Průběh funkce

A) $f(x) = \arcsin\left(\frac{1+x}{1-2x}\right)$ je spojitá na $D(f)$

1) $D(f) = (-\infty, 0] \cup [2, +\infty)$ protože

$$-1 \le \frac{1+x}{1-2x} \le 1 \quad \text{pokud}$$

α) $1-2x>0$: $2x-1 \le 1+x \le 1-2x$, $x \le 2$; $x \le 0$ $\Big\} \Rightarrow x \le 0$

β) $1-2x<0$: $x \ge 2$, $x \ge 0$ $\Rightarrow$ $x \ge 2$

$f(0) = \frac{\pi}{2}$, $f(2) = -\frac{\pi}{2}$, $\lim_{x\to\pm\infty} f(x) = \arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$, $f(-1)=0$

2) $x \in D(f)\setminus\{0,2\}$: $f'(x) = \frac{1}{\sqrt{1-\left(\frac{1+x}{1-2x}\right)^2}} \cdot \frac{(1-2x)+2(1+x)}{(1-2x)^2} =$

$$= \frac{3}{\sqrt{1-4x+4x^2-1-2x-x^2}\cdot|1-2x|} = \frac{3}{\sqrt{-6x+3x^2}\,|1-2x|} =$$

$\frac{3}{\sqrt{x^2-2x}\cdot\sqrt{3}\,|1-2x|}$ a tedy $f'>0$ na $D(f)\setminus\{0,2\}$

- $f'_-(0) = +\infty = \lim_{x\to 0-} f'(x)$; $f'_+(2) = \lim_{x\to 2+} f'(x) = +\infty$
- f je rostoucí na $(-\infty, 0]$ a $[2,+\infty)$

3) $x \in D(f)\setminus\{0,2\}$: $f''(x) = \frac{-3}{\left(\sqrt{-6x+3x^2}\,|1-2x|\right)^2}\left(\frac{1}{2\sqrt{\dots}}(-6+6x)|1-2x|\right.$

$$\left. + (-2)\,\mathrm{sgn}(1-2x)\cdot\sqrt{-6x+3x^2}\right) = \frac{-3}{\left(\sqrt{\ }\,|1-2x|\right)^2}\cdot\frac{1}{2\sqrt{\ }}\left(\ .\,1\,.\ \right)\mathrm{sgn}(1-2x)$$

$$\cdot\underbrace{\left(6x-6-12x^2+12x-4(-6x+3x^2)\right)}_{A}$$

kořeny: $\frac{-7\pm\sqrt{49-16}}{-8} = +\frac{7}{8} \pm \frac{\sqrt{33}}{8} \in (0,2)$

$$A = -24x^2 + 42x - 6 = 6\left(-4x^2+7x-1\right)$$

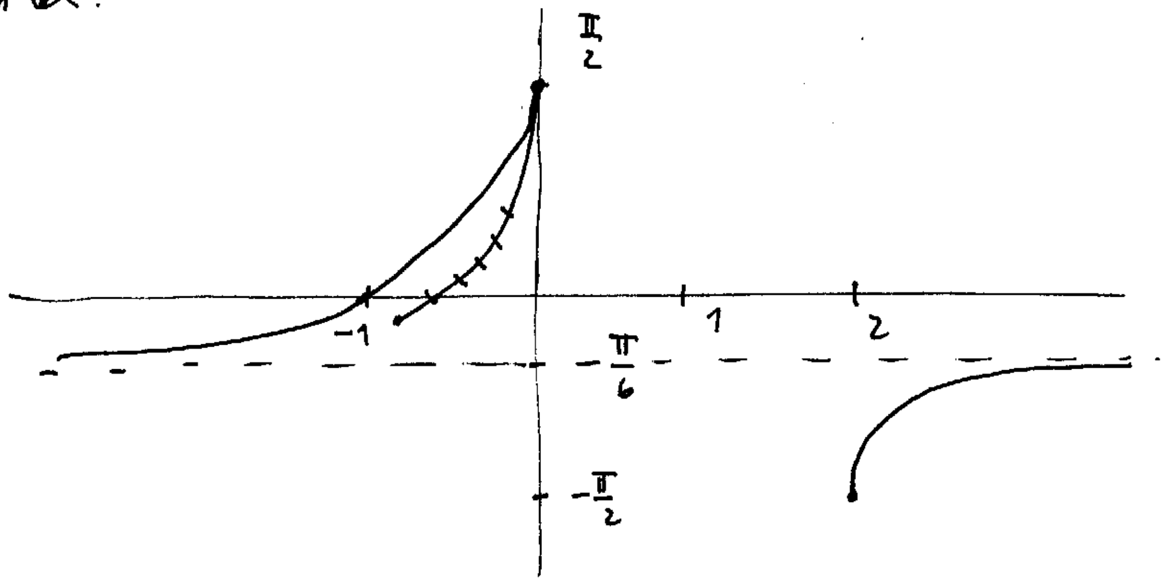
$\Rightarrow$ f'' má v $D(f)$ znaménko určené $\operatorname{sgn}(1-2x)$

$f'' > 0$ na $(-\infty, 0]$ a $f'' < 0$ na $[2, +\infty)$

f je konvexní na $(-\infty, 0]$ a konkávní na $[2, +\infty)$

4) $H(f) = \left[-\frac{\pi}{2}, -\frac{\pi}{6}\right) \cup \left(\frac{\pi}{6}, \frac{\pi}{2}\right]$

Obrázek:



B) $f(x) = \arcsin\left(\frac{x^2-1}{x^2+1}\right)$

1) $\left(-1 \le \frac{x^2-1}{x^2+1} \le 1 \iff -x^2-1 \le x^2-1 \le x^2+1 \iff x \in \mathbb{R}\right) \Rightarrow D(f) = \mathbb{R}$

$\lim\limits_{x\to\pm\infty} f(x) = \frac{\pi}{2}$; $f(0) = -\frac{\pi}{2}$, f je spojitá na $D(f)$ a sudá

2) Vzorec pro derivaci složené fce nelze použít v 0: $\arcsin'(1) = +\infty$

$$x \neq 0: \; f'(x) = \frac{1}{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)^2}} \cdot \frac{2x(x^2+1) - 2x(x^2-1)}{(x^2+1)^2} = \frac{4x}{\sqrt{x^4+2x^2+1-(x^4-2x^2+1)}\,|x^2+1|}$$

$$= \frac{4x}{(x^2+1)\sqrt{4x^2}} = \frac{2\operatorname{sgn} x}{x^2+1}$$

$\lim\limits_{x\to\pm\infty} f'(x) = 0$; $f'_{\pm}(0) = \pm 2$, $f' > 0$ na $(0, +\infty)$ a $f' < 0$ na $(-\infty, 0)$

$\Rightarrow$ f je rostoucí na $[0, +\infty)$ a klesající na $(-\infty, 0]$

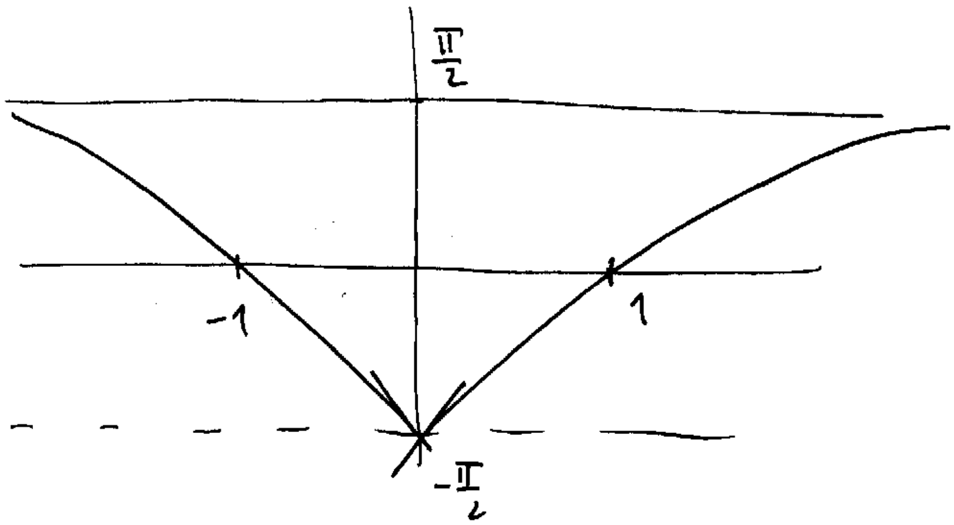
v 0 je globální minimum

3) $x \neq 0: \ f''(x) = 2 \operatorname{sgn} x \cdot (-1)(x^2+1)^{-2} \cdot 2x < 0$

$\Rightarrow f$ je konkávní na $(-\infty, 0]$ a $[0, +\infty)$

4) $\mathcal{H}(f) = [-\frac{\pi}{2}, \frac{\pi}{2})$

5)



6) $f(x) = \operatorname{arcsinh}\left(\frac{1+x}{1-2x}\right)$

1) $\mathcal{D}(f) = \mathbb{R} \setminus \{\frac{1}{2}\}$; $\lim_{x \to \frac{1}{2}\pm} f(x) = \begin{smallmatrix} -\infty \\ +\infty \end{smallmatrix}$; $\lim_{x \to \pm\infty} f(x) = \operatorname{arcsinh}(-\frac{1}{2})$

f je spojitá na $\mathcal{D}(f)$

2) $x \in \mathcal{D}(f): \ f'(x) = \dfrac{1}{\sqrt{1 + \left(\frac{1+x}{1-2x}\right)^2}} \cdot \dfrac{(1-2x) + 2(1+x)}{(1-2x)^2} =$

$$= \frac{1 \cdot 3}{\sqrt{1-4x+4x^2+1+2x+x^2}} \frac{1}{|1-2x|} > 0$$

$\Rightarrow f$ je rostoucí na $(-\infty, \frac{1}{2})$ a $(\frac{1}{2}, +\infty)$

$\lim_{x \to \frac{1}{2}\pm} f'(x) = +\infty$; $\lim_{x \to \pm\infty} f'(x) = 0$

3) $x \in \mathcal{D}(f)$: $f''(x) = 3\,\mathrm{sgn}(1-2x)\left(\frac{1}{\sqrt{2-2x+5x^2}}\,\frac{1}{(1-2x)}\right)' =$

$$= 3\,\mathrm{sgn}(1-2x)\,(-1)\,\frac{1}{\left[\sqrt{2-2x+5x^2}\,(1-2x)\right]^2}\left(\frac{-2+10x}{2\sqrt{\ }}(1-2x) - 2\sqrt{\ }\right)$$

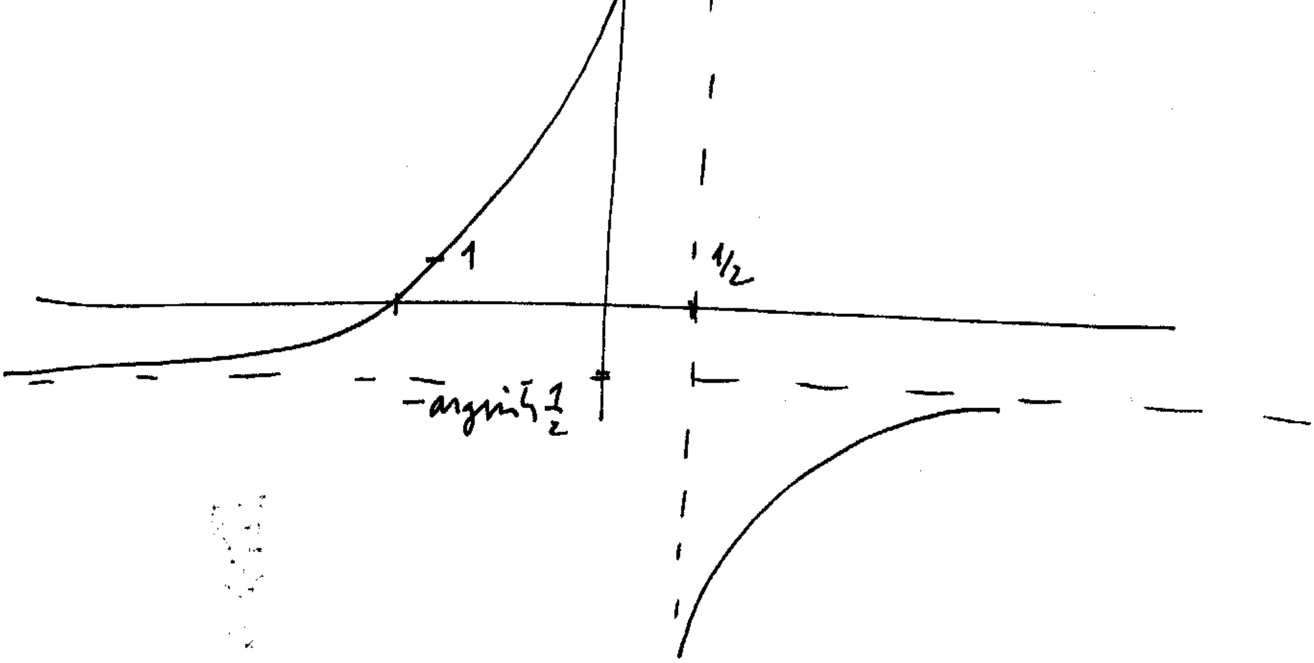
$$= 3\,\mathrm{sgn}(2x-1)\,\frac{1}{[\quad]^2}\cdot\frac{1}{\sqrt{\ }}\left(5x-1-10x^2+2x-2\left(2-2x+5x^2\right)\right)$$

$$= 3\,\mathrm{sgn}(2x-1)\,\frac{1}{[\ ]^2}\,\frac{1}{\sqrt{\ }}\,\underbrace{\left(-20x^2+11x-5\right)}_{<0}$$

$\Rightarrow$ $f'' < 0$ na $(\frac{1}{2}, +\infty)$ a $f'' > 0$ na $(-\infty, \frac{1}{2})$

f je konvexní na ~~$(\frac{1}{2}, +\infty)$~~ $(-\infty, \frac{1}{2})$ a konkávní na ~~$(-\infty, \frac{1}{2})$~~ $(\frac{1}{2}, +\infty)$

Obrázek:



4) $f(x) = \operatorname{argsinh}\left(\frac{x^2-1}{x^2+1}\right)$

1) $D(f) = \mathbb{R}$; $\lim\limits_{x\to\pm\infty} f(x) = \operatorname{argsinh} 1$; $f(\pm 1) = 0$

f je spojitá na $\mathbb{R}$ a sudá

2) $x \in \mathbb{R}$: $f'(x) = \dfrac{1}{\sqrt{1+\left(\frac{x^2-1}{x^2+1}\right)^2}} \cdot \dfrac{2x(x^2+1)-2x(x^2-1)}{(x^2+1)^2} =$

$$= \frac{1}{\sqrt{2x^4+2}} \cdot \frac{4x}{x^2+1}$$

$\Rightarrow \lim\limits_{x\to\pm\infty} f'(x) = \cancel{+\infty}\ 0$; $f' > 0$ na $(0,+\infty)$ a

$f' < 0$ na $(-\infty, 0)$

$\Rightarrow$ f je rostoucí na $[0,+\infty)$ a klesající na $(-\infty, 0]$

$\mathcal{H}(f) = [f(0), \operatorname{argsinh} 1) = [\operatorname{argsinh}(-1), \operatorname{argsinh} 1)$

3) $x \in \mathbb{R}$: $f''(x) = \dfrac{4\sqrt{2x^4+2}\,(x^2+1) - 4x\frac{1}{2\sqrt{2x^4+2}}\cdot(8x^3)(x^2+1) - 4x\sqrt{2x^4+2}\cdot 2x}{\left[\sqrt{2x^4+2}\ (x^2+1)\right]^2}$

čitatel $= \dfrac{8}{\sqrt{2x^4+2}}\left((x^4+1)(x^2+1) - 2x^4(x^2+1) - x^2(2x^4+2)\right) =$

$$= \frac{8}{\sqrt{2x^4+2}}\left(-3x^6 - x^4 - x^2 + 1\right)$$

značme: $F(x) = 1 - (3x^6 + x^4 + x^2)$. Zřejmě $F(0) = 1$,

$F(\pm 1) = -4$ a F je spojitá, tedy Darbouxovou vlastností

$\Rightarrow$ mezi $[0$ a $1]$ a $[(-1)$ a $0]$ ex. x_0 a $-x_0$ s $F(x_0) = 0$.
$x_0 > 0$

Navíc, protože F je klesající na $(0,+\infty)$ a rostoucí na $(-\infty,0)$ existuje takové x_0 právě 1 a $F > 0$ na $(-x_0, x_0)$ a $F < 0$ na $(-\infty, -x_0) \cup (x_0, +\infty)$

$\Rightarrow$ f je konvexní na $[-x_0, x_0]$ ($f'' > 0$) a f je konkávní na $(-\infty, -x_0]$ a $[x_0, +\infty)$
($f'' < 0$)

obrázek: