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EXISTENCE, UNIQUENESS AND OPTIMAL REGULARITY RESULTS FOR VERY WEAK SOLUTIONS TO NONLINEAR ELLIPTIC SYSTEMS

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We establish existence, uniqueness and optimal regularity results for very weak solutions to certain nonlinear elliptic boundary value problems. We introduce structural asymptotic assumptions of Uhlenbeck type on the nonlinearity, which are sufficient and in many cases also necessary for building such a theory. We provide a unified approach that leads qualitatively to the same theory as the one available for linear elliptic problems with continuous coefficients, e.g., the Poisson equation.

The result is based on several novel tools that are of independent interest: local and global estimates for (non)linear elliptic systems in weighted Lebesgue spaces with Muckenhoupt weights, a generalization of the celebrated div-curl lemma for identification of a weak limit in border line spaces and the introduction of a Lipschitz approximation that is stable in weighted Sobolev spaces.

1. Introduction

We study the following nonlinear problem: for a given n-dimensional domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$, a given $f : \Omega \to \mathbb{R}^{n\times N}$ with $N \in \mathbb{N}$ arbitrary and a given mapping $A : \Omega \times \mathbb{R}^{n\times N} \to \mathbb{R}^{n\times N}$, find $u : \Omega \to \mathbb{R}^N$ satisfying

$$\begin{aligned} -\operatorname{div}(A(x, \nabla u)) &= -\operatorname{div} f \quad &&\text{in } \Omega, \\ u &= 0 &&\text{on } \partial\Omega. \end{aligned} \tag{1-1}$$

Owing to a significant number of problems originating in various applications, it is natural to require that A is a Carathéodory mapping, satisfying the natural coercivity, growth and (strict) monotonicity conditions. It means that

$$A(\cdot\,, \eta) \text{ is measurable for any fixed } \eta \in \mathbb{R}^{n\times N}, \tag{1-2}$$

$$A(x, \cdot) \text{ is continuous for almost all } x \in \Omega, \tag{1-3}$$

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and there exist positive constants c_1 and c_2 such that for almost all $x \in \Omega$ and all $\eta_1, \eta_2 \in \mathbb{R}^{n\times N}$

$$c_1|\eta_1|^2 - c_2 \le A(x,\eta_1)\cdot\eta_1 \qquad \text{(coercivity)}, \tag{1-4}$$

$$|A(x,\eta_1)| \le c_2(1+|\eta_1|) \qquad \text{(growth)}, \tag{1-5}$$

$$0 \le (A(x,\eta_1) - A(x,\eta_2))\cdot(\eta_1-\eta_2) \qquad \text{(monotonicity)}. \tag{1-6}$$

If for all $\eta_1 \neq \eta_2$ the inequality (1-6) is strict, then A is said to be strictly monotone.

Under the assumptions (1-2)–(1-6), it is standard to show (with the help of the Minty method [1963]) that, for any $f \in L^2(\Omega;\mathbb{R}^{n\times N})$, there exists $u \in W_0^{1,2}(\Omega;\mathbb{R}^N)$ that solves (1-1) in the sense of distribution. In addition if A is strictly monotone, then this solution is unique in the class of $W_0^{1,2}(\Omega;\mathbb{R}^N)$-weak solutions.

An important question that immediately arises is whether such a result can be extended to a more general setting. Namely,

whether for any $f \in L^q(\Omega;\mathbb{R}^{n\times N})$ *with* $q \in (1,\infty)$
there exists a (unique) $u \in W_0^{1,q}(\Omega;\mathbb{R}^N)$ *solving (1-1) in the weak sense.* $\qquad (\mathcal{Q})$

If $q \neq 2$, then we call the problem of existence and uniqueness to (1-1) beyond the natural pairing. If $q > 2$ and $f \in L^q(\Omega;\mathbb{R}^{n\times N})$, then $f \in L^2(\Omega;\mathbb{R}^{n\times N})$ as well, and the standard monotone operator theory in the duality pairing provides a $W_0^{1,2}(\Omega;\mathbb{R}^N)$ solution to (1-1). Thus, in this case, $(\mathcal{Q})$ calls only for improvement of the integrability of ∇u. If $q < 2$, then the considered question is more challenging as the existence of an object with which to start any kind of analysis is unclear. This is the reason why, for $1 < q < 2$, $W_0^{1,q}(\Omega;\mathbb{R}^N)$-solutions are called *very weak solutions*.

Our general aim is to establish, for a given $f \in L^q(\Omega;\mathbb{R}^{n\times N})$ with $q \in (1,\infty)\setminus 2$, the existence of a (unique) $W_0^{1,q}(\Omega;\mathbb{R}^N)$ solution to (1-1)–(1-6), i.e., to give the affirmative answer to $(\mathcal{Q})$. However, for general operators, this is not possible due to the following two reasons:

(i) the way how the nonlinearity $A(x,\eta)$ depends on η,

(ii) the way how the nonlinearity $A(x,\eta)$ depends on x.

We shall discuss each of these points from two perspectives: the available counterexamples and so far established affirmative results (that were rather sporadic and had several limitations).

First, we consider (1-1) with A depending only on η. If $q \ge 2$, then there always exists a (unique) weak solution and the only difficult part is to obtain appropriate a priori estimates in the space $W_0^{1,q}(\Omega;\mathbb{R}^N)$. On the one hand, for general operators, such a priori estimates are not true for large $q \gg 2$. This follows from the counterexamples due to Nečas [1977] and Sverák and Yan [2002], where they found a mapping A that does not depend on x and satisfies[1] (1-2)–(1-6) and showed that the corresponding unique weak solution is not in $\mathcal{C}^1$ or is even unbounded for smooth f. This directly contradicts the general theory for $q \gg 2$. The singular behavior of solutions in the above-mentioned counterexamples is due to the fact that the mapping A depends highly nonlinearly on the *vectorial* variable η. On the other hand,

[1] Not only does the mapping A satisfy (1-2)–(1-6), it has even more structure. It is given as a derivative of a uniformly convex smooth potential F, which makes the counterexamples even stronger.

if $q \in [2, 2+\varepsilon)$, then the $W_0^{1,q}(\Omega; \mathbb{R}^N)$ theory can be built for general mappings fulfilling only (1-2)–(1-6), where $\varepsilon > 0$ depends on c_1 and c_2. For such q, it is known that, if $f \in L^q(\Omega; \mathbb{R}^{n\times N})$, then there exists a solution $u \in W_0^{1,q}(\Omega; \mathbb{R}^N)$ to (1-1). Such a result can be obtained by using the reverse Hölder inequality (see, e.g., [Giaquinta 1983]) and holds also for more general growth conditions, including operators of p-Laplacian type. For the p-Laplacian itself, $A(x, \eta) := |\eta|^{p-2}\eta$ with $p \in (1, \infty)$, various positive results are known for large exponents (in this case $q \in (p, \infty)$ or even BMO estimates) [Iwaniec 1983; Caffarelli and Peral 1998; Diening et al. 2012]. The theory is built on the seminal works of Uraltseva [1968] (the scalar case) and Uhlenbeck [1977] (the vectorial case).

For $q < 2$, the situation is even more delicate. In this case, the existence of any solution is not straightforward at all. Indeed, a general existence theory for operators satisfying (1-2)–(1-6) alone might be impossible to get. Up to now, the only general result holds for $q \in (2-\varepsilon, 2+\varepsilon)$ with ε depending only on c_1 and c_2 and A being uniformly monotone and also uniformly Lipschitz continuous, i.e., for all $\eta_1, \eta_2 \in \mathbb{R}^{n\times N}$ and almost all $x \in \Omega$,

$$|A(x, \eta_1) - A(x, \eta_2)| \le c_2 |\eta_1 - \eta_2|. \tag{1-7}$$

In this case, we know that for all $f \in L^q(\Omega; \mathbb{R}^{n\times N})$ there exists a unique solution $u \in W_0^{1,q}(\Omega; \mathbb{R}^N)$ to (1-1) whenever $q \in (2-\varepsilon, 2+\varepsilon)$ [Bulíček 2012], and we also recall [Greco et al. 1997] for the result in the so-called grand Lebesgue spaces $L^{(2)}(\Omega)$. Moreover, for a general operator satisfying only (1-4)–(1-5), it may be shown with the help of the technique developed in [Bulíček 2012] that any very weak solution to (1-1) satisfies the uniform estimate

$$\int_\Omega |\nabla u|^q \,dx \le C(c_1, c_2, q, \Omega) \int_\Omega |f|^q \,dx \quad \text{for all } q \in (2-\varepsilon, 2+\varepsilon). \tag{1-8}$$

However, any existence theory for q "away" from 2 is either missing or impossible.

More positive results are available in the scalar case $N = 1$ (and even for a more general class of operators including the p-Laplacian) but for the *smoother* right-hand side, i.e., the case when $f \in W^{1,1}(\Omega; \mathbb{R}^n)$ or at least $f \in \mathrm{BV}(\Omega; \mathbb{R}^n)$. Then the existence of a very weak solution is known; see the pioneering works [Boccardo and Gallouët 1992; Stampacchia 1965]. Furthermore, one can study further qualitative properties of such a solution [Mingione 2013]. Moreover, in case $f \in W^{1,1}(\Omega; \mathbb{R}^n)$, the uniqueness of a solution can be shown in the class of *entropy* solutions [Bénilan et al. 1995; Boccardo et al. 1996; Dal Maso et al. 1997; 1999]. On the other hand, in case $f \in \mathrm{BV}(\Omega; \mathbb{R}^n)$, or more precisely if $\operatorname{div} f$ is only a Radon measure, the uniqueness is not known. An exception is the case when $\operatorname{div} f$ is a finite sum of Dirac measures. In that case, the study on isolated singularities by Serrin implies the uniqueness for very general nonlinear operators including the p-Laplace equation; see [Serrin 1965; Friedman and Véron 1986] and references therein. To conclude this part, we would like to emphasize that all results for smoother right-hand side surely do not cover the full generality of the result we would like to have, which may be easily seen in the framework of the Sobolev embedding. Indeed, if $f \in W^{1,1}(\Omega; \mathbb{R}^n)$, then $f \in L^{n/(n-1)}(\Omega; \mathbb{R}^n)$ and we see that the case $q \in (2, n'(p-1))$ remains untouched even in the scalar case.[2]

[2] Throughout the paper, we use the notation of dual exponents $q' := q/(q-1)$.

The second obstacle, related to (ii), is the possible discontinuity of the operator with respect to the spatial variable. To demonstrate this in more detail, we consider the *linear* problem

$$\begin{aligned} -\operatorname{div}(a(x)\nabla u) &= -\operatorname{div} f \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad\quad \text{on } \partial\Omega, \end{aligned} \tag{1-9}$$

with a uniformly elliptic matrix a. Note here that (1-9) is a particular case of (1-1) with $A(x,\eta) := a(x)\eta$ and A fulfilling (1-2)–(1-6) with $N = 1$. In case a is continuous and Ω is a $\mathscr{C}^1$-domain, one can use the singular operator theory and show that for any $f \in L^q(\Omega;\mathbb{R}^n)$ there exists a unique weak solution $u \in W_0^{1,q}(\Omega)$ to (1-9) [Dolzmann and Müller 1995, Lemma 2]. This can be weakened to the case when a has coefficients with vanishing mean oscillations; see [Iwaniec and Sbordone 1998] or [Di Fazio 1996]. However, the same is not true in the case that a is uniformly elliptic with general measurable coefficients. Even worse, it was shown by Serrin [1964] that for any $q \in (1,2)$ and $f \equiv 0$ there exists an elliptic matrix a with measurable coefficients such that one can find a distributive solution (called a *pathological solution*) $v \in W_0^{1,q}(\Omega) \setminus W_0^{1,2}(\Omega)$ that satisfies (2-5). These pathological solutions should be excluded as only the zero function itself is the natural solution, which of course is the unique weak solution $u \in W_0^{1,2}(\Omega)$ in case $f \equiv 0$. This indicates that any reasonable theory for $q \in (1,2)$ must be able to avoid the existence of such pathological solutions.

Thus, to get a theory for all $q \in (1,\infty)$, the counterexamples mentioned above indicate that we need to assume more structural assumptions on A, which we shall describe in detail in the next section, where we recall our problem, introduce the structural assumptions on A and formulate the main results of this paper.

2. Results

As discussed above, we study the problem (1-1) with a mapping A fulfilling (1-2)–(1-6). Further, inspired by the counterexamples recalled in the previous section and also by the available positive results, we shall assume in what follows that the mapping A is *asymptotically Uhlenbeck*; i.e., we will assume that there exists a continuous mapping $\tilde{A} : \overline{\Omega} \to \mathbb{R}^{n\times N} \times \mathbb{R}^{n\times N}$ fulfilling the following:

$$\begin{aligned} &\text{for all } \varepsilon > 0, \text{ there exists } k > 0 \text{ such that,} \\ &\text{for almost all } x \in \Omega \text{ and all } \eta \in \mathbb{R}^{n\times N} \text{ satisfying } |\eta| \ge k, \end{aligned} \qquad |A(x,\eta) - \tilde{A}(x)\eta| \le \varepsilon|\eta|. \tag{2-1}$$

This assumption combined with (1-4)–(1-6) implies that $\tilde{A}$ necessarily satisfies

$$c_1|\eta|^2 \le \tilde{A}(x)\eta\cdot\eta \le c_2|\eta|^2 \quad \text{for all } \eta \in \mathbb{R}^{n\times N}. \tag{2-2}$$

Although the above assumption might seem to be restrictive, it enables us to cover many cases used in applications. The prototypical example is of the form

$$A(x,\eta) = a(x,|\eta|)\eta \quad \text{with } \lim_{\lambda\to\infty} a(x,\lambda) = \tilde{a}(x), \text{ where } \tilde{a} \in \mathscr{C}(\overline{\Omega}). \tag{2-3}$$

Note that a may be measurable with respect to x and the required continuity must hold only for $\tilde{a}$. The assumptions (1-4)–(1-6) are met if a is strictly positive and bounded and if the function $a(x,\lambda)\lambda$ is nondecreasing with respect to λ for almost all $x \in \Omega$. The fact that, besides (1-2)–(1-6), we will not assume anything more than (2-1) makes our approach general.

Moreover, to obtain the uniqueness of the solution, we will consider a stronger version of (2-1). Namely, we shall assume that A is *strongly asymptotically Uhlenbeck*; i.e., we will assume that there exists a continuous mapping $\tilde{A} : \overline{\Omega} \to \mathbb{R}^{n\times N} \times \mathbb{R}^{n\times N}$ fulfilling the following:

$$\begin{aligned}&\text{for all } \varepsilon > 0, \text{ there exists } k > 0 \text{ such that,}\\ &\text{for almost all } x \in \Omega \text{ and all } \eta \in \mathbb{R}^{n\times N} \text{ satisfying } |\eta| \geq k,\end{aligned} \quad \left|\frac{\partial A(x,\eta)}{\partial \eta} - \tilde{A}(x)\right| \leq \varepsilon. \tag{2-4}$$

Concerning the example (2-3), the condition (2-4) follows if $a(x, \lambda)$ is differentiable with respect to λ for $\lambda \gg 1$ and $\lim_{\lambda\to\infty} |a'(x, \lambda)\lambda| = 0$. This includes the approximations for the p-Laplace operator

$$\begin{aligned} a(x, |\eta|) &= \max\{\mu, |\eta|^{p-2}\} \quad &&\text{for } p \in (1, 2),\\ a(x, |\eta|) &= \min\{\mu^{-1}, |\eta|^{p-2}\} \quad &&\text{for } p \in (2, \infty),\end{aligned}$$

which are (for small μ) arbitrary close to the original setting.

The first main result of the paper giving the answer to ($\mathcal{Q}$) is the following:

Theorem 2.1. *Let Ω be a bounded $\mathscr{C}^1$-domain and A satisfy (1-2)–(1-6) and (2-1). Then for any $f \in L^q(\Omega; \mathbb{R}^{n\times N})$ with $q \in (1, \infty)$, there exists $u \in W_0^{1,q}(\Omega; \mathbb{R}^{n\times N})$ such that*

$$\int_\Omega A(x, \nabla u) \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega f \cdot \nabla\varphi \,\mathrm{d}x \quad \textit{for all } \varphi \in \mathscr{C}_0^{0,1}(\Omega; \mathbb{R}^N). \tag{2-5}$$

Moreover, every very weak solution $\tilde{u} \in W_0^{1,\tilde{q}}(\Omega, \mathbb{R}^N)$ to (2-5) with some $\tilde{q} > 1$ satisfies

$$\int_\Omega |\nabla\tilde{u}|^q \,\mathrm{d}x \leq C(A, q, \Omega)\left(1 + \int_\Omega |f|^q \,\mathrm{d}x\right). \tag{2-6}$$

In addition, if A is strictly monotone and strongly asymptotically Uhlenbeck, i.e., (2-4) holds, then the solution is unique in any class $W_0^{1,\tilde{q}}(\Omega; \mathbb{R}^N)$ with $\tilde{q} > 1$.

Notice here that (2-5) is nothing else than the weak formulation of (1-1). Next, we would like to emphasize the novelty of the above result. First, to derive the estimate (2-6), one can use the comparison of (2-5) with the system with $A(x, \eta)$ replaced by $\tilde{A}(x)\eta$ to end up with (2-6) provided that the left-hand side of (2-6) is finite a priori. From this point of view, the a priori estimate (2-6) is indeed clear. On the other hand, and what is not obvious, is that (2-6) holds for *all very weak solutions* to (2-5) that belong to some $W_0^{1,\tilde{q}}(\Omega; \mathbb{R}^N)$ for some $\tilde{q} > 1$.

Second, Theorem 2.1 implies that *we can construct solutions for the whole range* $q \in (1, \infty)$, which makes the existence theory identical to the theory for linear operators with continuous coefficients since we know that the linear theory is not true for $q = 1$ or $q = \infty$.

Third, Theorem 2.1 provides *the uniqueness of the very weak solution* for vector-valued nonlinear elliptic systems without any additional qualitative properties of a solution, e.g., the entropy inequality. In particular, the result of Theorem 2.1 directly leads to the uniqueness of a solution when div f is a general vector-valued Radon measure. As this is of independent interest, we formulate this result in the following corollary, where we shall denote by the symbol $\mathcal{M}(\Omega; \mathbb{R}^N)$ the space of $\mathbb{R}^N$-valued Radon measures.

Corollary 2.2. *Let Ω be a bounded $\mathscr{C}^1$-domain and A satisfy (1-2)–(1-6) and (2-1). Then for any $f \in \mathcal{M}(\Omega; \mathbb{R}^N)$, there exists $u \in W_0^{1,n'-\varepsilon}(\Omega; \mathbb{R}^{n\times N})$ with arbitrary $\varepsilon > 0$ such that*

$$\int_\Omega A(x, \nabla u) \cdot \nabla \varphi \, \mathrm{d}x = \langle f, \varphi \rangle \quad \textit{for all } \varphi \in \mathscr{C}_0^{0,1}(\Omega; \mathbb{R}^N). \tag{2-7}$$

Moreover, every very weak solution $\tilde{u} \in W_0^{1,\tilde{q}}(\Omega, \mathbb{R}^N)$ to (2-7) with some $\tilde{q} > 1$ satisfies for all $q \in (1, n')$

$$\int_\Omega |\nabla \tilde{u}|^q \, \mathrm{d}x \le C(A, q, \Omega)(1 + \|f\|_{\mathcal{M}}^q). \tag{2-8}$$

In addition, if A is strictly monotone and strongly asymptotically Uhlenbeck, i.e., (2-4) holds, then the solution is unique in any class $W_0^{1,\tilde{q}}(\Omega; \mathbb{R}^N)$ with $\tilde{q} > 1$.

Although Theorem 2.1 gives the final answer to ($\mathscr{Q}$), it is actually a consequence of the following stronger result. It shows the existence of a solution that is optimally smooth with respect to the right-hand side in weighted spaces. For $p \in [1, \infty)$, we denote by $\mathcal{A}_p$ the Muckenhoupt class of nonnegative weights on $\mathbb{R}^n$ (see Section 3 for the precise definition) and define the weighted Lebesgue space $L^p_\omega(\Omega) := \{f \in L^1(\Omega); \int_\Omega |f|^p \omega \, \mathrm{d}x < \infty\}$. Then we have the following result.

Theorem 2.3. *Let Ω be a bounded $\mathscr{C}^1$-domain, A satisfy (1-2)–(1-6) and (2-1) and $f \in L^{p_0}_{\omega_0}(\Omega; \mathbb{R}^{n\times N})$ for some $p_0 \in (1, \infty)$ and $\omega_0 \in \mathcal{A}_{p_0}$. Then there exists a $u \in W_0^{1,1}(\Omega; \mathbb{R}^N)$ solving (2-5) such that for all $p \in (1, \infty)$ and all weights $\omega \in \mathcal{A}_p$ the estimate*

$$\int_\Omega |\nabla u|^p \omega \, \mathrm{d}x \le C(A_p(\omega), \Omega, A, p)\left(1 + \int_\Omega |f|^p \omega \, \mathrm{d}x\right) \tag{2-9}$$

holds whenever the right-hand side is finite. Moreover, every very weak solution $\tilde{u} \in W_0^{1,\tilde{q}}(\Omega, \mathbb{R}^N)$ to (2-5) with some $\tilde{q} > 1$ satisfies (2-9). In addition, if A is strictly monotone and strongly asymptotically Uhlenbeck, i.e., (2-4) holds, then the solution is unique in any class $W_0^{1,\tilde{q}}(\Omega; \mathbb{R}^N)$ with $\tilde{q} > 1$.

Clearly, Theorem 2.1 is an immediate consequence of Theorem 2.3. Observe that (2-9) is an optimal existence result with respect to the weighted spaces. It cannot be generalized to more general weights, which is demonstrated by the theory for the Laplace equation in the whole $\mathbb{R}^n$, where one can prove that (2-9) holds in general if and only if $\omega \in \mathcal{A}_p$. This follows from the singular integral representation of the solution and the fundamental result of Muckenhoupt [1972] on the continuity of the maximal function in weighted spaces.

At this point, we wish to present the following corollary of Theorem 2.3. It shows that if $f \in L^q(\Omega; \mathbb{R}^{n\times N})$ the solution constructed by Theorem 2.3 implies an estimate in terms of a Hilbert space that therefore inherits the spirit of duality. Denoting by Mf the Hardy–Littlewood maximal function (see the Section 3 for the precise definition), we have the following corollary.

Corollary 2.4. *Let Ω be a bounded $\mathscr{C}^1$-domain and A satisfy (1-2)–(1-6) and (2-1). Then for any $f \in L^q(\Omega; \mathbb{R}^{n\times N})$ with $q \in (1, 2]$, there exists $u \in W_0^{1,q}(\Omega; \mathbb{R}^N)$ satisfying (2-5). Moreover, any very weak solution $\tilde{u} \in W_0^{1,\tilde{q}}(\Omega; \mathbb{R}^N)$ with some $\tilde{q} > 1$ fulfilling (2-5) satisfies the estimate*

$$\int_\Omega \frac{|\nabla \tilde{u}|^2}{(1 + Mf)^{2-q}} \, \mathrm{d}x \le C(A, q, \Omega, f)\left(1 + \int_\Omega |f|^q \, \mathrm{d}x\right). \tag{2-10}$$

As mentioned above, the estimate (2-10) preserves the natural duality pairing in terms of weighted L^2 spaces, and as will be seen in the proof, the estimate (2-10) plays the key role in the convergence analysis of approximate solutions to the desired one. Indeed, the weighted L^2 integrability is the key property of the system, and we wish to emphasize that the only L^q-a priori information (with $q < 2$) does not seem to be sufficient to pass to the limit with the nonlinearity of approximating sequences. The reason for such a speculation is that all known methods for identification of the weak limit in the nonlinearity $A(\nabla u)$ are based on the identification of the "weak" limit of $A(\nabla u) \cdot \nabla u$ on "large" sets. However, having only L^q-estimates with $q < 2$, any identification of this type is impossible. On the other hand, we believe (based on the result of the paper) that the key estimate should reflect the duality pairing with possibly Muckenhoupt weight exactly as in (2-10). Having such an estimate, the new technique developed in the paper allows us to reconstruct the nonlinearity, although it is governed by a weakly converging subsequence only. It highly relies on the weighted theory that allows us to use the *weighted biting div-curl lemma*; see Theorem 2.6. To support the conjecture about the only possible choice of estimates in the weighted spaces preserving the duality pairing and reflecting the right-hand side, we quote the recent result [Bulíček and Schwarzacher 2016]. Here the theory for general operators with measurable coefficients and having a p-Laplacian-like structure is developed for all $q \in (p - \varepsilon, p]$ with $\varepsilon > 0$ depending only on the nonlinearity. Observe that the L^q-estimates for these p-Laplacian-like operators and $q \in (p - \varepsilon, p]$ have been known for some time [Lewis 1993; Greco et al. 1997] but the existence even in that case was not possible. Moreover, we wish to mention that the proof for the a priori estimates by Lewis [1993] already relied on the characterization of Muckenhoupt weights via the maximal operator. Therefore, we strongly believe that the effort to establish the very weak solution for the p-Laplace problem should not be blindly focused on obtaining L^q-estimates for $q < p$ but we should rather focus on the weighted L^p-estimates.

Next, we formulate new results that are on the one hand essential for the proof of Theorems 2.3 and 2.1 but on the other hand of independent interest in the fields of harmonic analysis and the compensated compactness theory. These results are mainly related to two critical problems: first to the a priori estimate (2-9) and second to the stability of the nonlinearity $A(x, \nabla u)$ under the weak convergence of ∇u. To solve the first problem, we use the linear system as a comparison to provide (2-9). The weighted theory for linear problems is known for $\Omega = \mathbb{R}^n$ in the case of constant coefficients (see, e.g., [Coifman and Fefferman 1974, p. 244]) but seems to be missing for bounded domains and linear operators continuously depending on x. Therefore, another essential contribution of this paper is the following theorem.

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $\mathscr{C}^1$-domain, $\omega \in \mathscr{A}_p$ for some $p \in (1, \infty)$ be arbitrary and $\tilde{A} \in \mathscr{C}(\overline{\Omega}; \mathbb{R}^{n\times N\times n\times N})$ satisfy for all $z \in \mathbb{R}^{n\times N}$ and all $x \in \overline{\Omega}$*

$$c_1|\eta|^2 \le \tilde{A}(x)\eta \cdot \eta \le c_2|\eta|^2 \tag{2-11}$$

with some positive constants c_1 and c_2. Then for any $f \in L^p_\omega(\Omega; \mathbb{R}^{n\times N})$, there exists unique $v \in W^{1,1}_0(\Omega; \mathbb{R}^N)$ solving

$$\int_\Omega \tilde{A}(x)\nabla v(x) \cdot \nabla\varphi(x)\,\mathrm{d}x = \int_\Omega f(x) \cdot \nabla\varphi(x)\,\mathrm{d}x \quad \textit{for all } \varphi \in \mathscr{C}^{0,1}_0(\Omega; \mathbb{R}^N) \tag{2-12}$$

and fulfilling

$$\int_\Omega |\nabla v|^p \omega \, dx \le C(\Omega, \mathcal{A}_p(\omega), p, c_1, c_2) \int_\Omega |f|^p \omega \, dx. \tag{2-13}$$

In addition, if $\bar{v} \in W_0^{1,q}(\Omega; \mathbb{R}^N)$ *for some* $q > 1$ *fulfills* (2-12), *then* $\bar{v} = v$.

We wish to point out that we include natural local weighted estimates in the interior as well as on the boundary that are certainly of independent interest (see Lemmas 5.1 and 5.2).

The second obstacle we have to deal with is an identification of the weak limit, and for this purpose, we invent a generalization of the celebrated div-curl lemma.

Theorem 2.6 (weighted, biting div-curl lemma). *Let* $\Omega \subset \mathbb{R}^n$ *be an open bounded set. Assume that for some* $p \in (1, \infty)$ *and given* $\omega \in \mathcal{A}_p$ *we have a sequence of vector-valued measurable functions* $(a^k, b^k)_{k=1}^\infty : \Omega \to \mathbb{R}^n \times \mathbb{R}^n$ *such that*

$$\sup_{k \in \mathbb{N}} \int_\Omega |a^k|^p \omega + |b^k|^{p'} \omega \, dx < \infty. \tag{2-14}$$

Furthermore, assume that, for every bounded sequence $\{c^k\}_{k=1}^\infty$ *from* $W_0^{1,\infty}(\Omega)$ *that fulfills*

$$\nabla c^k \rightharpoonup^* 0 \quad \text{weakly}^* \text{ in } L^\infty(\Omega),$$

there holds

$$\lim_{k \to \infty} \int_\Omega b^k \cdot \nabla c^k \, dx = 0, \tag{2-15}$$

$$\lim_{k \to \infty} \int_\Omega a_i^k \partial_{x_j} c^k - a_j^k \partial_{x_i} c^k \, dx = 0 \quad \text{for all } i, j = 1, \dots, n. \tag{2-16}$$

Then there exists a subsequence (a^k, b^k) *that we do not relabel, and there exists a nondecreasing sequence of measurable subsets* $E_j \subset \Omega$ *with* $|\Omega \setminus E_j| \to 0$ *as* $j \to \infty$ *such that*

$$a^k \rightharpoonup a \qquad \text{weakly in } L^1(\Omega; \mathbb{R}^n), \tag{2-17}$$

$$b^k \rightharpoonup b \qquad \text{weakly in } L^1(\Omega; \mathbb{R}^n), \tag{2-18}$$

$$a^k \cdot b^k \omega \rightharpoonup a \cdot b \omega \quad \text{weakly in } L^1(E_j) \text{ for all } j \in \mathbb{N}. \tag{2-19}$$

The original version of this lemma, first invented by Murat [1978; 1981] and Tartar [1978; 1979], was designed to identify many types of nonlinearities appearing in many types of partial differential equations. However, they assumed stronger assumptions on a^k and b^k than (2-15)–(2-16), which lead to (2-19) for $E_j \equiv \Omega$. To be more specific, they did not assume weighted spaces and considered $\omega \equiv 1$ and they required that (2-15) hold for any c^k converging weakly in $W^{1,p}$ and (2-16) for any c^k converging weakly in $W^{1,p'}$. The first result more in the spirit of Theorem 2.6 is due to Conti et al. [2011], who worked with $\omega \equiv 1$ and kept (2-15)–(2-16) but assumed the equi-integrability of the sequence $a^k \cdot b^k$. Such a result is then based on the proper use of the Lipschitz approximation of Sobolev functions introduced in [Acerbi and Fusco 1984], which we shall use here as well. The first use of the biting version of this result is in [Bulíček 2015], where the very similar technique for identification of the nonlinearity as in this paper is used but yet without the presence of Muckenhoupt weights. In this paper, we finally use the full strength of the *weighted biting div-curl lemma*, which is able to cover a borderline case in two ways: the integrability assumptions on a^k and b^k are minimal with respect to Lebesgue spaces (2-14) and the convergence assumptions (2-15)–(2-16) on $\operatorname{div}(b^k)$ and $\operatorname{curl}(a^k)$ are minimal. In addition, exactly this version of the div-curl lemma was one of the key results of this manuscript used in the recent paper [Bulíček and Schwarzacher 2016] to treat the p-Laplacian problem.

The proof of Theorem 2.6 relies on the original proof but is completed by using the Chacon biting lemma [Brooks and Chacon 1980; Ball and Murat 1989] and also a very improved Lipschitz approximation method in the framework of weighted spaces, which is yet another essential result of the paper.

Theorem 2.7 (Lipschitz approximation). *Let $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary. Let $g \in W_0^{1,1}(\Omega; \mathbb{R}^N)$. Then for all $\lambda > 0$, there exists a* Lipschitz truncation $g^\lambda \in W_0^{1,\infty}(\Omega; \mathbb{R}^N)$ *such that*

$$g^\lambda = g \quad \text{and} \quad \nabla g^\lambda = \nabla g \qquad \text{in } \{M(\nabla g) \le \lambda\}, \tag{2-20}$$

$$|\nabla g^\lambda| \le |\nabla g| \chi_{\{M(\nabla g) \le \lambda\}} + C\lambda \chi_{\{M(\nabla g) > \lambda\}} \qquad \text{almost everywhere.} \tag{2-21}$$

Further, if $\nabla g \in L^p_\omega(\Omega; \mathbb{R}^{n\times N})$ for some $1 \le p < \infty$ and $\omega \in \mathcal{A}_p$, then

$$\begin{aligned} \int_\Omega |\nabla g^\lambda|^p \omega \,\mathrm{d}x &\le C(\mathcal{A}_p(\Omega), \Omega, N, p) \int_\Omega |\nabla g|^p \omega \,\mathrm{d}x, \\ \int_\Omega |\nabla (g - g^\lambda)|^p \omega \,\mathrm{d}x &\le C(\mathcal{A}_p(\Omega), \Omega, N, p) \int_{\Omega \cap \{M(\nabla g) > \lambda\}} |\nabla g|^p \omega \,\mathrm{d}x. \end{aligned} \tag{2-22}$$

This result has its origin in the paper [Acerbi and Fusco 1988]. The approach was considerably improved and successfully used for the existence theory in the context of fluid mechanics; see, e.g., [Frehse et al. 2000; Diening et al. 2008; 2013; Diening 2013] or [Breit et al. 2012; 2013] for divergence-free Lipschitz approximation. However, these results do not contain the weighted estimates (2-22) and for this reason we also provide its proof in this paper.

Finally, for the sake of completeness, we present straightforward generalizations of the above results. First, we establish the theory for the nonhomogeneous Dirichlet problem.

Theorem 2.8. *Let Ω be a bounded $\mathscr{C}^1$-domain, A satisfy (1-2)–(1-6) and (2-1), $f \in L^{p_0}_{\omega_0}(\Omega; \mathbb{R}^{n\times N})$ and $u_0 \in W^{1,1}(\Omega; \mathbb{R}^N)$ be such that $\nabla u_0 \in L^{p_0}_{\omega_0}(\Omega; \mathbb{R}^{n\times N})$ for some $p_0 \in (1, \infty)$ and $\omega_0 \in \mathcal{A}_{p_0}$. Then there exists a solution u of (2-5) such that $u - u_0 \in W_0^{1,1}(\Omega; \mathbb{R}^{n\times N})$, and for all $p \in (1, \infty)$ and all weights $\omega \in \mathcal{A}_p$, the estimate*

$$\int_\Omega |\nabla u|^p \omega \,\mathrm{d}x \le C(A_p(\omega), \Omega, A, p)\left(1 + \int_\Omega (|f|^p + |\nabla u_0|^p)\omega \,\mathrm{d}x\right) \tag{2-23}$$

holds whenever the right hand side is finite. Moreover, every very weak solution u of (2-5) fulfilling $\tilde{u} - u_0 \in W_0^{1,\tilde{q}}(\Omega, \mathbb{R}^N)$ with some $\tilde{q} > 1$ satisfies (2-23). In addition, if A is strictly monotone and strongly asymptotically Uhlenbeck, i.e., (2-4) holds, then the solution is unique in any class $W^{1,\tilde{q}}(\Omega; \mathbb{R}^N)$ with $\tilde{q} > 1$.

Second, we remark that, for the theory for (1-1), the assumptions (2-1)–(2-4) are not necessary and can be weakened.

Remark 2.9. At this point, we wish to discuss possible relaxations of the conditions (2-1) and (2-4) that might be useful for further application of the theory developed here. The proofs of existence or uniqueness do not require that the matrix $A(x, \eta)$ converge uniformly to a continuous target matrix $\tilde{A}(x)$ but rather that the two matrices are "close" for values $|\eta| > k$ for some k. Indeed, it is possible to quantify the necessary closeness in accordance with the ellipticity and continuity parameters of $\tilde{A}(x)$ and $\partial\Omega$. A different relaxation of (2-1) and (2-4) could be done in a nonpointwise manner by replacing the pointwise asymptotic conditions by asymptotic conditions in terms of vanishing mean oscillations (VMO).

We conclude this section by highlighting the essential novelties of this paper:

(1) A complete unified $W_0^{1,q}(\Omega; \mathbb{R}^N)$-theory for nonlinear elliptic systems with the asymptotic Uhlenbeck structure satisfying (1-2)–(1-6), (2-1) and (2-4) has been developed in such a way that the theory is identical with that for linear operators with continuous coefficients: Theorems 2.1 and 2.8. Moreover, the new estimate suitable for numerical purposes is established in Corollary 2.4.

(2) A maximal regularity in weighted spaces of any very weak solution is established as well as its uniqueness, which in particular leads to the uniqueness of very weak solutions to the problems with measure right-hand side: Theorem 2.3 for the nonlinear case and Theorem 2.5 for the linear setting.

(3) A new tool in harmonic analysis, the Lipschitz approximation method in weighted spaces, is developed: Theorem 2.7.

(4) A new tool for identification of a weak limit of the nonlinear operator, the biting weighted div-curl lemma, is invented: Theorem 2.6. Such a tool has a potential to improve the known methods in compensated compactness theory in significant manner.

To summarize, this paper proposes a new way to attack more general elliptic problems than those discussed in Section 2. Indeed, it seems that the only missing point in the analysis of more general problems, e.g., the p-Laplace equation, is the *formal* a priori estimates beyond the duality pairing. Once such a priori estimates are available, one can follow the method introduced in this paper and gain an existence and uniqueness theory for general problems beyond the natural duality. Indeed, the first step in this direction was already done in [Bulíček and Schwarzacher 2016], where more general operators having the p structure are treated.

The structure of the paper is somewhat in reversed order. After introducing some auxiliary tools and some necessary notation in Section 3, we first prove the main Theorems 2.1 and 2.3 in Section 4. For that result, we use the (technical) theorems, which are each independently proved in Sections 5–8. Finally Section 9 is dedicated to the proofs of the corollaries.

3. Auxiliary tools

3A. ***Muckenhoupt weights and the maximal function.*** We start this part by recalling the definition of the Hardy–Littlewood maximal function. For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we define

$$Mf(x) := \sup_{R>0} \fint_{B_R(x)} |f(y)|\,\mathrm{d}y \quad \text{with} \fint_{B_R(x)} |f(y)|\,\mathrm{d}y := \frac{1}{|B_R(x)|} \int_{B_R(x)} |f(y)|\,\mathrm{d}y,$$

where $B_R(x)$ denotes a ball with radius R centered at $x \in \mathbb{R}^n$. We shall use similar notation for vector- or tensor-valued functions as well. Note here that we could replace balls in the definition of the maximal function by cubes with sides parallel to the axes without any change. We will also use in what follows the standard notion for Lebesgue and Sobolev spaces. Further, we say that $\omega : \mathbb{R}^n \to \mathbb{R}$ is a weight if it is a measurable function that is almost everywhere finite and positive. For such a weight and arbitrary measurable $\Omega \subset \mathbb{R}^n$, we denote the space $L^p_\omega(\Omega)$ with $p \in [1, \infty)$ as

$$L^p_\omega(\Omega) := \left\{ u : \Omega \to \mathbb{R}^n;\ \|f\|_{L^p_\omega} := \left(\int_\Omega |u(x)|^p \omega(x)\,\mathrm{d}x \right)^{1/p} < \infty \right\}.$$

Note that our weights are defined on the whole space $\mathbb{R}^n$. Next, for $p \in [1, \infty)$, we say that a weight ω belongs to the Muckenhoupt class $\mathcal{A}_p$ if and only if there exists a positive constant A such that for every ball $B \subset \mathbb{R}^n$

$$\left(\fint_B \omega\,\mathrm{d}x \right) \left(\fint_B \omega^{-(p'-1)}\,\mathrm{d}x \right)^{1/(p'-1)} \le A \qquad \text{if } p \in (1, \infty), \tag{3-1}$$

$$M\omega(x) \le A\omega(x) \quad \text{if } p = 1. \tag{3-2}$$

In what follows, we denote by $A_p(\omega)$ the smallest constant A for which the inequality (3-1) or (3-2) holds. Due to the celebrated result of Muckenhoupt [1972], we know that $\omega \in \mathcal{A}_p$ is for $1 < p < \infty$ equivalent to the existence of a constant A' such that for all $f \in L^p(\mathbb{R}^n)$

$$\int |Mf|^p \omega\,\mathrm{d}x \le A' \int |f|^p \omega\,\mathrm{d}x. \tag{3-3}$$

Further, if $p \in [1, \infty)$ and $\omega \in \mathcal{A}_p$, then we have an embedding $L^p_\omega(\Omega) \hookrightarrow L^1_{\mathrm{loc}}(\Omega)$ since for all balls $B \subset \mathbb{R}^n$

$$\fint_B |f|\,\mathrm{d}x \le \left(\fint_B |f|^p \omega\,\mathrm{d}x \right)^{1/p} \left(\fint_B \omega^{-(p'-1)}\,\mathrm{d}x \right)^{1/p'} \le (\mathcal{A}_p(\omega))^{1/p} \left(\frac{1}{\omega(B)} \int_B |f|^p \omega\,\mathrm{d}x \right)^{1/p}.$$

In particular, the distributional derivatives of all $f \in L^p_\omega$ are well defined. Next, we summarize some properties of Muckenhoupt weights in the following lemma.

Lemma 3.1 [Turesson 2000, Lemma 1.2.12]. *Let $\omega \in \mathcal{A}_p$ for some $p \in [1, \infty)$. Then $\omega \in \mathcal{A}_q$ for all $q \ge p$. Moreover, there exists $s = s(p, A_p(\omega)) > 1$ such that $\omega \in L^s_{\mathrm{loc}}(\mathbb{R}^n)$ and we have the reverse Hölder inequality, i.e.,*

$$\left(\fint_B \omega^s\,\mathrm{d}x \right)^{1/s} \le C(n, A_p(\omega)) \fint_B \omega\,\mathrm{d}x. \tag{3-4}$$

Further, if $p \in (1, \infty)$, then there exists $\sigma = \sigma(p, A_p(\omega)) \in [1, p)$ such that $\omega \in \mathcal{A}_\sigma$. In addition, $\omega \in \mathcal{A}_p$ is equivalent to $\omega^{-(p'-1)} \in \mathcal{A}_{p'}$.

In the paper, we also use the following improved embedding $L^p_\omega(\Omega) \hookrightarrow L^q_{\mathrm{loc}}(\Omega)$ valid for all $\omega \in \mathcal{A}_p$ with $p \in (1, \infty)$ and some $q \in [1, p)$ depending only on $A_p(\omega)$. Such an embedding can be deduced by a direct application of Lemma 3.1. Indeed, since $\omega \in \mathcal{A}_p$, we have $\omega^{-(p'-1)} \in \mathcal{A}_{p'}$. Thus, using Lemma 3.1, there exists $s = s(A_p(\omega)) > 1$ such that

$$\left(\fint_B \omega^{-s(p'-1)}\,\mathrm{d}x\right)^{1/s} \le C(A_p(\omega)) \fint_B \omega^{-(p'-1)}\,\mathrm{d}x.$$

Consequently, for $q := sp/(p+s-1) \in (1, p)$, we can use the Hölder inequality to deduce that

$$\begin{aligned}\left(\fint_B |f|^q\,\mathrm{d}x\right)^{1/q} &\le \left(\fint_B |f|^p\omega\,\mathrm{d}x\right)^{1/p}\left(\fint_B \omega^{-s(p'-1)}\,\mathrm{d}x\right)^{1/(sp')}\\ &\le C(A_p(\omega))\left(\frac{1}{\omega(B)}\int_B |f|^p\omega\,\mathrm{d}x\right)^{1/p},\end{aligned} \tag{3-5}$$

which implies the desired embedding.

The next result makes another link between the maximal function and $\mathcal{A}_p$-weight.

Lemma 3.2 [Torchinsky 1986, p. 229–230; Turesson 2000, p. 5]. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be such that $Mf < \infty$ almost everywhere in $\mathbb{R}^n$. Then for all $\alpha \in (0, 1)$, we have $(Mf)^\alpha \in \mathcal{A}_1$. Furthermore, for all $p \in (1, \infty)$ and all $\alpha \in (0, 1)$, there holds $(Mf)^{-\alpha(p-1)} \in \mathcal{A}_p$.*

We would also like to point out that the maximum $\omega_1 \vee \omega_2$ and minimum $\omega_1 \wedge \omega_2$ of two $\mathcal{A}_p$-weights are again $\mathcal{A}_p$-weights. For $p = 2$, we even have $A_2(\omega_1 \wedge \omega_2) \le A(\omega_1) + A_2(\omega_2)$, which follows from the simple computation

$$\begin{aligned}\fint_B \omega_1 \wedge \omega_2\,\mathrm{d}x \fint_B \frac{1}{\omega_1 \wedge \omega_2}\,\mathrm{d}x &\le \left[\left(\fint_B \omega_1\,\mathrm{d}x\right) \wedge \left(\fint_B \omega_2\,\mathrm{d}x\right)\right] \fint_B \frac{1}{\omega_1} + \frac{1}{\omega_2}\,\mathrm{d}x\\ &\le A_2(\omega_1) + A_2(\omega_2).\end{aligned} \tag{3-6}$$

3B. *Convergence tools.* The results recalled in the previous sections shall give us a direct method for a priori estimates for an approximative problem (1-1). However, to identify the limit correctly, we use Theorem 2.6, which is based on the following biting lemma.

Lemma 3.3 (Chacon's biting lemma [Ball and Murat 1989]). *Let Ω be a bounded domain in $\mathbb{R}^n$, and let $\{v^n\}_{n=1}^\infty$ be a bounded sequence in $L^1(\Omega)$. Then there exists a nondecreasing sequence of measurable subsets $E_j \subset \Omega$ with $|\Omega \setminus E_j| \to 0$ as $j \to \infty$ such that $\{v^n\}_{n\in\mathbb{N}}$ is precompact in the weak topology of $L^1(E_j)$, for each $j \in \mathbb{N}$.*

Note here that precompactness of v^n is equivalent to the following: for every $j \in \mathbb{N}$ and every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $A \subset E_j$ with $|A| \le \delta$ and all $n \in \mathbb{N}$

$$\int_A |v^n|\,\mathrm{d}x \le \varepsilon. \tag{3-7}$$

3C. *L^q-theory for linear systems with continuous coefficients.* The starting point for getting all a priori estimates in the paper is the following:

Lemma 3.4 [Dolzmann and Müller 1995, Lemma 2]. *Let Ω be a $\mathscr{C}^1$-domain and $\boldsymbol{B} \in \mathscr{C}(\overline{\Omega}, \mathbb{R}^{n\times N\times n\times N})$ be a continuous, elliptic tensor that satisfies for all $\eta \in \mathbb{R}^{n\times N}$ and all $x \in \overline{\Omega}$*

$$c_1|\eta|^2 \le \boldsymbol{B}(x)\eta \cdot \eta \le c_2|\eta|^2 \tag{3-8}$$

for some $c_1, c_2 > 0$. *Then for any* $f \in L^q(\Omega; \mathbb{R}^{n\times N})$ *with* $q \in (1, \infty)$, *there exists unique* $w \in W_0^{1,q}(\Omega; \mathbb{R}^N)$ *solving*

$$-\operatorname{div}(\boldsymbol{B}\nabla w) = -\operatorname{div} f \quad \text{in } \Omega$$

in the sense of distribution. Moreover, there exists a constant C *depending only on* $\boldsymbol{B}$, q *and the shape of* Ω *such that*

$$\|\nabla w\|_{L^q(\Omega)} \le C(\boldsymbol{B}, q, \Omega)\|f\|_{L^q(\Omega)}. \tag{3-9}$$

4. Proof of Theorems 2.1 and 2.3

First, it is evident that Theorem 2.1 directly follows from Theorem 2.3 by setting $\omega \equiv 1$, which is surely an $\mathcal{A}_p$-weight. Therefore, we focus on the proof of Theorem 2.3. We split the proof into several steps. We start with the uniform estimates, which heavily rely on Theorem 2.5, then provide the existence proof, for which we use the result of Theorem 2.6, and finally show the uniqueness of the solution, again based on Theorem 2.5.

4A. ***Uniform estimates.*** We start the proof by showing the uniform estimate (2-9) for arbitrary $u \in W_0^{1,q}(\Omega; \mathbb{R}^N)$ with $q > 1$ solving (2-5). Without loss of generality, we can restrict ourselves to the case $q \in (1, 2)$. First, we consider the case when $f \in L^2_\omega(\Omega; \mathbb{R}^{n\times N})$ with some weight $\omega \in \mathcal{A}_2$. For $j \in \mathbb{N}$, we define the auxiliary weight $\omega_j := \omega \wedge j(1 + M|\nabla u|)^{q-2}$. Then it follows from Lemma 3.2 and the fact that $q \in (1, 2)$ that $w_j \in \mathcal{A}_2$. Moreover, we have

$$A_2(\omega_j) \le A_2(\omega) + A_2(j(1 + M|\nabla u|)^{q-2}) = A_2(\omega) + A_2((1 + M|\nabla u|)^{q-2}) \le C(u, \omega)$$

and also that $\nabla u, f \in L^2_{\omega_j}(\Omega; \mathbb{R}^{n\times N})$. Next, using (2-5), we see that for all $\varphi \in \mathscr{C}_0^{0,1}(\Omega; \mathbb{R}^N)$

$$\int_\Omega \tilde{A}(x)\nabla u \cdot \nabla\varphi \,dx = \int_\Omega (f - A(x, \nabla u) + \tilde{A}(x)\nabla u) \cdot \nabla\varphi \,dx. \tag{4-1}$$

Since the right-hand side belongs to $L^2_{\omega_j}(\Omega; \mathbb{R}^{n\times N})$, we can use Theorem 2.5 and the assumptions (1-5) and (2-2) to get the estimate

$$\begin{aligned}
\int_\Omega |\nabla u|^2\omega_j \,dx &\le C(\tilde{A}, A_2(\omega_j), \Omega, c_1, c_2)\int_\Omega |f - A(x, \nabla u) + \tilde{A}(x)\nabla u|^2\omega_j \,dx \\
&\le C(\tilde{A}, u, \omega, \Omega, c_1, c_2)\left(\int_\Omega |f|^2\omega_j \,dx + \int_\Omega |A(x, \nabla u) - \tilde{A}(x)\nabla u|^2\omega_j \,dx\right) \\
&\le C(\tilde{A}, u, \omega, \Omega, c_1, c_2)\int_\Omega (|f|^2 + k^2)\omega_j \,dx \\
&\qquad + C(\tilde{A}, u, \omega, \Omega, c_1, c_2)\int_{\{|\nabla u|\ge k\}} \frac{|A(x, \nabla u) - \tilde{A}(x)\nabla u|^2}{|\nabla u|^2}|\nabla u|^2\omega_j \,dx.
\end{aligned}$$

Finally, we set

$$\varepsilon^2 := \frac{1}{2C(\tilde{A}, u, \omega, \Omega, c_1, c_2)}$$

and according to (2-1) we can find k such that

$$\frac{|A(x,\nabla u)-\tilde{A}(x)\nabla u|^2}{|\nabla u|^2} \le \frac{1}{2C(\tilde{A},u,\omega,\Omega,c_1,c_2)},$$

provided that $|\nabla u| \ge k$. Inserting this inequality above, we deduce that

$$\int_\Omega |\nabla u|^2 \omega_j \,\mathrm{d}x \le C(\tilde{A},u,\omega,\Omega,c_1,c_2)\int_\Omega (|f|^2+k^2)\omega_j \,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u|^2\omega_j \,\mathrm{d}x.$$

Since we already know that $\nabla u \in L^2_{\omega_j}(\Omega;\mathbb{R}^{n\times N})$ and k is fixed independently of j, we can absorb the last term into the left-hand side to get

$$\int_\Omega |\nabla u|^2 \omega_j \,\mathrm{d}x \le C(\tilde{A},u,\omega,\Omega,c_1,c_2)\int_\Omega (|f|^2+1)\omega_j \,\mathrm{d}x.$$

Next, we let $j\to\infty$ in the above inequality. For the right-hand side, we use the fact that $\omega_j \le \omega$, and for the left-hand side, we use the monotone convergence theorem (notice here that $\omega_j \nearrow \omega$ since $M|\nabla u| < \infty$ almost everywhere) to obtain

$$\int_\Omega |\nabla u|^2 \omega \,\mathrm{d}x \le C(\tilde{A},u,\omega,\Omega,c_1,c_2)\left(1+\int_\Omega |f|^2\omega \,\mathrm{d}x\right).$$

Although this estimate is not uniform yet, since the right-hand side still depends on the A_2 constant of $(1+M|\nabla u|)^{q-2}$, it implies that $\nabla u \in L^2_\omega(\Omega;\mathbb{R}^{n\times N})$ for the original weight ω. Therefore, we can reiterate this procedure; i.e., going back to (4-1) and applying Theorem 2.5, we find that

$$\begin{aligned}\int_\Omega |\nabla u|^2\omega \,\mathrm{d}x &\le C(\tilde{A},A_2(\omega),\Omega,c_1,c_2)\int_\Omega |f-A(x,\nabla u)+\tilde{A}(x)\nabla u|^2\omega\,\mathrm{d}x\\ &\le C(\tilde{A},A_2(\omega),\Omega,c_1,c_2)\int_\Omega (|f|^2+k)\omega\,\mathrm{d}x\\ &\qquad + C(\tilde{A},A_2(\omega),\Omega,c_1,c_2)\int_{\{|\nabla u|\ge k\}} \frac{|A(x,\nabla u)-\tilde{A}(x)\nabla u|^2}{|\nabla u|^2}|\nabla u|^2\omega\,\mathrm{d}x.\end{aligned}$$

Since we already know that $\nabla u \in L^2_\omega(\Omega;\mathbb{R}^{n\times N})$, we can use the same procedure as above and absorb the last term into the left-hand side to get

$$\int_\Omega |\nabla u|^2\omega \,\mathrm{d}x \le C(c_1,c_2,A_2(\omega),\Omega,\tilde{A})\left(1+\int_\Omega |f|^2\omega\,\mathrm{d}x\right). \tag{4-2}$$

We would like to emphasize that the constant C in (4-2) depends on ω only through its A_2-constant. Therefore, by the *miracle of extrapolation* [Cruz-Uribe et al. 2006, Theorem 3.1] (see also [Rubio de Francia 1984]) applied to the couples $(\nabla u, f)$, we can extend this estimate valid for all $\mathcal{A}_2$-weights to all $\mathcal{A}_p$-weights. In particular, we find that

$$\int_\Omega |\nabla u|^p\omega \,\mathrm{d}x \le C(c_1,c_2,A_p(\omega),\Omega,\tilde{A})\left(1+\int_\Omega |f|^p\omega\,\mathrm{d}x\right) \quad \text{for all } 1<p<\infty \text{ and } \omega\in\mathcal{A}_p,$$

which is just (2-9) from our claim.

4B. ***Existence of a solution.*** Let $f \in L^p_\omega(\Omega; \mathbb{R}^{n\times N})$ with some $p \in (1, \infty)$ and $\omega \in \mathcal{A}_p$ be arbitrary. Then according to (3-5), there exists some $q_0 \in (1, 2)$ such that $L^p_\omega(\Omega) \hookrightarrow L^{q_0}(\Omega)$. Therefore, defining $\omega_0 := (1+Mf)^{q_0-2}$, we can use Lemma 3.2 to obtain that $\omega_0 \in \mathcal{A}_2$ and it is evident that $f \in L^2_{\omega_0}(\Omega; \mathbb{R}^{n\times N})$.

The construction of the solution is based on a proper approximation of the right-hand-side f and a limiting procedure. We first extend f outside of Ω by zero and define $f^k := f\chi_{\{|f|<k\}}$. Then f^k are bounded functions, $|f^k| \nearrow |f|$ and

$$f^k \to f \quad \text{strongly in } L^2_{\omega_0} \cap L^{q_0}(\mathbb{R}^n; \mathbb{R}^{n\times N}). \tag{4-3}$$

For such an approximative f^k, we can use the standard monotone operator theory to find a solution $u^k \in W^{1,2}_0(\Omega; \mathbb{R}^N)$ fulfilling

$$\int_\Omega A(x, \nabla u^k) \cdot \nabla\varphi \, dx = \int_\Omega f^k \cdot \nabla\varphi \, dx \quad \text{for all } \varphi \in W^{1,2}_0(\Omega; \mathbb{R}^N). \tag{4-4}$$

Hence, we can use the already proven estimate (2-9) to deduce that

$$\begin{aligned}\int_\Omega |\nabla u^k|^2\omega_0 \, dx &\le C(c_1, c_2, A_2(\omega_0), \Omega, \tilde{A})\left(1 + \int_\Omega |f^k|^2\omega_0 \, dx\right)\\ &\le C(c_1, c_2, q_0, f, A_2(\omega_0), \tilde{A})\left(1 + \int_\Omega |f|^2\omega_0 \, dx\right)\\ &\le C(c_1, c_2, \Omega, \tilde{A}, f, \omega).\end{aligned} \tag{4-5}$$

Using the estimate (4-5), the reflexivity of the corresponding spaces, the embedding $L^2_{\omega_0}(\Omega) \hookrightarrow L^{q_0}(\Omega)$ and the growth assumption (1-5), we can pass to a subsequence (still denoted by u^k) such that

$$u^k \rightharpoonup u \qquad \text{weakly in } W^{1,q_0}_0(\Omega; \mathbb{R}^N), \tag{4-6}$$

$$\nabla u^k \rightharpoonup \nabla u \qquad \text{weakly in } L^2_{\omega_0} \cap L^{q_0}(\Omega; \mathbb{R}^{n\times N}), \tag{4-7}$$

$$A(x, \nabla u^k) \rightharpoonup \bar{A} \qquad \text{weakly in } L^2_{\omega_0} \cap L^{q_0}(\Omega; \mathbb{R}^{n\times N}). \tag{4-8}$$

Next, using (4-5)–(4-7), the weak lower semicontinuity and the unique identification of the limit u in $W^{1,1}(\Omega)$, we obtain

$$\int_\Omega |\nabla u|^2\omega_0 \, dx \le C(c_1, c_2, A_2(\omega_0), \Omega, \tilde{A})\left(1 + \int_\Omega |f|^2\omega_0 \, dx\right). \tag{4-9}$$

The last step is to show that u is a solution to our problem, i.e., that it satisfies (2-5). Using (4-4), (4-3) and (4-8), it follows that

$$\int_\Omega \bar{A} \cdot \nabla\varphi \, dx = \int_\Omega f \cdot \nabla\varphi \, dx \quad \text{for all } \varphi \in \mathcal{C}^{0,1}_0(\Omega; \mathbb{R}^N). \tag{4-10}$$

Hence, to complete the existence part of the proof of Theorem 2.3, it remains to show that

$$\bar{A}(x) = A(x, \nabla u(x)) \quad \text{in } \Omega. \tag{4-11}$$

To do so, we use[3] Theorem 2.6. We denote $a^k := \nabla u^k$ and $b^k := A(x, \nabla u^k)$. By using (4-5) and (1-5), we find that (2-14) is satisfied with the weight ω_0. Also the assumption (2-15) holds, which follows from (4-3), (4-4) and (4-10). Finally, (2-16) is valid trivially since a^k is a gradient. Therefore, Theorem 2.6 can be applied, which implies the existence of a nondecreasing sequence of measurable sets E_j such that $|\Omega \setminus E_j| \to 0$ and

$$A(x, \nabla u^k) \cdot \nabla u^k \omega_0 \rightharpoonup \bar{A} \cdot \nabla u \omega_0 \qquad \text{weakly in } L^1(E_j). \tag{4-12}$$

For any $B \in L^2_{\omega_0}(\Omega; \mathbb{R}^{n\times N})$, we have that $B\omega_0$ and also $A(\,\cdot\,, B)\omega_0$ belong to $L^2_{1/\omega_0}(\Omega; \mathbb{R}^{n\times N})$, and therefore using (4-7) and (4-8), we can observe that

$$(A(x, \nabla u^k) - A(x, B)) \cdot (\nabla u^k - B)\omega_0 \rightharpoonup (\bar{A} - A(x, B)) \cdot (\nabla u - B)\omega_0 \quad \text{weakly in } L^1(E_j). \tag{4-13}$$

Due to the monotonicity of A, we see that the term on the left-hand side is nonnegative and consequently its weak limit is nonnegative as well and we have that

$$\int_{E_j} (\bar{A} - A(x, B)) \cdot (\nabla u - B)\omega_0 \,dx \geq 0 \quad \text{for all } B \in L^2_{\omega_0}(\Omega; \mathbb{R}^{n\times N}) \text{ and all } j \in \mathbb{N}. \tag{4-14}$$

Therefore, it follows that

$$\int_{\Omega} (\bar{A} - A(x, B)) \cdot (\nabla u - B)\omega_0 \,dx \geq \int_{\Omega\setminus E_j} (\bar{A} - A(x, B)) \cdot (\nabla u - B)\omega_0 \,dx,$$

and letting $j \to \infty$ (note that the integral is well defined due to (4-7) and (4-8)) and using the fact that $|\Omega \setminus E_j| \to 0$ as $j \to \infty$ and the Lebesgue dominated convergence theorem, we obtain

$$\int_{\Omega} (\bar{A} - A(x, B)) \cdot (\nabla u - B)\omega_0 \,dx \geq 0 \quad \text{for all } B \in L^2_{\omega_0}(\Omega; \mathbb{R}^{n\times N}).$$

Hence, setting $B := \nabla u - \varepsilon G$ where $G \in L^\infty(\Omega; \mathbb{R}^{n\times N})$ is arbitrary and dividing by ε, we get

$$\int_{\Omega} (\bar{A} - A(x, \nabla u - \varepsilon G)) \cdot G\omega_0 \,dx \geq 0 \quad \text{for all } G \in L^\infty(\Omega; \mathbb{R}^{n\times N}).$$

Finally, using the Lebesgue dominated convergence theorem, the assumption (1-5) and the continuity of A with respect to the second variable, we can let $\varepsilon \to 0_+$ to deduce

$$\int_{\Omega} (\bar{A} - A(x, \nabla u)) \cdot G\omega_0 \,dx \geq 0 \quad \text{for all } G \in L^\infty(\Omega; \mathbb{R}^{n\times N}).$$

Since ω_0 is strictly positive almost everywhere in Ω, the relation (4-11) easily follows by setting, e.g.,

$$G := -\frac{\bar{A} - A(x, \nabla u)}{1 + |\bar{A} - A(x, \nabla u)|}.$$

Thus, (4-10) follows and u is a very weak solution.

[3] Although Theorem 2.6 is formulated for vector-valued functions, it is an easy extension to use it also for matrix-valued functions, which is the case here.

4C. *Uniqueness.* Let $u_1, u_2 \in W_0^{1,q}(\Omega; \mathbb{R}^N)$ with $q > 1$ be two very weak solutions to (2-5) for some given $f \in L^p_\omega(\Omega; \mathbb{R}^{n\times N})$, where $p \in (1, \infty)$ and $\omega \in \mathcal{A}_p$. Then it directly follows that

$$\int_\Omega (A(x, \nabla u_1) - A(x, \nabla u_2)) \cdot \nabla\varphi \,\mathrm{d}x = 0 \quad \text{for all } \varphi \in \mathscr{C}_0^{0,1}(\Omega; \mathbb{R}^{n\times N}). \tag{4-15}$$

First, consider the case that $f \in L^2(\Omega; \mathbb{R}^{n\times N})$. Then using the result of the previous part, we see that $u_1, u_2 \in W_0^{1,2}(\Omega; \mathbb{R}^N)$, and therefore due to the growth assumption (1-5), we see that (4-15) is valid for all $\varphi \in W_0^{1,2}(\Omega; \mathbb{R}^N)$. Consequently, the choice $\varphi := u_1 - u_2$ is admissible, and due to the strict monotonicity of A, we conclude that $\nabla u_1 = \nabla u_2$ almost everywhere in Ω and due to the zero trace also that $u_1 = u_2$.

Thus, it remains to discuss the case $f \notin L^2(\Omega; \mathbb{R}^{n\times N})$. But since $f \in L^p_\omega(\Omega; \mathbb{R}^{n\times N})$ with $p > 1$ and ω being the $\mathcal{A}_p$-weight, we can deduce that $f \in L^{p_0}(\Omega; \mathbb{R}^{n\times N})$ for some $p_0 > 1$; see (3-5). Consequently, following Lemma 3.2, we can define the $\mathcal{A}_2$-weight $\omega_0 := (1+Mf)^{p_0-2}$ and we get that $f \in L^2_{\omega_0}(\Omega; \mathbb{R}^{n\times N})$. Therefore, the weighted a priori estimates imply that $\nabla u_i \in L^2_{\omega_0}(\Omega; \mathbb{R}^{n\times N})$ for $i = 1, 2$. Hence, defining a new weight $w^n := 1 \wedge (n\omega_0)$, which is bounded, we also get that for each n the solutions satisfy $\nabla u_i \in L^2_{\omega^n}(\Omega; \mathbb{R}^{n\times N})$. Moreover, we have the estimate $A_2(\omega^n) \le A_2(1) + A_2(n\omega_0) = 1 + A_2(\omega_0) \le C(f)$. Hence, rewriting the identity (4-15) into the form

$$\int_\Omega \tilde{A}(x)(\nabla u_1 - \nabla u_2) \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega \big(\tilde{A}(x)\nabla u_1 - A(x, \nabla u_1) - (\tilde{A}(x)\nabla u_2 - A(x, \nabla u_2))\big) \cdot \nabla\varphi \,\mathrm{d}x, \tag{4-16}$$

which is valid for all $\varphi \in \mathscr{C}_0^{0,1}(\Omega; \mathbb{R}^{n\times N})$, we can use Theorem 2.5 to obtain

$$\int_\Omega |\nabla u_1 - \nabla u_2|^2 \omega^n \,\mathrm{d}x \le C \int_\Omega |\tilde{A}(x)\nabla u_1 - A(x, \nabla u_1) - (\tilde{A}(x)\nabla u_2 - A(x, \nabla u_2))|^2 \omega^n \,\mathrm{d}x \tag{4-17}$$

with some constant C independent of n. Moreover, due to the properties of the solution and ω^n, we can deduce that the integral appearing on the right-hand side is finite. In order to continue, we first recall the following algebraic result, whose proof can be found at the end of this subsection.

Lemma 4.1. *Let A fulfill (1-4), (1-5), (2-1) and (2-4). Then for every $\delta > 0$, there exists C such that for all $x \in \Omega$ and all $\eta_1, \eta_2 \in \mathbb{R}^{n\times N}$*

$$|A(x, \eta_1) - A(x, \eta_2) - \tilde{A}(x)(\eta_1 - \eta_2)| \le \delta|\eta_1 - \eta_2| + C(\delta). \tag{4-18}$$

Next, using the estimate (4-18) in (4-17), we find that for all $\delta > 0$

$$\int_\Omega |\nabla u_1 - \nabla u_2|^2 \omega^n \,\mathrm{d}x \le C \int_\Omega \delta|\nabla u_1 - \nabla u_2|^2 \omega^n + C(\delta)\omega^n \,\mathrm{d}x. \tag{4-19}$$

Thus, setting $\delta := 1/(2C)$, we can deduce that

$$\int_\Omega |\nabla u_1 - \nabla u_2|^2 \omega^n \,\mathrm{d}x \le C(\delta) \int_\Omega \omega^n \,\mathrm{d}x \le C, \tag{4-20}$$

where the last inequality follows from the fact that Ω is bounded and $\omega^n \le 1$. Hence, letting $n \to \infty$ in (4-20), using that $\omega^n \nearrow 1$ (which follows from the fact that $\omega_0 > 0$ almost everywhere) and using the monotone convergence theorem, we find that

$$\int_\Omega |\nabla u_1 - \nabla u_2|^2 \, dx \le C.$$

Hence, we see that $u_1 - u_2 \in W_0^{1,2}(\Omega; \mathbb{R}^N)$. In addition, using (4-18) again,

$$\begin{aligned}\int_\Omega |A(x, \nabla u_1) - A(x, \nabla u_2)|^2 \, dx \\ &\le 2\int_\Omega |A(x,\nabla u_1) - \tilde{A}(x)\nabla u_1 - A(x,\nabla u_2) + \tilde{A}(x)\nabla u_2|^2 \, dx + 2\int_\Omega |\tilde{A}(x)\nabla u_1 - \tilde{A}(x)\nabla u_2|^2 \, dx \\ &\le C\Big(1 + \int_\Omega |\nabla u_1 - \nabla u_2|^2 \, dx\Big) \le C.\end{aligned}$$

Therefore, (4-15) holds for all $\varphi \in W_0^{1,2}(\Omega; \mathbb{R}^{n\times N})$ and consequently also for $\varphi := u_1 - u_2$ and the strict monotonicity finishes the proof of the uniqueness. It remains to prove Lemma 4.1.

Proof of Lemma 4.1. Let δ be given and fixed. According to (2-1) and (2-4), we can find $k > 0$ (depending on δ) such that for all $x \in \Omega$ and all $|\eta| \ge k$

$$\frac{|A(x,\eta) - \tilde{A}(x)\eta|}{|\eta|} + \left|\frac{\partial A(x,\eta)}{\partial \eta} - \tilde{A}(x)\right| \le \frac{\delta}{4}. \tag{4-21}$$

To prove (4-18), we shall discus all possible cases of values η_1 and η_2. Recall here that δ and k are already fixed.

The case $|\eta_1| \le 2k$ and $|\eta_2| \le 2k$. In this case, we can simply use (1-5) to show that

$$|A(x,\eta_1) - A(x,\eta_2) - \tilde{A}(x)(\eta_1 - \eta_2)| \le C(1 + |\eta_1| + |\eta_2|) \le C(1 + 4k)$$

and (4-18) follows.

The case $|\eta_1| \le 2k$ and $|\eta_2| > 2k$. In this case, we again use (1-5), which combined with (4-21) leads to

$$\begin{aligned}|A(x,\eta_1) - A(x,\eta_2) - \tilde{A}(x)(\eta_1 - \eta_2)| &\le C(1 + |\eta_1|) + \left|\frac{\tilde{A}(x)\eta_2 - A(x,\eta_2)}{|\eta_2|}\right||\eta_2| \le C(1+2k) + \frac{\delta|\eta_2|}{2} \\ &\le C(1 + 2k + |\eta_1|) + \frac{\delta|\eta_2 - \eta_1|}{2} \le C(1+4k) + \delta|\eta_2 - \eta_1|.\end{aligned}$$

Therefore, (4-18) holds. Moreover, the case $|\eta_1| \ge 2k$ and $|\eta_2| \le 2k$ is treated similarly.

The case $|\eta_1| > 2k$ and $|\eta_2| > 2k$. First, let us also assume that

$$|\eta_2| \le 2|\eta_1 - \eta_2| \quad \text{and} \quad |\eta_1| \le 2|\eta_1 - \eta_2|. \tag{4-22}$$

In this setting, we use (4-21) to conclude

$$\begin{aligned}|A(x,\eta_1) - A(x,\eta_2) - \tilde{A}(x)(\eta_1 - \eta_2)| &\le \left|\frac{\tilde{A}(x)\eta_1 - A(x,\eta_1)}{|\eta_1|}\right||\eta_1| + \left|\frac{\tilde{A}(x)\eta_2 - A(x,\eta_2)}{|\eta_2|}\right||\eta_2| \\ &\le \frac{\delta}{4}(|\eta_1| + |\eta_2|) \le \delta|\eta_1 - \eta_2|,\end{aligned}$$

which again directly implies (4-18). Finally, it remains to discuss the case when at least one of the inequalities in (4-22) does not hold. For simplicity, we consider only the case when $|\eta_1| > 2|\eta_1 - \eta_2|$ since the second case can be treated similarly. First of all, using the assumption on η_1 and η_2, we deduce that for all $t \in [0, 1]$

$$|t\eta_2 + (1-t)\eta_1| = |\eta_1 - t(\eta_1 - \eta_2)| \geq |\eta_1| - t|\eta_1 - \eta_2| \geq |\eta_1| - |\eta_1 - \eta_2| \geq \frac{|\eta_1|}{2} \geq k.$$

Hence, since any convex combination of η_1 and η_2 is outside of the ball or radius k, we can use the assumption (4-21) to conclude

$$\begin{aligned}
&|A(x,\eta_2) - A(x,\eta_1) - \tilde{A}(x)(\eta_2 - \eta_1)| \\
&\qquad = \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\big(A(x, t\eta_2 + (1-t)\eta_1) - \tilde{A}(x)(t\eta_2 + (1-t)\eta_1)\big)\,\mathrm{d}t \right| \\
&\qquad = \left| \int_0^1 \left(\frac{\partial A(x, t\eta_2 + (1-t)\eta_1)}{\partial(t\eta_2 + (1-t)\eta_1)} - \tilde{A}(x) \right)(\eta_2 - \eta_1)\,\mathrm{d}t \right| \leq \int_0^1 \frac{\delta}{4}|\eta_2 - \eta_1|\,\mathrm{d}t \leq \delta|\eta_2 - \eta_1|
\end{aligned}$$

and (4-18) follows. □

5. Proof of Theorem 2.5

We start the proof by getting the a priori estimate in the standard nonweighted Lebesgue spaces, which is available due to Lemma 3.4. Let us fix a ball Q_0 such that $\Omega \subset Q_0$. Since $\omega \in \mathcal{A}_p$, we can use (3-5) to show that for some $\tilde{q} > 1$ we have $L^p_\omega(Q_0) \hookrightarrow L^{\tilde{q}}(Q_0)$. Thus, $f \in L^p_\omega(\Omega; \mathbb{R}^{n\times N})$ implies that $f \in L^{\tilde{q}}(\Omega; \mathbb{R}^{n\times N})$. The starting point of further analysis is the use of Lemma 3.4, which leads to the existence of a unique solution $u \in W^{1,\tilde{q}}_0(\Omega; \mathbb{R}^N)$ to (2-12) with the a priori bound

$$\left(\int_\Omega |\nabla u|^{\tilde{q}}\,\mathrm{d}x \right)^{1/\tilde{q}} \leq C(A, \tilde{q}, \Omega) \left(\int_\Omega |f|^{\tilde{q}}\,\mathrm{d}x \right)^{1/\tilde{q}}.$$

Consequently, using (3-5), we deduce

$$\left(\frac{1}{|Q_0|} \int_\Omega |\nabla u|^{\tilde{q}}\,\mathrm{d}x \right)^{1/\tilde{q}} \leq C(A, p, \Omega, \mathcal{A}_p(\omega)) \left(\frac{1}{\omega(Q_0)} \int_\Omega |f|^p \omega\,\mathrm{d}x \right)^{1/p}. \tag{5-1}$$

It remains to prove the a priori estimate (2-13). We divide the proof into several steps. In the first one, we shall prove the local (in Ω) estimates. Then we extend such a result up to the boundary, and finally we combine them to get Theorem 2.5.

5A. *Interior estimates.* This part is devoted to the estimates that are local in Ω; i.e., we shall prove the following:

Lemma 5.1. *Let $B \subset \mathbb{R}^n$ be a ball, $\omega \in \mathcal{A}_p$ arbitrary with some $p \in (1, \infty)$ and $A \in L^\infty(2B; \mathbb{R}^{n\times N\times n\times N})$ arbitrary satisfying*

$$c_1|\eta|^2 \leq A(x)\eta \cdot \eta \leq c_2|\eta|^2 \quad \textit{for all } x \in 2B \textit{ and all } \eta \in \mathbb{R}^{n\times N}.$$

Then there exists $\delta > 0$ *depending only on* p, c_1, c_2 *and* $A_p(\omega)$ *such that, if*

$$|A(x) - A(y)| \le \delta \quad \textit{for all } x, y \in 2B,$$

then for arbitrary $f \in L^p_\omega(2B; \mathbb{R}^{n\times N})$ *and* $u \in W^{1,\tilde{q}}(2B; \mathbb{R}^N)$ *with some* $\tilde{q} > 1$ *satisfying*

$$\int_{2B} A(x)\nabla u(x) \cdot \nabla\varphi(x)\,\mathrm{d}x = \int_{2B} f(x)\cdot\nabla\varphi(x)\,\mathrm{d}x \quad \textit{for all } \varphi \in \mathscr{C}^{0,1}_0(2B; \mathbb{R}^N),$$

the following holds:

$$\left(\fint_B |\nabla u|^p \omega\,\mathrm{d}x\right)^{1/p} \le C\left(\fint_{2B} |f|^p \omega\,\mathrm{d}x\right)^{1/p} + C\left(\fint_{2B} \omega\,\mathrm{d}x\right)^{1/p}\left(\fint_{2B} |\nabla u|^{\tilde{q}}\,\mathrm{d}x\right)^{1/\tilde{q}}, \tag{5-2}$$

where the constant C *depends only on* p, c_1, c_2 *and* $A_p(\omega)$.

Proof. First, we introduce some more notation. For ω, we denote $\omega(S) := \int_S \omega\,\mathrm{d}x$. Next, using Lemma 3.1, we can find $q \in (1, \tilde{q})$ such that $\omega \in \mathcal{A}_{p/q}$. Note here that $u \in W^{1,q}(2B; \mathbb{R}^N)$, which follows from the fact that $2B$ is bounded. In what follows, we fix such q and introduce the centered maximal operator with power q

$$(M_q(g))(x) := \sup_{r>0}\left(\fint_{B_r(x)} |g|^q\,\mathrm{d}y\right)^{1/q}.$$

Since $M_q(g) = (M(|g|^q))^{1/q}$, we see from the definition and the choice of q (which leads to $\omega \in \mathcal{A}_{p/q}(\mathbb{R}^n)$) that the operator M_q is bounded in $L^p_\omega(\mathbb{R}^n)$. We shall also use the restricted maximal operator

$$(M_q^{<\rho}(g))(x) = \sup_{\rho\ge r>0}\left(\fint_{B_r(x)} |g|^q\,\mathrm{d}y\right)^{1/q},$$

and it directly follows that for every Lebesgue point x of g

$$|g(x)| \le (M_q^{<\rho}(g))(x) \le (M_q(g))(x).$$

The inequality (5-2) will be proven using the proper estimates on the level sets for $|\nabla u|$ defined through

$$O_\lambda := \{x \in \mathbb{R}^n;\ M_q(\chi_{2B}\nabla u)(x) > \lambda\}.$$

Please observe that O_λ are open. Next, we use the Calderón–Zygmund decomposition. Thus, for fixed $\lambda > 0$ and $x \in B \cap Q_\lambda$, using the continuity of the integral with respect to the integration domain, we can find a ball $Q_{r_x}(x)$ such that

$$\lambda^q < \fint_{Q_{r_x}(x)} |\chi_{2B}\nabla u|^q\,\mathrm{d}x \le 2\lambda^q \quad \text{and} \quad \fint_{Q_r(x)} |\chi_{2B}\nabla u|^q\,\mathrm{d}x \le 2\lambda^q \quad \text{for all } r \ge r_x. \tag{5-3}$$

Next, using the Besicovich covering theorem, we can extract a countable subset $Q_i := Q_{r_i}(x_i)$ such that the Q_i have finite intersection, i.e., there exists a constant C depending only on n such that for all $i \in \mathbb{N}$

$$\#\{j \in \mathbb{N};\ Q_i \cap Q_j \ne \varnothing\} \le C.$$

In addition, it follows from the construction that

$$O_\lambda \cap B = \bigcup_{i\in\mathbb{N}} (Q_i \cap B). \tag{5-4}$$

Then we set

$$\Lambda := \left(\fint_{2B} |\nabla u|^q \,\mathrm{d}x \right)^{1/q},$$

and it directly follows that for any $Q \subset \mathbb{R}^n$

$$\left(\fint_{Q} |\chi_{2B} \nabla u|^q \,\mathrm{d}x \right)^{1/q} \le \left(\frac{|2B|}{|Q|} \right)^{1/q} \Lambda.$$

Consequently, assuming that $\lambda \ge 2^{2n}\Lambda$, we can deduce for every Q_i that

$$2^{2n}\Lambda \le \lambda < \left(\fint_{Q_i} |\chi_{2B} \nabla u|^q dx \right)^{1/q} \le \left(\frac{|2B|}{|Q_i|} \right)^{1/q} \Lambda = 2^{2n/q} \left(\frac{|B|}{|2Q_i|} \right)^{1/q} \Lambda.$$

Since $q \ge 1$, this inequality directly leads to $|2Q_i| \le |B|$. Therefore, using the fact that $Q_i = Q_{r_i}(x_i)$ with some $x_i \in B$, we observe that $2Q_i \subset 2B$. Moreover, it is evident that for some constant C depending only on the dimension n

$$|Q_i| \le C(n)|Q_i \cap B|. \tag{5-5}$$

Since $\omega \in \mathcal{A}_p$, the above relation implies (see, e.g., [Stein 1993, §V.1.7])

$$\omega(Q_i) \le C(n, A_p(\omega))\omega(Q_i \cap B). \tag{5-6}$$

Next, for arbitrary $\varepsilon > 0$ and $k \ge 1$, we introduce the redistributional set

$$U^\lambda_{\varepsilon,k} := O_{k\lambda} \cap \{x \in \mathbb{R}^n;\ M_q(f\chi_{2B})(x) \le \varepsilon\lambda\}.$$

Finally, we shall assume the following (recall that δ comes from the assumption of Lemma 5.1):

there exists $k \ge 1$ depending only on c_1, c_2, n, p, and $A_p(\omega)$ such that for all $\varepsilon \in (0, 1)$ and all $\lambda \ge 2^{2^n}\Lambda$

$$|Q_i \cap U^\lambda_{\varepsilon,k} \cap B| \le C(c_1, c_2, n)(\varepsilon + \delta)|Q_i|. \tag{5-7}$$

We postpone the proof of (5-7) and continue assuming that it holds true with fixed k such that (5-7) is valid. Hence, using (5-7), the Hölder inequality and the reverse Hölder inequality (which follows for $\mathcal{A}_p$-weights from (3-4)) and (5-6), we obtain for some $r > 1$ depending only on n, p and $A_p(\omega)$

$$\begin{aligned}\omega(Q_i \cap U^\lambda_{\varepsilon,k} \cap B) &\le C(n)|Q_i| \left(\fint_{Q_i} \omega^r \,\mathrm{d}x \right)^{1/r} \left(\frac{|Q_i \cap U^\lambda_{\varepsilon,k} \cap B|}{|Q_i|} \right)^{1/r'} \\ &\le C(n, p, A_p(\omega), c_1, c_2)(\varepsilon + \delta)^{1/r'} \omega(Q_i) \le C(n, p, A_p(\omega), c_1, c_2)(\varepsilon + \delta)^{1/r'} \omega(Q_i \cap B).\end{aligned}$$

By using the finite intersection property of the Q_i, we find

$$\omega(U^\lambda_{\varepsilon,k} \cap B) \le C(n, A_p(\omega), c_1, c_2)(\varepsilon + \delta)^{1/r'} \omega(O_\lambda \cap B). \tag{5-8}$$

Finally, using the Fubini theorem, we obtain

$$\int_B |\nabla u|^p \omega \,\mathrm{d}x = p\int_0^\infty \omega(\{(\nabla u)\chi_B > \lambda\})\lambda^{p-1}\,\mathrm{d}\lambda \le \Lambda^p \omega(B) + p\int_\Lambda^\infty \lambda^{p-1}\omega(O_\lambda \cap B)\,\mathrm{d}\lambda. \tag{5-9}$$

Therefore, to get the estimate (5-2), we need to estimate the last term on the right-hand side. To do so, we use the definition of $U^\lambda_{\varepsilon,k}$ and the substitution theorem, which leads for all $m > k\Lambda$ to

$$\begin{aligned}\int_{k\Lambda}^m \lambda^{p-1}\omega(O_\lambda\cap B)\,\mathrm{d}\lambda &\le \int_{k\Lambda}^m \lambda^{p-1}\omega(U^{\lambda/k}_{\varepsilon,k}\cap B)\,\mathrm{d}\lambda + \int_{k\Lambda}^m \lambda^{p-1}\omega\Big(\Big\{M_q(f\chi_{2B}) > \varepsilon\frac{\lambda}{k}\Big\}\Big)\,\mathrm{d}\lambda\\ &\overset{(5\text{-}8)}{\le} C(\varepsilon+\delta)^{1/r'}\int_{k\Lambda}^m \lambda^{p-1}\omega(O_{\lambda/k}\cap B)\,\mathrm{d}\lambda + \frac{k^p}{p\varepsilon^p}\int_{\mathbb{R}^n}|M_q(f\chi_{2B})|^p\omega\,\mathrm{d}x\\ &\le C(p,q,\varepsilon,A_p(\omega))\int_{2B}|f|^p\omega\,\mathrm{d}x + Ck^p(\varepsilon+\delta)^{1/r'}\int_\Lambda^{m/k}\lambda^{p-1}\omega(O_\lambda\cap B)\,\mathrm{d}\lambda\\ &\le C(p,q,\varepsilon,A_p(\omega))\int_{2B}|f|^p\omega\,\mathrm{d}x + Ck^p(\varepsilon+\delta)^{1/r'}\int_\Lambda^{k\Lambda}\lambda^{p-1}\omega(O_\lambda\cap B)\,\mathrm{d}\lambda\\ &\qquad + Ck^p(\varepsilon+\delta)^{1/r'}\int_{k\Lambda}^m \lambda^{p-1}\omega(O_\lambda\cap B)\,\mathrm{d}\lambda,\end{aligned}$$

where we used the fact that $\omega \in \mathcal{A}_{p/q}$. Finally, assuming (note that k is already fixed by (5-7), and at this point, we fix the maximal value of δ arising in the assumption of Lemma 5.1) that δ is so small that $Ck^p\delta^{1/r'} \le \frac{1}{8}$, we can find $\varepsilon \in (0,1)$ such that $Ck^p(\varepsilon+\delta)^{1/r'} \le \frac{1}{2}$. Consequently, we absorb the last term into the left-hand side, and letting $m \to \infty$, we find that

$$\int_{k\Lambda}^\infty \lambda^{p-1}\omega(O_\lambda\cap B)\,\mathrm{d}\lambda \le C(k,p,q,A_p(\omega))\Big(\int_{2B}|f|^p\omega\,\mathrm{d}x + \Lambda^p\omega(B)\Big).$$

Substituting this into (5-9), we find (5-2). To finish the proof, it remains to find $k \ge 1$ such that (5-7) holds.

Hence, assume that $Q_i \cap B \cap U^\lambda_{\varepsilon,k} \ne \varnothing$. Then it follows from the definition of $U^\lambda_{\varepsilon,k}$ that

$$\Big(⨍_{2Q_i}|f|^q\,\mathrm{d}x\Big)^{1/q} \le 2^n\varepsilon\lambda. \tag{5-10}$$

For $\lambda \ge 2^{2n}\Lambda$ (which implies $2Q_i \subset 2B$), we compare the original problem with

$$\begin{aligned} -\operatorname{div}(A_i\nabla h) &= 0 \quad \text{in } 2Q_i,\\ h &= u \quad \text{on } \partial(2Q_i),\end{aligned} \tag{5-11}$$

where the matrix A_i is defined as $A_i := A(x_i)$. Lemma 3.4 ensures the existence of such a solution (just consider $u - h$ with zero boundary data). Moreover, the matrix A_i is constant and elliptic and therefore we have the local $L^\infty - L^1$ estimate for h, i.e.,

$$\sup_{(3/2)Q_i}|\nabla h| \le C ⨍_{2Q_i}|\nabla h|\,\mathrm{d}x, \tag{5-12}$$

where the constant C depends only on n, c_1 and c_2. Further, since u solves our original problem, we find

$$\begin{aligned} -\operatorname{div}(A_i \nabla(u-h)) &= -\operatorname{div}((A-A_i)\nabla u - f) && \text{in } 2Q_i, \\ u-h &= 0 && \text{on } \partial 2Q_i. \end{aligned}$$

Therefore, we can use Lemma 3.4 to observe

$$⨍_{2Q_i} |\nabla(u-h)|^q \,\mathrm{d}x \le C ⨍_{2Q_i} |A-A_i|^q |\nabla u|^q \,\mathrm{d}x + C ⨍_{2Q_i} |f|^q \,\mathrm{d}x \le C(\varepsilon^q + \delta^q)\lambda^q, \tag{5-13}$$

where for the second inequality we used (5-3), (5-10) and the assumption that $|A(x)-A(y)| \le \delta$ for all $x, y \in B$. Then using the definition of Q_i, we see that, for all $y \in Q_i$ and all $r > r_i/2$, we have that $B_r(y) \subset B_{3r}(x_i)$ and $Q_i \subset B_{3r}(x_i)$. Consequently,

$$⨍_{B_r(y)} |\chi_{2B} \nabla u|^q \,\mathrm{d}x \le 3^n ⨍_{B_{3r}(x_i)} |\chi_{2B} \nabla u|^q \,\mathrm{d}x \le 6^n \lambda^q,$$

where we used (5-3). Choosing $k \ge 6^n$ and assuming that $\varepsilon, \delta \le 1$, we get by the previous estimate, the sublinearity of the maximal operator and the weak Harnack inequality (5-12) that for all $x \in Q_i \cap \{M_q(\nabla u) > k\lambda\}$

$$\begin{aligned} M_q(\nabla u)(x) &= M_q^{<r_i/2}(\nabla u)(x) \le M_q^{<r_i/2}(\nabla h)(x) + M_q^{<r_i/2}(\nabla u - \nabla h)(x) \\ &\le C\left(⨍_{2Q_i} |\nabla h|^q \,\mathrm{d}x\right)^{1/q} + M_q^{<r_i/2}(\nabla u - \nabla h)(x) \le C\lambda + M_q^{<r_i/2}(\nabla u - \nabla h)(x). \end{aligned}$$

Hence, setting $k := \max\{C+1, 6^n\}$, we can use the weak L^q-estimate for the maximal functions and the estimate (5-13) to conclude

$$\begin{aligned} |\{M_q(\nabla u) > k\lambda\} \cap Q_i| \le |\{M_q^{<r_i/2}(\nabla u - \nabla h) \ge \lambda\} \cap Q_i| &\le \frac{C}{\lambda^q} \int_{2Q_i} |\nabla(u-h)|^q \,\mathrm{d}x \\ &\le C(\varepsilon + \delta)|Q_i|, \end{aligned}$$

which finishes the proof of (5-7) and Lemma 5.1. □

5B. ***Estimates near the boundary.*** In this part, we generalize the result from the previous paragraph and extend its validity also to the neighborhood of the boundary.

Lemma 5.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain with $\mathscr{C}^1$ boundary, $\omega \in \mathscr{A}_p$ be arbitrary with some $p \in (1, \infty)$ and $A \in L^\infty(\Omega; \mathbb{R}^{n\times N\times n\times N})$ be arbitrary satisfying*

$$c_1|\eta|^2 \le A(x)\eta\cdot\eta \le c_2|\eta|^2 \quad \textit{for all } x \in 2B \textit{ and all } \eta \in \mathbb{R}^{n\times N}.$$

Then there exists $r^* > 0$ and $\delta > 0$ depending only on Ω, p, c_1, c_2 and $A_p(\omega)$ such that, if

$$\sup_{x,y\in\Omega;\,|x-y|\le r^*} |A(x)-A(y)| \le \delta,$$

then for arbitrary $f \in L^p_\omega(\Omega; \mathbb{R}^{n\times N})$ *and* $u \in W^{1,\tilde q}_0(\Omega; \mathbb{R}^N)$ *with some* $\tilde q > 1$ *satisfying*

$$\int_\Omega A\nabla u \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega f \cdot \nabla\varphi \,\mathrm{d}x \quad \textit{for all } \varphi \in \mathscr{C}^{0,1}_0(\Omega; \mathbb{R}^N), \tag{5-14}$$

we have for all $x_0 \in \overline{\Omega}$ *and all* $r \le r^*$ *the estimate*

$$\fint_{B_r(x_0)\cap\Omega} |\nabla u|^p \omega \,\mathrm{d}x \le \fint_{B_{2r}(x_0)\cap\Omega} C|f|^p\omega \,\mathrm{d}x + \fint_{B_{2r}(x_0)\cap\Omega} \omega \,\mathrm{d}x \left(\fint_{B_{2r}(x_0)\cap\Omega} C|\nabla u|^{\tilde q} \,\mathrm{d}x\right)^{p/\tilde q}. \tag{5-15}$$

First notice that in case $B_{2r}(x_0) \subset \Omega$ the inequality (5-15) follows from Lemma 5.1. Therefore, we focus only on the behavior near the boundary. Hence, let $x_0 \in \partial\Omega$ be arbitrary. Since $\Omega \in \mathscr{C}^1$, we know that there exist $\alpha, \beta > 0$ and $r_0 > 0$ such that (after a possible change of coordinates)

$$B^+_{r_0} := \{(x', x_n);\ |x'| < \alpha,\ a(x') - \beta < x_n < a(x')\} \subset \Omega,$$
$$B^-_{r_0} := \{(x', x_n);\ |x'| < \alpha,\ a(x') < x_n < a(x') + \beta\} \subset \Omega^c.$$

Here, we abbreviated $(x_1, \dots, x_n) := (x', x_n)$. Moreover, we know that for all $r \le r_0/2$ it holds that $B_{2r}(x_0) \cap \Omega \subset B^+_{r_0}$ and $B_{2r}(x_0) \cap \Omega^c \subset B^-_{r_0}$. In addition, we have $a \in \mathscr{C}^1([-\alpha, \alpha]^{n-1})$ and $\nabla a(0) \equiv 0$. For later purposes, we also denote

$$B_{r_0} := B^+_{r_0} \cup B^-_{r_0} \cup \{(x, x_n);\ |x'| < \alpha,\ a(x') = x_n\}$$

and define a mapping $T : B^+_{r_0} \to B^-_{r_0}$ as

$$T(x', x_n) := (x', 2a(x') - x_n) \quad \text{with } J(x) := \nabla T(x), \text{ i.e., } (J(x))_{ij} := \partial_{x_j}(T(x))_i.$$

It directly follows from the definition that $|\det J(x)| \equiv 1$ and also that T and T^{-1} are $\mathscr{C}^1$ mappings. Finally, we extend all quantities into $B^-_{r_0}$ as follows:

$$\tilde u(x) := \begin{cases} u(x) & \text{for } x \in B^+_{r_0},\\ -u(T^{-1}(x)) & \text{for } x \in B^-_{r_0},\end{cases}$$
$$\tilde A(x) := \begin{cases} A(x) & \text{for } x \in B^+_{r_0},\\ J(T^{-1}x)A(T^{-1}x)J^T(T^{-1}x) & \text{for } x \in B^-_{r_0},\end{cases}$$
$$\tilde f(x) := \begin{cases} f(x) & \text{for } x \in B^+_{r_0},\\ -J(T^{-1}x)f(T^{-1}(x)) & \text{for } x \in B^-_{r_0},\end{cases}$$
$$\tilde\omega(x) := \begin{cases} \omega(x) & \text{for } x \in B^+_{r_0},\\ \omega(T^{-1}(x)) & \text{for } x \in B^-_{r_0}.\end{cases}$$

It also directly follows from the definition and the fact that u has zero trace on $\partial\Omega$ that $\tilde u \in W^{1,q}(B_{r_0}; \mathbb{R}^N)$. Finally, we show that for all $\varphi \in \mathscr{C}^{0,1}_0(B_{r_0}; \mathbb{R}^N)$ the following identity holds:

$$\int_{B_{r_0}} \tilde A\nabla\tilde u \cdot \nabla\varphi \,\mathrm{d}x = \int_{B_{r_0}} \tilde f \cdot \nabla\varphi \,\mathrm{d}x. \tag{5-16}$$

For this, we observe that for any $\varphi \in \mathscr{C}_0^{0,1}(B_{r_0}^-;\mathbb{R}^N)$ and $\hat{\varphi} := \varphi \circ T \in \mathscr{C}_0^{0,1}(B_{r_0}^+;\mathbb{R}^N)$

$$\begin{aligned}
\int_{B_{r_0}^-} (\tilde{A}\nabla\tilde{u} - \tilde{f}) \cdot \nabla\varphi \,\mathrm{d}x &= \int_{B_{r_0}^-} \left(\tilde{A}_{ij}^{\mu\nu}(x) \frac{\partial \tilde{u}^\nu(x)}{\partial x_j} - \tilde{f}_i^\mu(x) \right) \frac{\partial \varphi^\mu(x)}{\partial x_i} \,\mathrm{d}x \\
&= \int_{B_{r_0}^-} \left(-\tilde{A}_{ij}^{\mu\nu}(x) \frac{\partial (u^\nu(T^{-1}x))}{\partial x_j} - \tilde{f}_i^\mu(x) \right) \frac{\partial (\hat{\varphi}^\mu(T^{-1}(x)))}{\partial x_i} \,\mathrm{d}x \\
&= \int_{B_{r_0}^-} \left(-\tilde{A}_{ij}^{\mu\nu}(x) \frac{\partial u^\nu(T^{-1}x)}{\partial (T^{-1}(x))_k} J_{kj}^{-1}(T^{-1}(x)) - \tilde{f}_i^\mu(x) \right) \frac{\partial \hat{\varphi}^\mu(T^{-1}(x))}{\partial (T^{-1}(x))_m} J_{mi}^{-1}(T^{-1}(x)) \,\mathrm{d}x \\
&= \int_{B_{r_0}^+} \left(-\tilde{A}_{ij}^{\mu\nu}(Tx) \frac{\partial u^\nu(x)}{\partial x_k} J_{kj}^{-1}(x) J_{mi}^{-1}(x) - \tilde{f}_i^\mu(Tx) J_{mi}^{-1}(x) \right) \frac{\partial \hat{\varphi}^\mu(x)}{\partial x_m} \,\mathrm{d}x \\
&= -\int_{B_{r_0}^+} (A(x)\nabla u(x) - f(x)) \cdot \nabla\hat{\varphi}(x) \,\mathrm{d}x.
\end{aligned}$$

In particular, for all $\varphi \in \mathscr{C}_0^{0,1}(B_{r_0}^+;\mathbb{R}^N)$

$$\int_{B_{r_0}^-} (\tilde{A}\nabla\tilde{u} - \tilde{f}) \cdot \nabla(\varphi \circ T^{-1}) \,\mathrm{d}x = -\int_{B_{r_0}^+} (A\nabla u - f) \cdot \nabla\varphi \,\mathrm{d}x. \tag{5-17}$$

Thus, if we define for $\varphi \in \mathscr{C}_0^{0,1}(B_{r_0};\mathbb{R}^N)$ the function

$$\bar{\varphi} := \begin{cases} \varphi \circ T^{-1} & \text{on } B_{r_0}^-, \\ \varphi & \text{on } B_{r_0}^+, \end{cases}$$

then $\bar{\varphi} \in \mathscr{C}_0^{0,1}(B_{r_0};\mathbb{R}^N)$ and (5-17) implies

$$\int_{B_{r_0}} (\tilde{A}\nabla\tilde{u} - \tilde{f}) \cdot \nabla\bar{\varphi} \,\mathrm{d}x = 0.$$

Therefore,

$$\int_{B_{r_0}} (\tilde{A}\nabla\tilde{u} - \tilde{f}) \cdot \nabla\varphi \,\mathrm{d}x = \int_{B_{r_0}} (\tilde{A}\nabla\tilde{u} - \tilde{f}) \cdot \nabla(\varphi - \bar{\varphi}) \,\mathrm{d}x = \int_{B_{r_0}^-} (\tilde{A}\nabla\tilde{u} - \tilde{f}) \cdot \nabla(\varphi - \bar{\varphi}) \,\mathrm{d}x.$$

Using (5-17) again, we get

$$\int_{B_{r_0}} (\tilde{A}\nabla\tilde{u} - \tilde{f}) \cdot \nabla\varphi \,\mathrm{d}x = -\int_{B_{r_0}^+} (A\nabla u - f) \cdot \nabla((\varphi - \bar{\varphi}) \circ T^{-1}) \,\mathrm{d}x.$$

Since $(\varphi - \bar{\varphi}) \circ T^{-1} = 0$ on $\partial\Omega$, we finally deduce with the help of (5-14) that

$$\int_{B_{r_0}} (\tilde{A}\nabla\tilde{u} - \tilde{f}) \cdot \nabla\varphi \,\mathrm{d}x = 0$$

for all $\varphi \in \mathscr{C}_0^{0,1}(B_{r_0};\mathbb{R}^N)$, which proves (5-16).

Consequently, we see that (5-16) holds, and therefore, we shall apply the local result stated in Lemma 5.1. To do so, we need to check the assumptions. First, the ellipticity of $\tilde{A}$ can be shown directly from the definition and the fact that J is a regular matrix. Moreover, the constant of ellipticity of $\tilde{A}$ depends only on the same constant for A and on the shape of Ω. Further, to be able to use (5-2), we need to show small oscillations of $\tilde{A}$. Since T is $\mathscr{C}^1$,

$$\begin{aligned}\sup_{x,y\in B^-_{r_0}} |\tilde{A}(x)-\tilde{A}(y)| &\le \sup_{x,y\in B^+_{r_0}} |J(x)A(x)J^T(x)-J(y)A(y)J^T(y)| \\ &\le C \sup_{x,y\in B^+_{r_0}} |A(x)-A(y)| + C \sup_{x,y\in B^+_{r_0}} |J(x)-J(y)|.\end{aligned}$$

Similarly, we can also deduce that

$$\begin{aligned}\sup_{x\in B^-_{r_0},\, y\in B^+_{r_0}} |\tilde{A}(x)-\tilde{A}(y)| &\le \sup_{x,y\in B^+_{r_0}} |J(x)A(x)J^T(x)-A(y)| \\ &\le C \sup_{x,y\in B^+_{r_0}} |A(x)-A(y)| + C \sup_{x\in B^+_{r_0}} |J(x)A(x)J^T(x)-A(x)| \\ &\le C \sup_{x,y\in B^+_{r_0}} |A(x)-A(y)| + C \sup_{x\in B^+_{r_0}} |\nabla a(x')|.\end{aligned}$$

Therefore, due to the continuity of J and the fact that $\nabla a(0)=0$, we see that for any $\delta>0$ we can find $r^*>0$ such that

$$C \sup_{x,y\in B^+_{r^*}} |J(x)-J(y)| + C \sup_{x\in B^+_{r^*}} |\nabla a(x')| < \frac{\delta}{2}.$$

Thus, assuming that

$$\sup_{x,y\in\Omega;\, C|x-y|\le r^*} |A(x)-A(y)| \le \frac{\delta}{2},$$

we can conclude that

$$\sup_{x,y\in B_{r^*}} |\tilde{A}(x)-\tilde{A}(y)| \le \delta.$$

We find $\delta>0$ and fix r^* such that all assumptions of Lemma 5.1 are satisfied and we consequently have

$$\left(\fint_{B_{r^*}(x_0)} |\nabla\tilde{u}|^p \widetilde{\omega}\,\mathrm{d}x\right)^{1/p} \le C\left(\fint_{B_{2r^*}(x_0)} |\tilde{f}|^p \widetilde{\omega}\,\mathrm{d}x\right)^{1/p} + C\left(\fint_{B_{2r^*}(x_0)} \widetilde{\omega}\,\mathrm{d}x\right)^{1/p} \left(\fint_{B_{2r^*}(x_0)} |\nabla\tilde{u}|^{\tilde{q}}\,\mathrm{d}x\right)^{1/\tilde{q}}$$

and (5-15) follows directly.

5C. *Global estimates.* Finally, we focus on the proof of Theorem 2.5. Recall that the ball Q_0 is a superset of Ω. Since A is continuous, we can find for any $\delta>0$ some r^* such that

$$\sup_{x,y\in\Omega;\, |x-y|\le r^*} |A(x)-A(y)| \le \delta.$$

Therefore on any sufficiently small ball, we can use the estimate (5-15). Since Ω has $\mathscr{C}^1$ boundary, we can find a finite covering of Ω by balls B_i of radius at most equal to r^* such that $|B_i\cap\Omega|\ge c|B_i|$. Then it follows from (5-15) and (5-1) that

$$\int_\Omega |\nabla u|^p \omega \,\mathrm{d}x \le C\int_\Omega |f|^p\omega\,\mathrm{d}x + C\sum_i \frac{\omega(2B_i)}{|2B_i|^{p/\tilde{q}}}\left(\int_\Omega |\nabla u|^{\tilde{q}}\,\mathrm{d}x\right)^{p/\tilde{q}}$$

$$\le C\int_\Omega |f|^p\omega\,\mathrm{d}x + C(p,\tilde{q},A,\Omega)\omega(Q_0)\left(\int_\Omega |\nabla u|^{\tilde{q}}\,\mathrm{d}x\right)^{p/\tilde{q}} \le C(A,\Omega,A_p(\omega))\int_\Omega |f|^p\omega\,\mathrm{d}x,$$

which finishes the proof of Theorem 2.5.

6. Proof of Theorem 2.6

We start the proof by observing that (2-14) leads to the estimate

$$\int_\Omega |a^k\cdot b^k|\omega\,\mathrm{d}x \le \int_\Omega |a^k|^p\omega + |b^k|^{p'}\omega\,\mathrm{d}x \le C.$$

Consequently, we can use Lemma 3.3 to conclude that there is a nondecreasing sequence of measurable sets $E_j\subset\Omega$ fulfilling $|\Omega\setminus E_j|\to 0$ as $j\to\infty$ such that for any $j\in\mathbb{N}$ and any $\varepsilon>0$ there exists a $\delta>0$ such that for each $U\subset E_j$ fulfilling $|U|\le\delta$

$$\sup_{k\in\mathbb{N}}\int_U |a^k\cdot b^k|\omega\,\mathrm{d}x \le \sup_{k\in\mathbb{N}}\int_U |a^k|^p\omega + |b^k|^{p'}\omega\,\mathrm{d}x \le \varepsilon. \tag{6-1}$$

Consequently, for any E_j, we can extract a subsequence that we do not relabel such that

$$a^k\cdot b^k\omega \rightharpoonup \overline{a\cdot b\omega} \quad \text{weakly in } L^1(E_j), \tag{6-2}$$

where $\overline{a\cdot b\omega}$ denotes in our notation the weak limit. Further, since $L^p_\omega(\Omega)$ and $L^{p'}_\omega(\Omega)$ are reflexive, we can pass to a (nonrelabeled) subsequence with

$$\begin{aligned} a_k &\rightharpoonup a \quad \text{weakly in } L^p_\omega(\Omega;\mathbb{R}^n),\\ b_k &\rightharpoonup b \quad \text{weakly in } L^{p'}_\omega(\Omega;\mathbb{R}^n).\end{aligned} \tag{6-3}$$

Our goal is to show that

$$\overline{a\cdot b\omega} = a\cdot b\omega \quad \text{almost everywhere in } \Omega. \tag{6-4}$$

Indeed, if this is the case, then it follows that not only a subsequence but the whole sequence fulfills (6-2).

Since $\omega\in\mathcal{A}_p$, we can find by (3-5) some $q>1$ such that $L^p_\omega(\Omega)\hookrightarrow L^q(\Omega)$. This implies

$$a^k \rightharpoonup a \quad \text{weakly in } L^q(\Omega;\mathbb{R}^n). \tag{6-5}$$

Moreover, since the mapping $g\mapsto g\omega^{1/s}$ is an isometry from $L^s_\omega(\Omega)$ to $L^s(\Omega)$, we also have

$$a^k\omega^{1/p} \rightharpoonup a\omega^{1/p} \quad \text{weakly in } L^p(\Omega;\mathbb{R}^n), \tag{6-6}$$

$$b^k\omega^{1/p'} \rightharpoonup b\omega^{1/p'} \quad \text{weakly in } L^{p'}(\Omega;\mathbb{R}^n). \tag{6-7}$$

Then, extending a^k by zero outside Ω, we can introduce d^k such that

$$\Delta d^k = a^k \quad \text{in } \mathbb{R}^n;$$

i.e., we set $d^k := a^k * G$, where G denotes the Green function of the Laplace operator on the whole $\mathbb{R}^n$. Then, using (6-5), we see that

$$d^k \rightharpoonup d \quad \text{weakly in } W^{2,q}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n), \tag{6-8}$$

where

$$\Delta d = a \quad \text{in } \mathbb{R}^n.$$

In addition, using (2-14) and the weighted theory for Laplace equation on $\mathbb{R}^n$ [Coifman and Fefferman 1974, p. 244], we can deduce

$$\nabla^2 d^k \rightharpoonup \nabla^2 d \quad \text{weakly in } L^p_\omega(\mathbb{R}^n;\mathbb{R}^{n\times n\times n}). \tag{6-9}$$

Hence, to show (6-4), it is enough to check whether

$$b^k\cdot(a^k-\nabla\operatorname{div} d^k)\omega \rightharpoonup b\cdot(a-\nabla\operatorname{div} d)\omega \quad \text{weakly in } L^1(E_j), \tag{6-10}$$

$$b^k\cdot\nabla(\operatorname{div} d^k)\omega \rightharpoonup b\cdot\nabla(\operatorname{div} d)\omega \quad \text{weakly in } L^1(E_j), \tag{6-11}$$

for all $j\in\mathbb{N}$.

First, we focus on (6-10). Assume for a moment that we know

$$\lim_{k\to\infty}\int_\Omega |a^k-a+\nabla(\operatorname{div}(d-d^k))|\tau\,\mathrm{d}x = 0 \tag{6-12}$$

for all nonnegative $\tau\in\mathfrak{D}(\Omega)$. Then for any $\varphi\in L^\infty(E_j)$,

$$\begin{aligned}
&\lim_{k\to\infty}\int_{E_j} b^k\cdot(a^k-\nabla\operatorname{div} d^k)\omega\varphi\,\mathrm{d}x\\
&\qquad=\lim_{k\to\infty}\int_{E_j} b^k\cdot(a-\nabla\operatorname{div} d)\omega\varphi\,\mathrm{d}x+\lim_{k\to\infty}\int_{E_j} b^k\cdot(a^k-a+\nabla\operatorname{div}(d-d^k))\omega\varphi\,\mathrm{d}x\\
&\qquad\overset{(6\text{-}7)}{=}\int_{E_j} b\cdot(a-\nabla\operatorname{div} d)\omega\varphi\,\mathrm{d}x+\lim_{k\to\infty}\int_{E_j} b^k\cdot(a^k-a+\nabla\operatorname{div}(d-d^k))\omega\varphi\,\mathrm{d}x
\end{aligned}$$

and (6-10) follows provided that the second limit in the above formula vanishes. However, we first notice that (for a subsequence) (6-12) implies that

$$b^k\cdot(a^k-a+\nabla\operatorname{div}(d-d^k))\omega\varphi\to 0 \quad \text{almost everywhere in } \Omega. \tag{6-13}$$

Second, using (6-8) and (6-6), we see that for any $U\subset E_j$

$$\int_U |b^k\cdot(a^k-a+\nabla\operatorname{div}(d-d^k))\omega\varphi|\,\mathrm{d}x \le C\|\varphi\|_\infty\|a^k-a\|_{L^p_\omega(\Omega)}\Big(\int_U |b^k|^{p'}\omega\,\mathrm{d}x\Big)^{1/p'}.$$

Then the equi-integrability (6-1) also guarantees the equi-integrability of the sequence (6-13), and consequently, the Vitali theorem leads to

$$\lim_{k\to\infty}\int_{E_j} b^k\cdot(a^k-a+\nabla\operatorname{div}(d-d^k))\omega\varphi\,\mathrm{d}x = 0,$$

which finishes the proof of (6-10) provided we show (6-12). First, it follows from (2-16) and (6-5) that for a subsequence that we do not relabel $\partial_{x_i} a_j^k - \partial_{x_j} a_i^k \to \partial_{x_i} a_j - \partial_{x_j} a_i$ strongly in $(W_0^{1,r}(\Omega))^*$ for all $i, j = 1, \dots, n$. Therefore, by the regularity theory for Poisson's equation, we find that

$$\partial_{x_i} d_j^k - \partial_{x_j} d_i^k \to \partial_{x_i} d_j - \partial_{x_j} d_i \quad \text{strongly in } W^{1,r}_{\mathrm{loc}}(\Omega) \tag{6-14}$$

for all $i, j = 1, \dots, n$ and all $r \in [1, q)$, where $q > 1$ comes from (6-5). Moreover, using the definition of d^k,

$$a_j^k - \partial_{x_j} \operatorname{div} d^k = \sum_{m=1}^{n} \partial^2_{x_m^2} d_j^k - \partial_{x_j} \partial_{x_m} d_m^k = \sum_{m=1}^{n} \partial_{x_m} (\partial_{x_m} d_j^k - \partial_{x_j} d_m^k),$$

and with the help of (6-14), we see that (6-12) directly follows and the proof of (6-10) is complete.

The rest of this section is devoted to the most difficult part of the proof, which is the validity of (6-11). For simplicity, we denote $e^k := \operatorname{div} d^k$, and due to (6-8) and (6-9),

$$e^k \rightharpoonup e \qquad \text{weakly in } W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n), \tag{6-15}$$

$$\nabla e^k \rightharpoonup \nabla e \quad \text{weakly in } L^p_\omega(\mathbb{R}^n; \mathbb{R}^n), \tag{6-16}$$

where $e = \operatorname{div} d$. Since we are interested only in the convergence result in Ω, we localize e^k by a proper cutting outside Ω. To be more precise on the ball B (recall that it is a ball such that $\Omega \subsetneq B$), we set

$$e_B^k := e^k \tau$$

with $\tau \in \mathcal{D}(B)$ being identically one in Ω. In addition, we can observe that

$$e_B^k \rightharpoonup e_B \qquad \text{weakly in } W_0^{1,q}(B), \tag{6-17}$$

$$\nabla e_B^k \rightharpoonup \nabla e_B \quad \text{weakly in } L^p_\omega(B; \mathbb{R}^n). \tag{6-18}$$

Indeed, the relation (6-17) is a trivial consequence of (6-15), and for the validity of (6-18), it is enough to show that

$$\int_B |\nabla e_B^k|^p \omega \, \mathrm{d}x \le C.$$

Since $|\nabla e_B^k| \le C|\nabla e^k| + C|e^k - (e^k)_B| + C|(e^k)_B|$, where e_B^k denotes the mean value of e^k over B, it follows from (6-15) and (6-16) that we just need to estimate the term involving $|e^k - (e^k)_B|$. But using the pointwise estimate $|e^k - (e^k)_B| \le C(B) M(\nabla e^k)$,

$$\int_B |e_B^k - (e^k)_B|^p \omega \, \mathrm{d}x \le C \int_{\mathbb{R}^n} |M(\nabla e^k)|^p \omega \, \mathrm{d}x \le C A_p(\omega) \int_{\mathbb{R}^n} |\nabla e^k|^p \omega \, \mathrm{d}x \le C,$$

where we used the properties of $\mathcal{A}_p$-weights. Finally, since $e_B^k \in W_0^{1,1}(B)$, we can apply the Lipschitz approximation (Theorem 2.7), which implies that for arbitrary fixed $\lambda > \lambda_0$ and for any k we find the Lipschitz approximation of e_B^k on the set B and denote it by $e_B^{k,\lambda}$. Then thanks to Theorem 2.7, for any λ, we can find a subsequence (that is not relabeled) such that

$$\nabla e_B^{k,\lambda} \rightharpoonup^* \nabla e_B^\lambda \quad \text{weakly* in } L^\infty(B;\mathbb{R}^n), \tag{6-19}$$

$$\nabla e_B^{k,\lambda} \rightharpoonup \nabla e_B^\lambda \quad \text{weakly in } L^p_\omega(B;\mathbb{R}^n), \tag{6-20}$$

$$e_B^{k,\lambda} \to e_B^\lambda \quad \text{strongly in } \mathscr{C}(B). \tag{6-21}$$

Please notice that we do not have any a priori knowledge of how the limit e_B^λ can be found; we just know that it exists.

In the next step, we identify the weak limit of $b^k \cdot \nabla e_B^{k,\lambda}$. Due to (6-3) and (6-19), we see that this sequence is equi-integrable and consequently poses a weakly converging (in the topology of L^1) subsequence. Therefore, to identify it, it is enough to show that for all $\eta \in \mathscr{D}(\Omega)$

$$\lim_{k\to\infty} \int_\Omega b^k \cdot \nabla e_B^{k,\lambda} \eta \,\mathrm{d}x = \int_\Omega b \cdot \nabla e_B^\lambda \eta \,\mathrm{d}x.$$

However, using (2-15), (6-19) and (6-21), we can deduce that

$$\lim_{k\to\infty} \int_\Omega b^k \cdot \nabla e_B^{k,\lambda} \eta \,\mathrm{d}x = \lim_{k\to\infty} \int_\Omega b^k \cdot (\nabla e_B^{k,\lambda} - \nabla e_B^\lambda)\eta \,\mathrm{d}x + \int_\Omega b \cdot \nabla e_B^\lambda \eta \,\mathrm{d}x = \int_\Omega b \cdot \nabla e_B^\lambda \eta \,\mathrm{d}x$$

and therefore

$$b^k \cdot \nabla e_B^{k,\lambda} \rightharpoonup b \cdot \nabla e_B^\lambda \quad \text{weakly in } L^1(\Omega). \tag{6-22}$$

Finally, let $\varphi \in L^\infty(E_j)$ be arbitrary and $C := C(\|\varphi\|_\infty)$. Then we check the validity of (6-11) as follows:

$$\begin{aligned}
\lim_{k\to\infty} \left| \int_{E_j} (b^k \cdot \nabla(\operatorname{div} d^k) - b \cdot \nabla(\operatorname{div} d))\omega\varphi \,\mathrm{d}x \right| &= \lim_{k\to\infty} \left| \int_{E_j} (b^k \cdot \nabla e_B^k - b \cdot \nabla e_B)\omega\varphi \,\mathrm{d}x \right| \\
\le \lim_{k\to\infty} \left| \int_{E_j} (b^k \cdot \nabla e_B^{k,\lambda} - b \cdot \nabla e_B^\lambda)\omega\varphi \,\mathrm{d}x \right| &+ C \limsup_{k\to\infty} \int_{E_j} |b^k||\nabla(e_B^k - e_B^{k,\lambda})|\omega \,\mathrm{d}x \\
&+ \left| \int_{E_j} b \cdot \nabla(e_B - e_B^\lambda)\omega\varphi \,\mathrm{d}x \right| \\
\le \lim_{k\to\infty} \left| \int_{E_j} \frac{(b^k \cdot \nabla e_B^{k,\lambda} - b \cdot \nabla e_B^\lambda)\varphi\omega}{1+\varepsilon\omega} \,\mathrm{d}x \right| &+ C \limsup_{k\to\infty} \left| \int_{E_j} \frac{\varepsilon\omega^2(|b^k||\nabla e_B^{k,\lambda}| + |b||\nabla e_B^\lambda|)}{1+\varepsilon\omega} \,\mathrm{d}x \right| \\
&+ C \limsup_{k\to\infty} \int_{E_j} |b^k||\nabla(e_B^k - e_B^{k,\lambda})|\omega \,\mathrm{d}x + \left| \int_{E_j} b \cdot \nabla(e_B - e_B^\lambda)\omega\varphi \,\mathrm{d}x \right| \\
\le C \limsup_{k\to\infty} \int_{E_j} \frac{\varepsilon\omega^2 |b^k||\nabla e_B^{k,\lambda}|}{1+\varepsilon\omega} &+ C \limsup_{k\to\infty} \int_{E_j} |b^k||\nabla(e_B^k - e_B^{k,\lambda})|\omega \,\mathrm{d}x \\
&+ \left| \int_{E_j} b \cdot \nabla(e_B - e_B^\lambda)\omega\varphi \,\mathrm{d}x \right| + C \int_{E_j} \frac{\varepsilon\omega^2 |b||\nabla e_B^\lambda|}{1+\varepsilon\omega} \,\mathrm{d}x =: \text{(I)} + \text{(II)} + \text{(III)} + \text{(IV)},
\end{aligned} \tag{6-23}$$

where the last identity follows from (6-22) since $\varphi\omega/(1+\varepsilon\omega)$ is a bounded function whenever $\varepsilon > 0$. In the next step, we show that all terms on the right-hand side vanish when we let $\varepsilon \to 0_+$ and $\lambda \to \infty$. To do so, we first observe that thanks to Theorem 2.7 and the weak lower semicontinuity

$$\nabla e_B^{k,\lambda} \rightharpoonup \nabla e_B^{\lambda} \quad \text{weakly in } L^p_\omega(\Omega;\mathbb{R}^n), \tag{6-24}$$

$$e_B^{k,\lambda} \rightharpoonup e_B^{\lambda} \quad \text{weakly in } W^{1,q}(\Omega), \tag{6-25}$$

$$\int_\Omega |\nabla e_B^\lambda|^q + |\nabla e_B^\lambda|^p \omega \,\mathrm{d}x \le C \liminf_{k\to\infty} \int_B |\nabla e_B^k|^q + |\nabla e_B^k|^p \omega \,\mathrm{d}x \le C. \tag{6-26}$$

Therefore, applying the Hölder inequality, we have the estimate

$$\int_{E_j} |b||\nabla e_B^\lambda|\omega \,\mathrm{d}x \le C.$$

Consequently, using the Lebesgue dominated convergence theorem (and also the fact that ω is finite almost everywhere), we deduce

$$\lim_{\varepsilon\to 0_+} (\mathrm{IV}) = C \lim_{\varepsilon\to 0_+} \int_{E_j} |b||\nabla e_B^\lambda| \frac{\varepsilon\omega^2}{1+\varepsilon\omega} \,\mathrm{d}x = 0. \tag{6-27}$$

For the second term involving ε the key property is the uniform equi-integrability of b^k stated in (6-1). Indeed, applying the Hölder inequality and (6-26) we have

$$\begin{aligned}
\lim_{\varepsilon\to 0_+} (\mathrm{I}) &= C \limsup_{\varepsilon\to 0_+} \limsup_{k\to\infty} \int_{E_j} |b^k||\nabla e_B^{k,\lambda}| \frac{\varepsilon\omega^2|\varphi|}{1+\varepsilon\omega} \,\mathrm{d}x \\
&\le C \limsup_{\varepsilon\to 0_+} \limsup_{k\to\infty} \biggl(\int_{E_j} |b^k|^{p'} \omega \frac{\varepsilon\omega}{1+\varepsilon\omega} \,\mathrm{d}x\biggr)^{1/p'} \biggl(\int_{E_j} |\nabla e_B^{k,\lambda}|^p \omega \,\mathrm{d}x\biggr)^{1/p} \\
&\le C \limsup_{\varepsilon\to 0_+} \limsup_{k\to\infty} \biggl(\int_{E_j\cap\{\omega>\lambda\}} |b^k|^{p'} \omega \frac{\varepsilon\omega}{1+\varepsilon\omega} \,\mathrm{d}x\biggr)^{1/p'} \\
&\qquad + C \limsup_{\varepsilon\to 0_+} \limsup_{k\to\infty} \biggl(\int_{E_j\cap\{\omega\le\lambda\}} |b^k|^{p'} \omega \frac{\varepsilon\omega}{1+\varepsilon\omega} \,\mathrm{d}x\biggr)^{1/p'} \\
&\le C \limsup_{k\to\infty} \biggl(\int_{E_j\cap\{\omega>\lambda\}} |b^k|^{p'} \omega \,\mathrm{d}x\biggr)^{1/p'}.
\end{aligned}$$

Since $|\{\omega > \lambda\}| \le C/\lambda$, we can use (6-1) and let $\lambda \to \infty$ in the last inequality to deduce

$$\limsup_{\lambda\to\infty} \limsup_{\varepsilon\to 0_+} \limsup_{k\to\infty} \int_{E_j} |b^k||\nabla e_B^{k,\lambda}| \frac{\varepsilon\omega^2|\varphi|}{1+\varepsilon\omega} \,\mathrm{d}x = 0. \tag{6-28}$$

Next, we let $\lambda \to \infty$ in all remaining terms on the right-hand side of (6-23). Using (2-22) and the Hölder inequality,

$$\begin{aligned}\limsup_{\lambda\to\infty}(\mathrm{II}) &= C \limsup_{\lambda\to\infty}\limsup_{k\to\infty}\int_{E_j}|b^k||\nabla(e_B^k - e_B^{k,\lambda})|\omega\,\mathrm{d}x\\ &= C \limsup_{\lambda\to\infty}\limsup_{k\to\infty}\int_{E_j\cap\{M(\nabla e_B^k)>\lambda\}}|b^k||\nabla(e_B^k - e_B^{k,\lambda})|\omega\,\mathrm{d}x\\ &\le C \limsup_{\lambda\to\infty}\limsup_{k\to\infty}\biggl(\int_{E_j\cap\{M(\nabla e_B^k)>\lambda\}}|b^k|^{p'}\omega\,\mathrm{d}x\biggr)^{1/p'} = 0,\end{aligned}\tag{6-29}$$

where the last inequality follows from the fact that $|\{M(\nabla e_B^k) > \lambda\}| \le C/\lambda$ and (6-1). Finally, we are left to show

$$\lim_{\lambda\to\infty}(\mathrm{III}) = \lim_{\lambda\to\infty}\left|\int_{E_j} b\cdot\nabla(e_B - e_B^\lambda)\omega\varphi\,\mathrm{d}x\right| = 0.\tag{6-30}$$

However, to get (6-30), it is enough to show that

$$\nabla e_B^\lambda \rightharpoonup \nabla e_B \quad \text{weakly in } L^p_\omega(\Omega;\mathbb{R}^n).$$

Due to (6-26), we however have that there is some $\overline{e_B} \in W^{1,q}(\Omega)$ such that

$$\begin{aligned} e_B^\lambda &\rightharpoonup \overline{e_B} && \text{weakly in } W^{1,q}(\Omega),\\ \nabla e_B^\lambda &\rightharpoonup \nabla\overline{e_B} && \text{weakly in } L^p_\omega(\Omega;\mathbb{R}^n).\end{aligned}$$

Hence, due to the uniqueness of the weak limit, it is enough to check that $\bar{e}_B = e_B$. To do so, we use the compact embedding $W^{1,1}(\Omega) \hookrightarrow\hookrightarrow L^1(\Omega)$ to get

$$\begin{aligned}\|\bar{e}_B - e_B\|_1 &= \lim_{\lambda\to\infty}\int_\Omega |e_B^\lambda - e_B|\,\mathrm{d}x = \lim_{\lambda\to\infty}\lim_{k\to\infty}\int_\Omega |e_B^{k,\lambda} - e_B^k|\,\mathrm{d}x\\ &= \lim_{\lambda\to\infty}\lim_{k\to\infty}\int_{\Omega\cap\{M(\nabla e_B^k)>\lambda\}}|e_B^{k,\lambda} - e_B^k|\,\mathrm{d}x\\ &\le \lim_{\lambda\to\infty}\lim_{k\to\infty}\|e_B^{k,\lambda} - e_B^k\|_q|\Omega\cap\{M(\nabla e_B^k) > \lambda\}|^{1/q'} \le C\lim_{\lambda\to\infty}\lambda^{-1/q'} = 0,\end{aligned}$$

and consequently (6-30) holds. Hence, using (6-27)–(6-30) in (6-23), we deduce (6-11) and the proof is complete.

7. Proof of Theorem 2.7

This part of the paper is devoted to the proof of Theorem 2.7. All statements except (2-22) are already contained in [Diening et al. 2013, Theorem 13] (see also [Diening 2013] for a survey on the Lipschitz truncation). The first inequality of (2-22) follows directly from the second one, so it is enough to prove the second estimate.

It follows from (2-20) and (2-21) that

$$\begin{aligned}\|\nabla(g - g^\lambda)\|_{L^p_\omega} &\le \|\nabla(g - g^\lambda)\chi_{\{M(\nabla g)>\lambda\}}\|_{L^p_\omega}\\ &\le \|\nabla g\chi_{\{M(\nabla g)>\lambda\}}\|_{L^p_\omega} + c\|\lambda\chi_{\{M(\nabla g)>\lambda\}}\|_{L^p_\omega}.\end{aligned}$$

We need to control the second term in the last estimate. Let us consider the open set $\{M(\nabla g) > \lambda\}$. For every $x \in \{M(\nabla g) > \lambda\}$, there exists a ball $B_{r(x)}(x)$ with

$$\lambda < \fint_{B_r(x)} |\nabla g| \,\mathrm{d}x \le 2\lambda. \tag{7-1}$$

These balls cover $\{M(\nabla g) > \lambda\}$. Next, using the Besicovich covering theorem, we can extract from this cover a countable subset B_i that is locally finite, i.e.,

$$\#\{j \in \mathbb{N};\ B_i \cap B_j \neq \varnothing\} \le C(n). \tag{7-2}$$

Using (7-1) and (7-2), we have the estimate

$$\begin{aligned}
\|\lambda \chi_{\{M(\nabla g)>\lambda\}}\|^p_{L^p_\omega} &= \lambda^p \omega(\{M(\nabla g) > \lambda\}) \le \sum_i \lambda^p \omega(B_i) \\
&\le \sum_i \Big(\fint_{B_i} |\nabla g| \,\mathrm{d}x \Big)^p \omega(B_i) \le \sum_i \fint_{B_i} |\nabla g|^p \omega \,\mathrm{d}x \Big(\fint_{B_i} \omega^{-(p'-1)} \,\mathrm{d}x \Big)^{1/(p'-1)} \omega(B_i) \\
&\le \mathcal{A}_p(\omega) \sum_i \int_{B_i} |\nabla g|^p \omega \,\mathrm{d}x \le C(n) \mathcal{A}_p(\omega) \int_{\{M(\nabla g)>\lambda\}} |\nabla g|^p \omega \,\mathrm{d}x.
\end{aligned}$$

This directly leads to the inequality

$$\|\lambda \chi_{\{M(\nabla g)>\lambda\}}\|_{L^p_\omega} \le C(n) \mathcal{A}_p(\omega)^{1/p} \|\nabla g \chi_{\{M(\nabla g)>\lambda\}}\|_{L^p_\omega},$$

which proves the desired estimate (2-22).

8. Proof of Theorem 2.8

We present only a sketch of the proof here since all steps were already justified in the proof of Theorem 2.3. Hence, to obtain the a priori estimate (2-23), we observe that

$$\int_\Omega \tilde{A}(x)(\nabla u - \nabla u_0) \cdot \nabla \varphi \,\mathrm{d}x = \int_\Omega \big(f - \tilde{A}(x) \nabla u_0 + \tilde{A}(x) \nabla u - A(x, \nabla u) \big) \cdot \nabla \varphi \,\mathrm{d}x,$$

which by the use of Theorem 2.5 (note here that $u - u_0$ has zero trace) and (2-1) leads to

$$\begin{aligned}
\int_\Omega |\nabla u - \nabla u_0|^p \omega \,\mathrm{d}x &\le C \int_\Omega \big(|f|^p + |\nabla u_0|^p + |\tilde{A}(x) \nabla u - A(x, \nabla u)|^p \big) \omega \,\mathrm{d}x \\
&\le C(\varepsilon) \int_\Omega (|f|^p + |\nabla u_0|^p + 1) \omega \,\mathrm{d}x + \varepsilon \int_\Omega |\nabla u|^p \omega \,\mathrm{d}x.
\end{aligned}$$

Consequently, choosing ε small enough and using the triangle inequality, we find (2-9). The existence is then identically the same; we just also need to approximate u_0 by a sequence of smooth functions such that

$$u_0^k \to u_0 \quad \text{strongly in } W^{1,\tilde{q}}(\Omega; \mathbb{R}^N).$$

Finally, for the uniqueness proof, we use a similar procedure and see that if u_1 and u_2 are two solutions then

$$\int_\Omega \tilde{A}(x)(\nabla u_1 - \nabla u_2)\cdot\nabla\varphi\,dx = \int_\Omega \big(\tilde{A}(x)(\nabla u_1 - \nabla u_2) + A(x,\nabla u_1) - A(x,\nabla u_2)\big)\cdot\nabla\varphi\,dx,$$

and since $u_1 - u_2 \in W_0^{1,\tilde{q}}(\Omega;\mathbb{R}^N)$, we may now follow step by step the proof of Theorem 2.3.

9. Proofs of corollaries

Proof of Corollary 2.2. The proof of Corollary 2.2 is rather straightforward. Indeed, for a given measure $f \in \mathcal{M}(\Omega;\mathbb{R}^N)$, we can use the classical theory and find $v \in W_0^{1,n'-\varepsilon}(\Omega;\mathbb{R}^N)$ for all $\varepsilon > 0$ solving

$$\int_\Omega \nabla v\cdot\nabla\varphi\,dx = \langle f,\varphi\rangle \quad \text{for all } \varphi \in \mathscr{C}_0^{0,1}(\Omega;\mathbb{R}^N).$$

Then it follows that u is a solution to (2-7) if and only if it solves

$$\int_\Omega A(x,\nabla u)\cdot\nabla\varphi\,dx = \int_\Omega \nabla v\cdot\nabla\varphi\,dx \quad \text{for all } \varphi \in \mathscr{C}_0^{0,1}(\Omega;\mathbb{R}^N). \tag{9-1}$$

Thus, we can now apply Theorem 2.1 with $f := \nabla v$ and all statements in Corollary 2.2 directly follow. □

Proof of Corollary 2.4. We show that Corollary 2.4 can be directly proved by using Theorem 2.3. Indeed, by setting

$$\omega := (1+Mf)^{q-2} = (M(1+|f|))^{q-2},$$

where we extended f by zero outside Ω, we can use Lemma 3.2 to deduce that $\omega \in \mathcal{A}_2$ provided that $|q-2| < 1$. Since $q \in (1,2)$, we always have $|q-2| < 1$ and therefore $\omega \in \mathcal{A}_2$. Consequently, we can construct a solution u according to Theorem 2.3. Next, using (2-9) and the continuity of the maximal function, we can deduce

$$\begin{aligned}\int_\Omega \frac{|\nabla u|^2}{(1+Mf)^{2-q}}\,dx &= \int_\Omega |\nabla u|^2\omega\,dx \le C(A_2(\omega),\Omega)\left(1+\int_\Omega |f|^2\omega\,dx\right)\\ &= C(A_2(\omega),\Omega)\left(1+\int_\Omega \frac{|f|^2}{(1+Mf)^{2-q}}\,dx\right)\\ &\le C(A_2(\omega),\Omega)\left(1+\int_\Omega (Mf)^q\,dx\right) \le C(f,\Omega,q)\left(1+\int_\Omega |f|^q\,dx\right),\end{aligned}$$

which is nothing else than (2-10). □

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