using this estimate repeatedly with B replaced by $2^{-m_0(l-2)}B$ with $l \in \{2, 3, \ldots\}$ and using $\|f\|_{\mathrm{BMO}\omega(2^{-lm_0}2B)} \le \|f\|_{\mathrm{BMO}\omega(2B)}$ we get

$$\max_{m_0\le m\le lm_0} M^\sharp_{\omega,2^{-m}B}(A(\nabla u)) \le \frac{1}{2} \max_{0\le i\le lm_0} M^\sharp_{\omega,2^{1-i}B}(A(\nabla u)) + c_{m_0} \|f\|_{\mathrm{BMO}\omega(2B)}.$$

This estimate implies by induction

$$\max_{m_0 \le m \le l m_0} M^\sharp_{\omega, 2^{-m}B}(A(\nabla u)) \le \max_{0 \le i \le m_0} M^\sharp_{\omega, 2^{1-i}B}(A(\nabla u)) + c_{m_0} \|f\|_{\mathrm{BMO}\omega(2B)}.$$

The estimate $\max_{0 \le i \le m_0} M^\sharp_{2^{1-i}B}(A(\nabla u)) \le c_{m_0} M^\sharp_{2B}(A(\nabla u))$ proves the first claim of the theorem. A standard covering argument proves the second claim. □

Corollary 5.4. *Let B be a ball in $\mathbb{R}^n$, u be a solution of (1.1) on $2B$ and φ satisfy Assumption 1.1. If $f \in \mathrm{VMO}(2B)$, then $A(\nabla u) \in \mathrm{VMO}(B)$.*

Proof. Since $f \in \mathrm{VMO}(2B)$, there exists a non-decreasing function $\tilde{\omega} : (0, \infty) \to (0, \infty)$ with $\lim_{r \to 0} \tilde{\omega}(r) = 0$, such that $\|f\|_{\mathrm{BMO}(B_r)} \le \tilde{\omega}(r)$, for all $B_r \subset 2B$. The result follows by Theorem 5.3 by defining $\omega(r) = \min\left\{\tilde{\omega}(r), r^{\frac{\alpha}{p'}}\right\}$. □

The next result is a direct consequence of Theorem 5.3 with the choice of $\omega(r) = r^\beta$ and the equivalence of $\mathrm{BMO}_\beta := \mathrm{BMO}_{t^\beta}$ and $C^{0,\beta}$.

Corollary 5.5. *Let φ hold Assumption 1.1. Let u be a solution of (1.1) on a ball $2B \subset \mathbb{R}^n$. Let α be the Hölder coefficient (defined in Theorem 4.1) for φ-harmonic gradients.*

If $f \in C^{0,\beta}(2B)$ for $\beta < \frac{2\alpha}{p'}$, then $A(\nabla u) \in C^{0,\beta}(B)$. Moreover,

$$\|A(\nabla u)\|_{\mathrm{BMO}_\beta(B)} \le c\|f\|_{\mathrm{BMO}_\beta(2B)} + cR^{-\beta} \fint_{2B} |A(\nabla u) - \langle A(\nabla u)\rangle_{2B}|.$$

The constant depends on β and the characteristics of φ.

Let us remark that the result in Corollary 5.5 is optimal in the sense that any improvement of α in the decay estimate Theorem 4.1 transfers directly to the inhomogeneous case in the best possible way.

Remark 5.6. If h is φ-harmonic on the open set $\Omega \subset \mathbb{R}^n$, then for any ball $B \subset \Omega$ we have the following decay estimate for $A(\nabla h)$. For any $\beta < \frac{2\alpha}{p'}$ (where α is from Theorem 4.1) and any $\lambda \in (0, 1]$ there holds

$$\begin{aligned}\fint_{\lambda B} |A(\nabla h) - \langle A(\nabla h)\rangle_{\lambda B}| &\le c_\beta (\lambda R)^\beta \|A(\nabla h)\|_{\mathrm{BMO}_\beta(B)} \\ &\le c_\beta \lambda^\beta \fint_B |A(\nabla h) - \langle A(\nabla h)\rangle_B|.\end{aligned}$$

Remark 5.7. Let us consider the system

$$-\operatorname{div}(A(\nabla u)) = g \quad \text{with } A(\nabla u) = \varphi'(|\nabla u|)\frac{\nabla u}{|\nabla u|},$$

where the right-hand side function g is not in divergence form. If $g \in L^n$, then there exists locally $f \in W^{1,n}$ with $\operatorname{div} f = g$ by solving the Laplace equation. Since $W^{1,n}$ embeds to VMO, it follows by Corollary 5.4 that $A(\nabla u) \in \mathrm{VMO}$ locally.

Let us compare this to the situation of [3,4], who studied the case $g \in L^{n,1}$ (Lorentz space) and proved $A(\nabla u) \in L^\infty$. Since $L^{n,1}$ embeds to L^n, we conclude that for such g additionally holds $A(\nabla u) \in \mathrm{VMO}$ locally.

Certainly, if $g \in L^s$ with $s > n$, then we find $f \in W^{1,s}$ and therefore $f \in C^{0,\sigma}$ with $\sigma = 1 - \frac{n}{s}$. Hence, by Corollary 5.5 we get Hölder continuity of $A(\nabla u)$.

Remark 5.8. Let us explain that our result includes the estimates of [2] in the super-quadratic case $p \ge 2$ with $\varphi(t) = t^p$. Let $A(Q) := \langle A(\nabla u)\rangle_B$. Then $p \ge 2$ implies $\varphi(t) = t^p \le \varphi_{|Q|}(t)$ and $(\varphi^*)_{|A(Q)|}(t) \le \varphi^*(t) = c_p t^{p'}$. Hence, with Lemmas 2.2, A.1 and Theorem 1.2 we estimate

$$\begin{aligned}\fint_B |\nabla u - Q|^p \, dx &\le \fint_B \varphi_{|Q|}(|\nabla u - Q|)\, dx \\ &\le c\fint_B (\varphi^*)_{|A(Q)|}(|A(\nabla u) - A(Q)|)\, dx \\ &\le c\fint_B (\varphi^*)(|A(\nabla u) - A(Q)|)\, dx \\ &\le c\|A(\nabla u)\|^{p'}_{\mathrm{BMO}(B)} \\ &\le c\|f\|^{p'}_{\mathrm{BMO}(2B)} + c\left(M^\sharp_{2B}(A(\nabla u))\right)^{p'}.\end{aligned}$$

Now, the estimate

$$\left(\fint_B |\nabla u - \langle \nabla u\rangle_B|\,dx\right)^p \leq \left(2\fint_B |\nabla u - Q|\,dx\right)^p$$

implies

$$\fint_B |\nabla u - Q|\,dx \leq c\|f\|_{\mathrm{BMO}}^{\frac{1}{p-1}} + c\big(M_{2B}^{\sharp}(A(\nabla u))\big)^{\frac{1}{p-1}}.$$

This is the same result as that of Manfredi and DiBenedetto [2]. Only the last, lower order term is expressed by Manfredi and DiBenedetto in terms of u rather that ∇u. This is just due to another application of the Caccioppoli estimate.

Remark 5.9. Our result also generalizes to the case of differential forms on $\Omega \subset \mathbb{R}^n$. In this Euclidean setting, we have the isometry $\Lambda^k \cong \mathbb{R}^{\binom{n}{k}}$, so the case of differential forms is just a special case of the vectorial situation. In particular, if $g \in \mathrm{BMO}(\Omega;\Lambda^k)$ and $d^*A(du) = d^*g$, with $u \in W^{1,\varphi}(\Omega;\Lambda^{k-1})$, then Theorem 5.3 (same ω) provides

$$\|A(du)\|_{\mathrm{BMO}_\omega(B)} \leq c\|g\|_{\mathrm{BMO}_\omega(2B)} + cM_{\omega,2B}^{\sharp}(A(du)). \tag{5.5}$$

Let us show that a simple conjugation argument (see also [13,14]) provides another interesting result: We start with a solution $v \in W^{1,\varphi}(\Omega;\Lambda^{k-1})$ of

$$d^*(A(dv+g)) = 0$$

which is a local minimizer of $\int \varphi(|dv+g|)\,dx$. By Hodge theory we find $w \in W^{1,\varphi^*}(\Omega,\Lambda^{k+1})$ such that

$$A(dv+g) = d^*w.$$

Applying A^{-1} and then d we get the dual equation

$$dg = d(A^{-1}(d^*w)).$$

If we define $A^* := (-1)^{k(n-k)} * A^{-1}*$, then we can rewrite this equation as

$$d^*(A^*(dw)) = \pm d^*(*g).$$

Moreover, we have (see [14]) that $A^*(dw) = (\varphi^*)'(|dw|)\frac{dw}{|dw|}$. In particular, we are in the same situation as with u if we replace φ by φ^* and dw by du. Therefore, by (5.5)

$$\|A^*(dw)\|_{\mathrm{BMO}_\omega(B)} \leq c\|g\|_{\mathrm{BMO}_\omega(2B)} + cM_{\omega,2B}^{\sharp}(A^*(dw)).$$

This and $A(dv+g) = d^*w$ implies

$$\|dv+g\|_{\mathrm{BMO}_\omega(B)} \leq c\|g\|_{\mathrm{BMO}_\omega(2B)} + cM_{\omega,2B}^{\sharp}(dv+g).$$

The triangle inequality gives

$$\|dv\|_{\mathrm{BMO}_\omega(B)} \leq c\|g\|_{\mathrm{BMO}_\omega(2B)} + cM_{\omega,2B}^{\sharp}(dv). \tag{5.6}$$

In particular, we can apply this argument to the φ-harmonic function h. Then (5.6) (with $g = 0$) implies the decay estimate

$$\fint_{\lambda B} |\nabla h - \langle \nabla h\rangle_{\lambda B}| \leq c\lambda^\beta \fint_{2B} |\nabla h - \langle \nabla h\rangle_B| \tag{5.7}$$

for all $\lambda \in (0,1]$ with $\beta = \frac{2\alpha}{q}$.

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Appendix

The classical John–Nirenberg estimate [15] proves the following lemma in the case $\psi(t) = t^p$. We give an extension to N-functions ψ.

Lemma A.1. *If ψ is an N-function, which satisfies the Δ_2 condition, $B \subset \mathbb{R}^n$ a ball and $g \in \mathrm{BMO}(B)$, then*

$$\fint_B \psi(|g - \langle g\rangle_B|)\,dx \le c\,\psi(\|g\|_{\mathrm{BMO}(B)}),$$

where c only depends on $\Delta_2(\psi)$.

Proof of Lemma A.1. Because $\psi \in \Delta_2$, there exists $q < \infty$ only depending on $\Delta_2(\varphi)$ such that

$$\psi'(st) \le c_1 \max\{1, s^{q-1}\}\psi'(t),$$

where c_1 only depends on $\Delta_2(\psi)$.

Since $g \in \mathrm{BMO}(B)$ we find by the classical John–Nirenberg estimate which can be found in [15]:

$$\frac{|\{x \in B : |g(x) - \langle g\rangle| > \lambda\}|}{|B|} \le \exp\left(\frac{-c_2\lambda}{\|g\|_{\mathrm{BMO}(B)}}\right),$$

where $c_2 \in (0, 1]$ only depends on the dimension. This implies

$$\begin{aligned}
\fint_B \psi(|g - \langle g\rangle|)\,dx &= \int_0^\infty \frac{|\{x \in B : |g(x) - \langle g\rangle| > \lambda\}|}{|B|}\psi'(\lambda)\,d\lambda \\
&\le \int_0^\infty \exp\left(\frac{-c_2\lambda}{\|g\|_{\mathrm{BMO}(B)}}\right)\psi'(\lambda)\,d\lambda \\
&= \frac{\|g\|_{\mathrm{BMO}(B)}}{c_2}\int_0^\infty \exp(-s)\psi'\left(\frac{s\|g\|_{\mathrm{BMO}(B)}}{c_2}\right)ds \\
&\le \frac{\|g\|_{\mathrm{BMO}(B)}}{c_2}\psi'\left(\frac{\|g\|_{\mathrm{BMO}(B)}}{c_2}\right)\int_0^\infty \exp(-s)\max\{1, s^{q-1}\}\,ds \\
&\le \frac{\|g\|_{\mathrm{BMO}(B)}}{c_2}\psi'\left(\frac{\|g\|_{\mathrm{BMO}(B)}}{c_2}\right)(1 + \Gamma(q)) \\
&\le (1 + \Gamma(q))\psi\left(\frac{2\|g\|_{\mathrm{BMO}(B)}}{c_2}\right) \\
&\le (1 + \Gamma(q))\left(\frac{2}{c_0}\right)^q \psi(\|g\|_{\mathrm{BMO}(B)}). \quad \square
\end{aligned}$$

Proof of Lemma 2.1. It has been shown in [16] that if $\varphi \in T(p, q, K)$, then $\varphi^{-1} \in T(1/q, 1/p, K_1)$, where K_1 only depends on p, q and K. From this, (2.4) and

$$t \le \varphi^{-1}(t)(\varphi^*)^{-1}(t) \le 2t$$

it follows, that $(\varphi^*)^{-1} \in T(1 - 1/p, 1 - 1/q, 2K_1)$ and as a consequence $\varphi^* \in T(q', p', K_2)$ with $K_2 = K_2(p, q, K)$. $\square$

Proof of Lemma 2.3. Let $\varphi \in T(p, q, K)$. Then φ_a is of type $T(\bar{p}, \bar{q}, K_5)$, where K_5 only depends on K, p, q. Recall that every N-function ψ satisfies $\psi(t) \le \psi'(t)\,t \le \psi(2t)$, see for example [6]. This and $\varphi \in T(p, q, K)$ implies

$$\varphi'(st) \le \frac{\varphi(2st)}{st} \le K2^q \max\{s^p, s^q\}\frac{\varphi(t)}{st} \le K2^q \max\{s^{p-1}, s^{q-1}\}\varphi'(t).$$

We define $\tau = \frac{a+st}{a+t}$. This implies

$$\begin{aligned}
\varphi_a'(st) &= \frac{\varphi'(\tau(a+t))}{a+st}st \le K2^q \max\{\tau^{p-1}, \tau^{q-1}\}\varphi'(a+t)\frac{st}{a+st} \\
&= K2^q s \max\{\tau^{p-2}, \tau^{q-2}\}\varphi_a'(t) \\
&\le K2^q s \max\{\tau^{\bar{p}-2}, \tau^{\bar{q}-2}\}\varphi_a'(t)
\end{aligned}$$

for all $s, t \geq 0$. Now we split the cases $s \geq 1$ and $s \in (0, 1)$ and apply $\overline{p} \leq 2 \leq \overline{q}$. It follows

$$\max\{\tau^{\overline{p}-2}, \tau^{\overline{q}-2}\} \leq \max\{s^{\overline{p}-2}, s^{\overline{q}-2}\}.$$

This and the previous estimate proves the claim for $\varphi_{|P|}$. Since $\varphi \in T(p, q, K)$, we have $\varphi^*(q', p', K_2)$ by Lemma 2.1. This proves the claim for $(\varphi^*)_{|A(P)|}$. Now, the equivalence $(\varphi_{|P|})^*(t) \sim (\varphi^*)_{|A(P)|}(t)$ of Lemma 2.2 concludes the proof. □

In the following equivalence lemma is used in the proof of Proposition 5.1. It allows us to express the mean oscillation of $V(\nabla u)$ in terms of different mean values.

Lemma A.2. *Let φ satisfy Assumption 1.1. Let $B \subset \mathbb{R}^n$ be a ball and $g \in L^\varphi(B; \mathbb{R}^{N\times n})$. Define $g_A \in \mathbb{R}^{N\times n}$ by $A(g_A) := \langle A(g)\rangle_B$. Then*

$$\fint_B |V(g) - \langle V(g)\rangle_B|^2\,dx \sim \fint_B |V(g) - V(\langle g\rangle_B)|^2\,dx \sim \fint_B |V(g) - V(g_A)|^2\,dx$$

holds. The constants are independent of B and g; they only depend on the characteristics of φ.

Proof. Define $g_V \in \mathbb{R}^{N\times n}$ by $V(g_V) := \langle V(g)\rangle_B$. We denote the three terms by (I), (II) and (III). Note that

$$(\mathrm{I}) = \inf_{P\in\mathbb{R}^{N\times n}} \fint_B |V(g) - P|^2\,dx,$$

which proves (I) $\leq$ (II) and (I) $\leq$ (III).

We calculate with Lemma 2.4 and $\langle A(g) - A(g_A)\rangle_B = 0$

$$(\mathrm{II}) \sim \fint_B \big(A(g) - A(g_A)\big)\cdot(g - g_A)\,dx = \fint_B \big(A(g) - A(g_A)\big)\cdot(g - g_V)\,dx.$$

Again, by Lemma 2.4, Young's inequality with $\varphi_{|g|}$ in combination with (2.2) (second part) and again Lemma 2.4 we estimate

$$\begin{aligned}(\mathrm{II}) &\leq c\fint_B \varphi'_{|g|}(|g - g_A|)|g - g_V|\,dx\\ &\leq \delta\fint_B \varphi_{|g|}(|g - g_A|)\,dx + c_\delta\fint_B \varphi_{|g|}(|g - g_V|)\,dx\\ &\leq \delta\,c\fint_B |V(g) - V(g_A)|^2\,dx + c_\delta\fint_B |V(g) - V(g_V)|^2\,dx\\ &\leq \delta\,c(\mathrm{II}) + c_\delta(\mathrm{I}).\end{aligned}$$

It follows that (II) $\leq c$(I).

On the other hand with Lemma 2.4 and $\langle g - \langle g\rangle_B\rangle_B = 0$ there follows

$$(\mathrm{III}) \sim \fint_B \big(A(g) - A(\langle g\rangle_B)\big)\cdot(g - \langle g\rangle_B)\,dx = \fint_B \big(A(g) - A(g_V)\big)\cdot(g - \langle g\rangle_B)\,dx.$$

By Young's inequality with $\varphi_{|g|}$ it follows analogously to the estimates of (II) that (III) $\leq c_\delta$(I) $+ \delta\,c$(III). Now, (III) $\leq c$(I) follows. □

Lemma A.3. *Let ψ be of type $T(p, q, K)$ and let $\gamma \in (0, 1)$ such that $\gamma q \leq 1$. Then the function $(\psi^\gamma)^{-1}$ is quasi-convex, i.e. there exists a convex function $\kappa : [0, \infty) \to [0, \infty)$ such that $(\psi^\gamma)^{-1}(t) \sim \kappa(t)$. The implicit constant only depends on q and K.*

Proof. Define $\rho(t) := \psi^\gamma(t)$. Since ψ is of type $T(p, q, K)$, there holds $\psi(st) \leq Ks^q\psi(t)$ for all $t \geq 0$ and $s \geq 1$. This implies $s\psi^{-1}(u) \leq \psi^{-1}(Ks^q u)$ for all $u \geq 0$ and $s \geq 1$. From $\rho^{-1}(u) = \psi^{-1}(u^{1/\gamma})$ and $\psi^{-1}(t) = \rho^{-1}(t^\gamma)$ we get $s\rho^{-1}(u) \leq \rho^{-1}(K^\gamma s^{\gamma q}u)$. In particular, with $\gamma q \leq 1$ there follows

$$\frac{\rho^{-1}(u)}{u} \leq \frac{\rho^{-1}(K^\gamma s^{\gamma q-1} su)}{su} \leq \frac{\rho^{-1}(K^\gamma su)}{su}$$

for all $u \geq 0$ and $s \geq 1$. Therefore, Lemma 1.1.1 of [17] implies that ρ^{-1} is quasi-convex. □

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