Prove all the following exercises.
Ex.1 Prove Lemma 38: If c_0-semigroups $S(t)$ and $\tilde{S}(t)$ have the same generator, then $S(t) = \tilde{S}(t)$ for all $t \geq 0$.
Hint. For a fixed $\tau > 0$ and $x \in \mathcal{D}(A)$, show that the function $y(t) = S(\tau - t)\tilde{S}(t)x$ is constant on $[0, \tau]$.

Ex.2 Fill the gap in proof of Lemma 39: show that $(A, \mathcal{D}(A))$ is the generator of $S(t)$ if and only if $(\tilde{A}, \mathcal{D}(\tilde{A}))$ is the generator of $\tilde{S}(t)$, where $\tilde{S}(t) = e^{-\omega t}S(t)$ and $\tilde{A} = A - \omega I$ with $\mathcal{D}(\tilde{A}) = \mathcal{D}(A)$.
Also show that in this situation $R(\lambda, \tilde{A}) = R(\lambda + \omega, A)$, whenever $\lambda \in \rho(\tilde{A}) \iff \lambda + \omega \in \rho(A)$.
Hint. It is enough to prove just one implication.

Ex.3 Verify the resolvent identity

$$R(\lambda, A)x - R(\mu, A)x = (\mu - \lambda)R(\lambda, A)R(\mu, A)x$$

for all $x \in X$, and λ, $\mu \in \rho(A)$.
Hint. Deduce first formally using $R(\lambda, A) = \frac{1}{\lambda - A}$.