Lecture NDIR247

The aim of this course is to study qualitative properties of weak solutions to a system of PDE's

$$-\operatorname{div}(T(\nabla u)) = -\operatorname{div} f \quad \text{in } \Omega.$$

The smooth domain $\Omega \subset \mathbb{R}^n$ is given as well as $f : \Omega \to \mathbb{R}^{N\times n}$ and $T : \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$. We assume that T has a special form

$$\forall A \in \mathbb{R}^{N\times n} : T(A) = \frac{\varphi'(|A|)}{|A|} A$$

for a given N-function $\varphi : [0, +\infty) \to [0, +\infty)$.

Plan of the lecture. 1) Basic properties of N-functions 2) Existence of a weak solutions 3) Higher integrability of the weak solutions 4) Differentiability 5) Boundedness of gradients 6) Hölder continuity of gradients 7) L^p/BMO theory

We will follow articles:

- Diening, Lars; Ettwein, Frank Fractional estimates for non-differentiable elliptic systems with general growth. Forum Math. 20 (2008), no. 3, 523–556.
- Diening, Lars; Stroffolini, Bianca; Verde, Anna Everywhere regularity of functionals with φ–growth. Manuscripta Math. 129 (2009), no. 4, 449–481.
- Diening, L.; Kaplický, P.; Schwarzacher, S. BMO estimates for the p-Laplacian. Nonlinear Anal. 75 (2012), no. 2, 637–650.

Definition. A real function $\varphi : [0, +\infty) \to [0, +\infty)$ is N-function if there is a function $\psi : (0, +\infty) \to (0, +\infty)$ such that ψ is increasing, right continuous, $\psi(0+) = 0$, $\psi(+\infty-) = +\infty$ and $\varphi(t) = \int_0^t \psi(s)ds$. We denote $\varphi' = \psi$.

Example.
- $\varphi(t) = t^p/p$, $\varphi'(t) = t^{p-1}$
- $\varphi(t) = (t + 1)\lg(t + 1) - t$
- $\varphi(t) = e^t - t - 1$

Jensen's inequality. Let μ be a probability measure on Ω, $f \in L^1(\Omega, \mu)$, φ convex. $\varphi(\int_\Omega f d\mu) \leq \int_\Omega \varphi \circ f d\mu$.

Definition. If φ' is strictly increasing and continuous, we define $(\varphi^*)' = (\varphi')_{-1}$. Otherwise we take $(\varphi^*)'(t) = \sup\{s \in [0, +\infty), \varphi'(s) \leq t\}$, the so-called right continuous inverse. We define *complementary N-function to* φ by $\varphi^*(t) = \int_0^t (\varphi^*)'(s)ds$.

Example. $\varphi(t) = t^p/p, p > 1$, $\varphi^*(t) = t^{p'}/p'$.

Lemma. (Young's inequality) $\forall t, s > 0 : st \leq \varphi(s) + \varphi^*(t)$. The equality holds if and only if $t = \varphi'(s)$ or $s = (\varphi^*)'(t)$.

Corollary. $\varphi^*(t) = \sup_{s>0}(st - \varphi(s))$.

Definition. We say that an N-function φ satisfies Δ_2 condition if: $\exists C > 0, \forall t > 0 : \varphi(2t) \leq C\varphi(t)$. We say that φ satisfies ∇_2 condition if: $\exists C > 1, \forall t > 0 : \varphi(t) \leq \varphi(ct)/2c$.

Remark. We write $\varphi \sim \psi$ for $t > 0$ if: $\exists C, D > 0, \forall t > 0 : \varphi(t) \in [C\psi(t), D\psi(t)]$. Then φ satisfies $\Delta_2$condition iff $\varphi(t) \sim \varphi(2t)$ for $t > 0$.

Lemma 1. Let φ be an N-function, denote ψ its conjugate function. TFAE 1) φ satisfies Δ_2 condition 2) $\exists \alpha > 1, \forall x > 0 : x\varphi'(x) \leq \alpha\varphi(x)$ 3) $\exists \beta > 1, \forall y > 0 : y\psi'(y) \geq \beta\psi(y)$ 4) ψ satisfied ∇_2 condition.

End of the lecture 1 (8.10.2012)

Assumption 1. From now on we will always assume $\Delta_2(\varphi, \varphi^*) < +\infty$.

Definice. Let $\Delta_2(\varphi, \varphi^*) < +\infty$. We define

$$L^\varphi(\Omega) = \{f \in L^1(\Omega); \int_\Omega \varphi(|f|) < +\infty\}$$
$$\|f\|_\varphi = \inf\{\lambda > 0; \int_\Omega \varphi(\frac{|f|}{\lambda} < 1\}$$
$$W^{1,\varphi}(\Omega) = \{f \in W^{1,\varphi}(\Omega); f \in L^\varphi(\Omega), \nabla f \in L^\varphi(\Omega)\}$$
$$\|f\|_{1,\varphi} = \|f\|_\varphi + \|\nabla\varphi\|_\varphi.$$

Remark. If $\Delta_2(\varphi, \varphi^*) < +\infty$ then $L^\varphi(\Omega)$, $W^{1,\varphi}(\Omega)$ are separable, reflexive Banach spaces.

Lemma 2. Let $\Delta_2(\varphi, \varphi^*) < +\infty$. There are $p, q \in (0, +\infty)$ and $K > 0$ (they depend only on $\Delta_2(\varphi, \varphi^*)$) such that

$$\forall s, t > 0 : \varphi(st) \leq K \max(s^p, s^q)\varphi(t).$$

Corollary (Young's inequality). Let $\Delta_2(\varphi, \varphi^*) < +\infty$. Then

$$\forall \varepsilon > 0, \exists C > 0, \forall s, t > 0 : st \leq \varepsilon\varphi(s) + C\varphi^*(t).$$

Corollary. Under the assumptions of the previous lemma

$$\forall s, t > 0 : \varphi(st) \geq K \min(s^p, s^q)\varphi(t).$$

Corollary. If $p < q$ in the previous lemma, Ω bounded, then $W^{1,p}(\Omega) \subset W^{1,\varphi}(\Omega) \subset W^{1,q}(\Omega)$.

Lemma 3. For all $t > 0$ $\varphi(t) \sim \varphi'(t)t$ and $\varphi^*(\varphi'(t)) \sim \varphi(t)$ uniformly in $t > 0$.

Assumption 2. From now on we will assume that $\Delta_2(\varphi, \varphi^*) < +\infty$, $\varphi \in C^1([0, +\infty)) \cap C^2((0, +\infty))$ and $\varphi'(t) \sim \varphi''(t)t$ uniformly in $t > 0$.

Remark. It holds $\varphi(2t) \sim \varphi(t)$, $\varphi'(2t) \sim \varphi'(t)$, $\varphi''(2t) \sim \varphi''(t)$ uniformly in $t > 0$.

Definition. We say that $\varphi : [0 + \infty) \to \mathbb{R}$ is quasiconvex if there is a convex function ω and a constant $c > 1$ such that $\omega(t) \le \varphi(t) \le c\omega(ct)$ for all $t \ge 0$.

Lemma 4. Let $\varphi : (0 + \infty) \to \mathbb{R}$ be a non-negative, increasing, $\varphi(0+) = 0$, $\varphi(+\infty-) = +\infty$. Then φ is quasiconvex iff

$$\exists C > 1, \forall 0 < t_1 < t_2 : \frac{\varphi(t_1)}{t_1} \le C \frac{\varphi(Ct_2)}{t_2}.$$

End of the lecture 2 (15.10.2012)

Lemma 5. There is $\theta \in (0,1)$ and an N-function ρ with $\varphi^\theta \sim \rho$ and $\Delta_2(\rho, \rho^*) < +\infty$. The constants θ and $\Delta_2(\rho, \rho^*)$ only depend on $\Delta_2(\varphi, \varphi^*)$.

Lemma 6. If $\alpha > -1$ then

$$(|\mathbf{P}| + |\mathbf{Q}|)^\alpha \sim \int_0^1 |(1-\theta)\mathbf{P} + \theta\mathbf{Q}|^\alpha d\theta$$

uniformly in $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N \times n}$ such that $|\mathbf{P}| + |\mathbf{Q}| > 0$.

Lemma 7. It holds

$$\int_0^1 \frac{\varphi'(|(1-\theta)\mathbf{P} + \theta\mathbf{Q}|)}{|(1-\theta)\mathbf{P} + \theta\mathbf{Q}|} d\theta \sim \frac{\varphi'(|\mathbf{P}| + |\mathbf{Q}|)}{|\mathbf{P}| + |\mathbf{Q}|}$$

uniformly in $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N \times n}$ such that $|\mathbf{P}| + |\mathbf{Q}| > 0$. The constants depend only on $\Delta_2(\varphi, \varphi^*)$.

Lemma 8. Let $\Phi : \mathbb{R}^{N \times n} \to [0, +\infty)$ be given by $\Phi(\mathbf{Q}) := \varphi(|\mathbf{Q}|)$ and let $\mathbf{A}(\mathbf{Q}) := (\nabla_{N \times n}\Phi)(\mathbf{Q})$. Then $\mathbf{A}(\mathbf{Q}) = \varphi'(|\mathbf{Q}|)\mathbf{Q}/|\mathbf{Q}|$ for $\mathbf{Q} \neq 0$, $\mathbf{A}(0) = 0$, and $\mathbf{A}$ satisfies for all $\mathbf{P}, \mathbf{Q}$

$$\begin{aligned}\big(\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})\big) \cdot \big(\mathbf{P} - \mathbf{Q}\big) &\ge c\,\varphi''(|\mathbf{P}| + |\mathbf{Q}|)\,|\mathbf{P} - \mathbf{Q}|^2, \\ |\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})| &\le C\,\varphi''(|\mathbf{P}| + |\mathbf{Q}|)\,|\mathbf{P} - \mathbf{Q}|.\end{aligned}$$

End of the lecture 3 (22.10.2012)

Lemma 9. Let $u_n, u \in L^\varphi(\Omega)$, 1) $\exists L > 0 : \int_\Omega \varphi(|u_n|) < L$, is equivalent to 2) $\exists K > 0 : \|u_n\|_\varphi < K$. 3) $u_n \to u$ in $L^\varphi(\Omega)$ is equivalent to 4) $\int_\Omega \varphi(|u_n - u|) \to 0$ as $n \to +\infty$.

Lemma 10. Let $\mathbf{f}, \mathbf{g} \in L^\varphi(\Omega)$ and $\mathbf{A}$ be from previous Lemma. Then $\mathbf{A}(\mathbf{f} + s\mathbf{g}) \to \mathbf{A}(\mathbf{f})$ in $L^{\varphi^*}(\Omega)$ as $s \to 0+$.

Lemma 11. $C^\infty(\Omega)$ is dense in $L^\varphi(\Omega)$ and in $W^{1,\varphi}(\Omega)$.

Lemma 12 (Fridrichs-Sobolev-Poincaré inequality). Let $\Omega \subset \mathbb{R}^n$, $|\Omega| \sim R^n$ with $R = \operatorname{diam}(\Omega)$.
1) There is $C > 0$ and $\theta \in (0,1)$ such that for all $f \in W_0^{1,\varphi}(\Omega)$ the inequality $\left(⨍_\Omega \varphi(\frac{|f|}{R})\right)^\theta \leq C ⨍_\Omega \varphi^\theta(|\nabla f|)$ holds.
2) Let moreover Ω be convex and $\omega \in L^\infty(\Omega)$ then there is $C > 0$ and $\theta \in (0,1)$ such that for all $f \in W^{1,\varphi}(\Omega)$ the inequality $\left(⨍_\Omega \varphi(|\frac{f-(f)_\omega}{R}|)\right)^\theta \leq C ⨍_\Omega \varphi^\theta(|\nabla f|)$ holds.

End of the lecture 4 (29.10.2012)

Theorem 1. Let Assumption 2 hold, $\mathbf{A}$ be given in Lemma 8, $\mathbf{f} \in L^{\varphi^*}(\Omega)$. Then there exists a unique weak solution of the problem

$$-\operatorname{div}(\mathbf{A}(\nabla\mathbf{u})) = -\operatorname{div}(\mathbf{f}) \tag{1}$$

in Ω with the homogeneous Dirichlet boundary condition $\mathbf{u} = 0$ on $\partial\Omega$.

Definition. We say that $\mathbf{u} \in W^{1,\varphi}(\Omega)$ is a local weak solution of (1) if

$$\forall \boldsymbol{\xi} \in W_0^{1,\varphi}(\Omega) : \int_\Omega (\mathbf{A}(\nabla\mathbf{u}) - \mathbf{f}) : \nabla\boldsymbol{\xi} = 0.$$

Theorem 2 ([DE08, Theorem 4]). Let $\mathbf{u}$ be a local weak solution of system (1). Then there exists $c > 1$ such that for all balls Q with $2Q \Subset \Omega$ holds

$$⨍_Q \varphi(|\nabla\mathbf{u}|) \leq c\left(⨍_{2Q} \varphi(\frac{\mathbf{u} - (\mathbf{u})_{2Q}}{R}) + ⨍_{2Q} \varphi^*(|\mathbf{f}|)\right),$$

where R is the radius of the ball Q. The constant c only depends on $\Delta_2(\varphi, \varphi^*)$.

Remark. We can prove that for all $\mathbf{v} \in \mathbb{R}^N$, $\mathbf{P} \in \mathbb{R}^{N \times n}$

$$⨍_Q \varphi(|\nabla\mathbf{u}|) \leq c\left(⨍_{2Q} \varphi(\frac{\mathbf{u} - \mathbf{v}}{R}) + ⨍_{2Q} \varphi^*(|\mathbf{f} - \mathbf{P}|)\right)$$

Theorem 3 (Gehring-Giaquinta-Modica, [Gia82, Proposition V.1.1]). Let $Q_0 \subset \mathbb{R}^n$ be a ball, $G \in L^1(Q_0)$, and $H \in L^{q_0}(Q_0)$ for some $q_0 > 1$. Suppose that for some $q \in (0,1)$, $c_1 > 0$, and all balls Q with $2Q \subset Q_0$

$$\fint_Q |G| \le c_1 (\fint_{2Q} |G|^\theta)^{\frac{1}{\theta}} + \fint_{2Q} |H|.$$

Then there exist $q_1 > 1$ and $c_2 > 1$ such that $G \in L^{q_1}_{loc}(Q)$ and for all $q \in [1, q_1]$

$$(\fint_Q |G|^q)^{\frac{1}{q}} \le c_2 \fint_{2Q} |G| + c_2 (\fint_{2Q} |H|^q)^{\frac{1}{q}}.$$

Corollary ([DE08, Theorem 9]). Let $\mathbf{u}$ be a local weak solution of system (1). Then there exists $c > 1$, $q_0 > 1$ such that for all balls Q with $2Q \Subset \Omega$, $\mathbf{P} \in \mathbb{R}^{N\times n}$ and $q \in [1, q_0)$ holds

$$\left(\fint_Q \varphi^q(|\nabla \mathbf{u}|)\right)^{\frac{1}{q}} \le c \fint_{2Q} \varphi\Big(\frac{\mathbf{u} - (\mathbf{u})_{2Q}}{R}\Big) + c\left(\fint_{2Q} \varphi^*(|\mathbf{f} - \mathbf{P}|)^q\right)^{\frac{1}{q}},$$

where R is the radius of the ball Q. The constants c, q_0 only depends on $\Delta_2(\varphi, \varphi^*)$.

Definition. Let φ be an N-functions with $\Delta_2(\varphi, \varphi^*) < +\infty$. Then for $a \ge 0$ we define $(\varphi_a)' : [0, +\infty) \to [0, +\infty)$ by $(\varphi_a)'(t) = t\varphi'(t+a)/(t+a)$. Further we define $\varphi_a(t) = \int_0^t (\varphi_a)'(s)ds$.

Lemma 13. ([DE08, Lemma 23]) Let φ satisfy $\Delta_2(\varphi, \varphi^*) < +\infty$. Then for all $a \ge 0$ the function φ_a is an N-function and $\Delta_2(\{\varphi_a\}_{a\ge 0}) < +\infty$, i. e. the family φ_a satisfies the Δ_2-condition uniformly in $a \ge 0$.

End of the lecture 5 (5.11.2012)

Lemma 14. ([DE08, Lemma 26]) Let φ be as in the previous Lemma. Then $(\varphi_a)^*(t) \sim (\varphi^*)_{\varphi'(a)}(t)$ uniformly in $a, t \ge 0$.

Lemma 15. ([DE08, Lemma 27]) Let φ satisfy $\Delta_2(\varphi, \varphi^*) < +\infty$. Then $\Delta_2(\{\varphi_a\}_{a\ge 0}, \{(\varphi_a)^*\}_{a\ge 0}) < +\infty$, i. e. the families φ_a and $(\varphi_a)^*$ satisfy the Δ_2-condition uniformly in $a \ge 0$.

Definition. For a given N-function φ we define ψ by

$$\frac{\psi'(t)}{t} = \left(\frac{\varphi'(t)}{t}\right)^{\frac{1}{2}},$$

and $\mathbf{V}(\mathbf{P}) = \psi'(\mathbf{P})\mathbf{P}/|\mathbf{P}|$ for $\mathbf{P} \in \mathbb{R}^{N\times n}$.

Lemma 16. ([DE08, Lemma 25]) Let φ hold Assumption 2. Then ψ also satisfies Assumption 2 and Lemma 8 applies to $\varphi, \mathbf{A}$ replaced with $\psi, \mathbf{V}$.

End of the lecture 6 (12.11.2012)

Lemma 17.([DE08, Lemma 3]) Let $\mathbf{A}$ be given in Lemma 8. Then $\mathbf{A}$ satisfies uniformly in $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N\times n}$

$$\begin{aligned}\big(\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})\big) \cdot \big(\mathbf{P} - \mathbf{Q}\big) &\sim \varphi''(|\mathbf{P}| + |\mathbf{Q}|)\,|\mathbf{P} - \mathbf{Q}|^2, \\ &\sim \varphi_{|\mathbf{P}|}(|\mathbf{Q} - \mathbf{P}|) \\ &\sim |\mathbf{V}(\mathbf{P}) - \mathbf{V}(\mathbf{Q})|^2\end{aligned}$$

Moreover, $\mathbf{A}(\mathbf{Q}) \cdot \mathbf{Q} \sim |\mathbf{V}(\mathbf{Q})|^2 \sim \varphi(|\mathbf{Q}|)$.

Lemma 18. ([DKS12, Lemma 3.2]) Let $\mathbf{u}$ be a local weak solution of (1). There exists $\theta \in (0,1)$ such that for all $P, f_0 \in \mathbb{R}^{N\times n}$ and all balls B satisfying $2B \subset \Omega$

$$\begin{aligned}\fint_B |V(\nabla u) - V(P)|^2\,dx &\le c\left(\fint_{2B} |V(\nabla u) - V(P)|^{2\theta}\,dx\right)^{\frac{1}{\theta}} \\ &\quad + c\fint_{2B} (\varphi^*)_{|A(P)|}(|f - f_0|)\,dx\end{aligned}$$

holds. The constants c and θ only depend on the characteristics of φ.

End of the lecture 7 (19.11.2012)

Lemma 19. Let $f : [a,b] \to [0,+\infty)$ be a bounded function satisfying the condition

$$\exists A, B, \alpha > 0, \varepsilon \in (0,1), \forall r, R \in [a,b] : f(r) \le \varepsilon f(R) + A(R-r)^{-\alpha} + B.$$

There exists $c = c(\alpha, \varepsilon)$ such that

$$\forall r, R \in [a,b] : f(r) \le c[A(R-r)^{-\alpha} + B].$$

Lemma 20. Let $\alpha, C > 0$, $\gamma > 1$ and $\varepsilon \in (0,1)$. Let for all cubes Q such that $2Q \Subset \Omega$ the inequality (R denotes the sidelength of the cube)

$$\int_Q |f| \le \varepsilon \int_{2Q} |f| + (\frac{C}{R})^\alpha (\int_{2Q} |g|^\gamma)^{\frac{1}{\gamma}}$$

holds. Then there is $N>0$ such that for all $0<r<\rho<1$, $Q_r \subset Q_R \Subset \Omega$

$$\int_{Q_r} |f| \le N\varepsilon \int_{Q_\rho} |f| + C(\rho-r)^{-\alpha}\Big(\int_{Q_\rho} |g|^\gamma\Big)^{\frac{1}{\gamma}}.$$

Corollary 21. Under the assumptions of Lemma 18 for any $\gamma \in (0,1)$ exists $c>0$ such that such that for all $P, f_0 \in \mathbb{R}^{N\times n}$ and all balls B satisfying $2B \subset \Omega$

$$\fint_B |V(\nabla u) - V(P)|^2\,dx \le c\bigg(\fint_{2B} |V(\nabla u)-V(P)|^{2\gamma}\,dx\bigg)^{\frac{1}{\gamma}} + c\fint_{2B} (\varphi^*)_{|A(P)|}(|f-f_0|)\,dx$$

holds. The constants c and θ only depend on the characteristics of φ.

Corollary 22. Under the assumptions of Corollary 21 there exists $\alpha > 1$ such that for all $P, f_0 \in \mathbb{R}^{N\times n}$ and all balls B satisfying $2B \subset \Omega$

$$\Big(\fint_B |V(\nabla u) - V(P)|^{2\alpha}\,dx\Big)^{\frac{1}{\alpha}} \le c\bigg(\fint_{2B} |V(\nabla u)-V(P)|^{2\gamma}\,dx\bigg)^{\frac{1}{\gamma}} + c\Big(\fint_{2B} (\varphi^*)_{|A(P)|}(|f-f_0|)^\alpha\,dx\Big)^{\frac{1}{\alpha}}$$

holds. The constant c only depends on the characteristics of φ.

Lemma 23. Let for an increasing function ψ, $K>0$, $q>1$ the inequality $\psi(st) \le Ks^q\psi(t)$ holds for all $t \ge 0$, $s \ge 1$. Then there is a $\gamma \in (0,1)$ (actually any γ such that $\gamma q \le 1$) that $(\psi^\gamma)_{-1}$ is quasiconvex.

Corollary 24. Under the assumptions of Lemma 18 and Corollary 21 we have that for all $P, f_0 \in \mathbb{R}^{N\times n}$ and all balls B satisfying $2B \subset \Omega$

$$\fint_B |V(\nabla u) - V(P)|^2\,dx \le c\fint_{2B} (\varphi^*)_{|A(P)|}(|f-f_0|)\,dx + c\begin{cases} \varphi_{|P|}\Big(\fint_{2B} |\nabla u - P|\Big) \\ \big(\varphi_{|A(P)|}\big)^*\Big(\fint_{2B} |A(\nabla u) - A(P)|\Big) \end{cases}$$

Differentiability

Lemma 25 [DE, Lemma 29]. Let Assumption 2 hold, $M \in \mathbb{N}$. Then a constant $C > 0$ exists such that for all $a, b, c \in \mathbb{R}^M$

$$\varphi'_{|a|}(|b-a|) \le C(\varphi'_{|c|}(|b-c|) + \varphi'_{|c|}(|a-c|).$$

Lemma 26 [DE, Lemma 30]. Let Assumption 2 hold. Then a constant $C > 0$ exists such that for all $a > 0$, $h \in (0,1)$, $\varphi_a(ha) \le Ch^2\varphi(a)$.

Definition. For $h \in \mathbb{R}^n \setminus \{0\}$, $u : \mathbb{R}^n \to \mathbb{R}^N$ we define $T_h(x) = x + h$ and $\tau_h u(x) = u(T_h(x)) - u(x)$.

Lemma 27. Let u be a weak solution of the system (1) with $\operatorname{div} f = F$. Then there exists $C > 0$ such that: if $Q \subset \Omega$ satisfies $20Q \Subset \Omega$ and $s, h \in \mathbb{R}^n \setminus \{0\}$, $|s| \le |h| < R$ sidelength of Q then

$$\begin{aligned}\fint_Q |\tau_s V(\nabla u)|^2 dx \le & C|h|^2 \big(R^{-2} \fint_{20Q} |V(\nabla u)|^2 dx + \fint_{20Q} \varphi^*(|\nabla F|) dx\big) \\ & + \delta \frac{|s|}{|h|} \fint_0^{|s|} \fint_{20Q} |\tau_{\frac{\lambda s}{|s|}} V(\nabla u)|^2 dx d\lambda.\end{aligned}$$

The constant C may depend only on $\Delta_2(\varphi, \varphi^*)$ and constants from Assumption 2.

Theorem 28 [DE, Theorem 11]. Under the assumptions of Lemma 27 it holds

$$\fint_Q |\tau_h V(\nabla u)|^2 dx \le C|h|^2 \big(R^{-2} \fint_{20Q} |V(\nabla u)|^2 dx + \fint_{20Q} \varphi^*(|\nabla F|) dx\big).$$

Corollaries of differentiability.

Assumption 3. Let φ satisfies $\Delta(\varphi, \varphi^*)$ and moreover $\lim_{t\to 0+} \varphi''(t) \in (0, +\infty)$.

Remark. We excluded $\varphi(t) = t^p$ but $\varphi(t) = (1+t^2)^{p/2}$ is still allowed. Further we always assume Assumption 3 and $F = 0$ ($f = 0$).

Lemma 29 [DSV09]. Let u be a solution of (1). There is $s > 1$ such that $u \in W^{2,s}_{loc}(\Omega)$.

Lemma 30 [DSV09]. Let $w \in W^{2,s}(\Omega)$ with $s > 1$ and $V(\nabla w) \in W^{1,2}(\Omega)$. Then $\varphi''(|\nabla w|)|\partial_i \nabla w|^2 \sim |\partial_i V(\nabla w)|^2$ for $i = 1, \dots, n$ a.e. in Ω. Consequently, $\int_B \varphi''(|\nabla w|)|\partial_i \nabla w|^2 dx \sim \int_B |\partial_i V(\nabla w)|^2 dx$ for any $B \Subset \Omega$.

Lemma 31 [DSV09] Let u be a solution to (1), B a ball such that $2B \subset \Omega$ and $V(\nabla u) \in L^{2q}(2B)$ for a $q \geq 1$. Then $\varphi(|\nabla u|) \in W^{1,2q/(1+q)}(B)$, $\varphi'(|\nabla u|)|\nabla^2 u|^2 \in L^{2q/(1+q)}(B)$ and

$$\Big(\fint_B |\nabla\varphi(|\nabla u|)|^{\frac{2q}{1+q}}\Big)^{\frac{1+q}{2q}} \leq c\Big(\fint_{2B} (\varphi'(|\nabla u|)|\nabla^2 u|^2)^{\frac{2q}{1+q}}\Big)^{\frac{1+q}{2q}} \leq cR^{-1}\Big(\fint_{2B} |V(\nabla u)|^{2q}\Big)^{\frac{1}{q}}.$$

Moreover, for any $j = 1, \dots, n$, $\partial_j \varphi(|\nabla u|) = \varphi'(|\nabla u|)\partial_j|\nabla u|$ a.e. in 2B.

Corollary 32. Let u be a solution to (1), B a ball such that $2B \subset \Omega$. Then $\varphi(|\nabla u|) \in W^{1,n/(n-1)}(B)$ and

$$\Big(\fint_B |\nabla\varphi(|\nabla u|)|^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}} \leq cR^{-1}\Big(\fint_{2B} |V(\nabla u)|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \leq cR^{-1}\Big(\fint_{2B} |V(\nabla u)|^2\Big).$$

Lemma 32. There is $s_2 > 1$ such that for all u, solutions of (1), holds $A(\nabla u) \in W^{1,s_2}_{loc}(\Omega)$.

Lemma 33. Let G satisfy $G \in W^{1,s}(B)$, $A(G) \in W^{1,s}(B)$ for some $s > 1$ and ball B. Then a.e. in B

$$\partial_i A_{jk}(G) = \frac{\varphi'(|G|)}{|G|}\partial_i G_{jk} + (\varphi''(|G|) - \frac{\varphi'(|G|)}{|G|})\frac{G_{jk}}{|G|}\partial_i|G|.$$

Theorem 34 [DSV09]. Let Assumption 3 hold. Let u be a local solution to the problem (1) and let B be a ball with $2B \Subset \Omega$. Then there exists $G : 2B \to \mathbb{R}^{n\times n}$ which is uniformly elliptic and $c_3 > 0$ such that

$$\int \sum_{kl} \big[G^{kl}(\nabla u)\partial_l(\varphi(|\nabla u|))\big]\partial_k \eta dx \leq -c\int \eta|\nabla V(\nabla u)|^2 dx \leq 0$$

holds for all $\eta \in C^1_0(2B)$, $\eta \geq 0$. Moreover,

$$c_0|\xi|^2 \leq \sum_{kl} G^{kl}(Q)\xi_k\xi_l \leq c_1|\xi|^2$$

for all $Q \in \mathbb{R}^{n\times N}$ and all $\xi \in \mathbb{R}^n$, where $c_0, c_1 > 0$ are suitably chosen.

End of the lecture (25.2.2013)

Theorem 35.(see [MP06]) Let Assumption 3 hold. Let u be a local solution to the problem (1) and let B be a ball with $3B \Subset \Omega$ and $R > R_0 > 0$. Then $u \in W^{1,\infty}_{loc}(\Omega)$ and there exists a constant $C > 0$ that may depend only on characteritics of φ and R_0 such that

$$\sup_B \varphi(|\nabla u|) \leq C \fint_{2B} \varphi(|\nabla u|).$$

End of the lecture (4.3.2013)

Theorem 36. (see [DSV09, Lemmas 5.7 and 5.8]) Under the assumptions of the previous theorem, $u \in W^{2,2}_{loc}(\Omega)$, $\varphi(|\nabla u|) \in W^{1,2}_{loc}(\Omega)$ and

$$\exists C > 0, \forall B, 2B \Subset \Omega : \fint_B |\nabla \varphi(|\nabla u|)|^2 \leq cR^{-2} \fint_{2B} |\varphi(|\nabla u|)|^2.$$

Assumption 4. Let Assumption 3 hold and moreover there is $\beta > 0$, $c > 0$ such that for all $\mathbf{H}, \mathbf{Q} \in \mathbb{R}^{N \times n}$ with $|\mathbf{H}| < |\mathbf{Q}|/2$

$$|\nabla \mathbf{A}(\mathbf{Q} + \mathbf{H}) - \nabla \mathbf{A}(\mathbf{Q})| \leq c\varphi''(|\mathbf{Q}|) \left(\frac{|\mathbf{H}|}{|\mathbf{Q}|} \right)^{\beta}.$$

Definition. For a ball B and a function u we define *excess functional* as

$$\varPhi(u, B) = \fint_B |\mathbf{V}(\nabla u) - (\mathbf{V}(\nabla u))_B|^2.$$

Theorem 37. (see [DSV09, Lemma 6.2]) Let Assumption 4 hold, and u be a local weak solution to (1) with $f = 0$. Then there is a $c > 1$ such that for every $\tau \in (0, 1)$ there is a $\varepsilon_0 > 0$ such that for every ball $B \Subset \Omega$:

$$\varPhi(u, B) \leq \varepsilon_0 \sup_{B/2} \varphi(|\nabla u|) \implies \varPhi(u, \tau B) \leq c\tau^2 \varPhi(u, B).$$

Exam equestions

1. properties of N-functions
2. existence of a weak solution
3. 1st apriori estimate and its consequences
4. 2nd apriori estimate
5. consequences of the differentiability of V

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