MA341, SS2020

Tests

Test 1

SS2020, MA341, Test 1

1) Determine if the function $\theta(t) = 2e^{3t} - e^{2t}$ is a solution to the equation

$$\theta'' - \theta\theta' + 3\theta = -2e^{2t}.$$

2) Consider the problem

$$y' = x^2(1+y).$$

Do the complete theoretical analysis of the problem. Sketch the direction fields. Integrate the equation. Find all solutions on maximal intervals. Draw their graphs. Determine all solutions with the initial value $y(0) = 3$.

3) Find all solutions to the problem on maximal intervals

$$xy' + 3(y + x^2) = \frac{\sin(x)}{x}.$$

SS 2019 - MA341 - Test 1 - solution

1) $\theta(t) = 2e^{3t} - e^{2t}$

$\theta'(t) = 6e^{3t} - 2e^{2t}$

$\theta''(t) = 18e^{3t} - 4e^{2t}$

$\theta'' - \theta\theta' + 3\theta = 18e^{3t} - 4e^{2t} - (2e^{3t} - e^{2t})(6e^{3t} - 2e^{2t}) +$
$3(2e^{3t} - e^{2t}) = 18e^{3t} - 4e^{2t} - (12e^{6t} - 10e^{5t} + 2e^{4t}) + 6e^{3t} - 3e^{2t}$
$= -12e^{6t} + 10e^{5t} - 2e^{4t} + 24e^{3t} - 7e^{2t}$

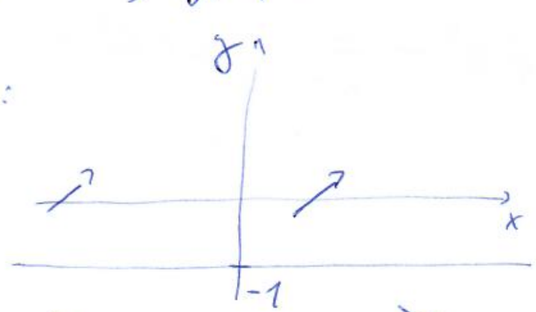
at $t=0$: $\theta''(0) - \theta(0)\theta'(0) + 3\theta(0) = 13$

$-2e^{2t}\big|_{t=0} = -2$

$13 \neq -2 \Rightarrow \theta(t)$ is not a solution on $\mathbb{R}$ (any interval containing 0)

2) $y' = x^2(1+y)$ .. separable eqn. $g(x) = x^2$, $h(y) = 1+y$; $\frac{\partial h}{\partial y}(y) = 1$
$\Rightarrow (x,y) \in \mathbb{R}^2$; g, h, h' continuous on $\mathbb{R}^2 \Rightarrow$ uniqueness and ex. of solution

direction fields:



$y(x) = -1$ on $\mathbb{R}$ is equilibrium (unstable)

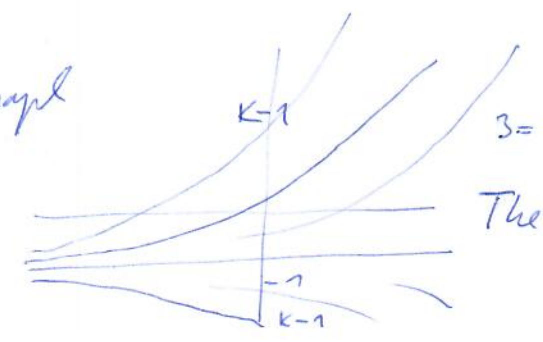
$\frac{y'}{1+y} = x^2$; $(\lg|1+y|)' = \left(\frac{x^3}{3} + c\right)'$; $\lg|1+y| = \frac{x^3}{3} + c$;

$|1+y| = \exp\left(\frac{x^3}{3} + c\right) \left(= e^{\frac{x^3}{3}} \cdot K\right)$; $K > 0$

$1+y = e^{x^3/3} \cdot K$, $K \in \mathbb{R}$; $\boxed{y(x) = -1 + Ke^{\frac{x^3}{3}},\ K \in \mathbb{R} \text{ in } \mathbb{R}}$

general formula for sol.

graph



$3 = y(0) = -1 + K \Rightarrow K = 4$.

The unique sol of the initial value problem is
$y(x) = -1 + 4e^{x^3/3}$ in $\mathbb{R}$.

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3) $$xy' + 3y = \frac{\sin x}{x} - 3x^2$$

for $x \neq 0$: $y' + 3\frac{y}{x} = \frac{\sin x}{x^2} - 3x$, i.f.: $e^{3\ln|x|} = |x|^3$

we consider it on $(-\infty, 0)$ and $(0, +\infty)$

i.f. x^3:

$x \in \mathbb{R} \setminus \{0\}$

$$x^3 y' + 3x^2 y = x \sin x - 3x^4 ; \quad (x^3 y)' = \left(\sin x - x\cos x - \frac{3}{5}x^5\right)'$$

$$\int x \sin x \, dx = -x\cos x + \int \cos x \, dx = -x\cos x + \sin x \text{ in } \mathbb{R}$$

$\downarrow d \quad \downarrow i$

$1 \quad -\cos x$

$$x^3 y = \sin x - x\cos x - \frac{3}{5}x^5 + C , \quad C \in \mathbb{R}$$

$$y(x) = \frac{\sin x}{x^3} - \frac{\cos x}{x^2} - \frac{3}{5}x^2 + \frac{C}{x^3} ; \quad C \in \mathbb{R}, \text{ in } (-\infty, 0) \text{ and } (0, +\infty).$$

Homeworks

Homework 1

1. Show that $\Phi(x) = x^2$ is an explicit solution to $xy' = 2y$ on the interval $(-\infty, \infty)$.[1.2 2a]

2. Show that $\Phi(x) = x^2 - x$ is an explicit solution to $y' + y^2 = e^{2x} + (1-2x)e^x + x^2 - 1$ on the interval $(-\infty, \infty)$.[1.2 2b]

3. Show that $\Phi(x) = x^2 - x^{-1}$ is an explicit solution to $x^2 y'' = 2y$ on the interval $(0, \infty)$.[1.2 2c]

4. Classify next equations as ordinary differential equation or partial differential equation, linear or nonlinear. Determine its degree.

$$y' = \frac{y(2-3x)}{x(1-3y)}$$

$$\sqrt{1-y}y'' + 2xy' = 0$$

$$\frac{\partial N}{\partial t} = \frac{\partial^2 N}{\partial r^2} + \frac{1}{r}\frac{\partial N}{\partial r} + kN, \quad \text{for } k > 0.$$

HW1

1) Set $y(x)=x^2$. $LS = x y'(x) = x\cdot 2x = 2x^2$

$RS = 2y(x) = 2x^2$

$LS = RS$ for all $x\in\mathbb{R}$ $\Rightarrow$ $y(x)=x^2$ is solution on $\mathbb{R}$.

2) Set $y(x)=x^2-x$.

$LS = y'(x)-y^2(x) = 2x-1-(x^2-x)^2 = 2x-1-x^4+2x^3-x^2$

$RS = e^{2x}+(1-2x)e^{x}+x^2-1$

(at $x=0$: $LS(0) = -1 \neq RS(0) = 1$) $\Rightarrow$ $y(x)=x^2-x$ is not a solution on $\mathbb{R}$

3) Set $y(x)=x^2-x^{-1}$. $LS = x^2y''(x) = x^2\left(2x+\frac{1}{x^2}\right)' = x^2\left(2-\frac{2}{x^3}\right) =$

$= 2x^2-\frac{2}{x}$, $RS = 2y(x) = 2x^2-\frac{2}{x}$. $LS=RS$ on $(0,+\infty)$

$\Rightarrow$ $y(x):=x^2-x^{-1}$ is a solution on $(0,+\infty)$

4) $A: y' = \dfrac{y(2-3x)}{x(1-3y)}$, $B: \sqrt{1-y}\,y''+2xy' = 0$, $C: \dfrac{\partial N}{\partial t} = \dfrac{\partial^2 N}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial N}{\partial r}+kN$, $k>0$.

- A,B: ordinary differential equations

  C: partial differential eqn

- degree of $A = 1$, degree of B is 2, degree of C is 2

A is nonlinear since it is written as $F(x,y,y')=0$ with

$F(x,y,z) = z-\dfrac{y(2-3x)}{x(1-3y)}$ and this function is not linear since

$$F(x,\alpha y,\alpha z) = \alpha z-\frac{\alpha y(2-3x)}{x(1-3\alpha y)} = \alpha F(x,y,z)-\frac{\alpha y(2-3x)}{x(1-3\alpha y)}+\frac{\alpha y(2-3x)}{x(1-3y)}$$

$\neq \alpha F(x,y,z)$ for example if $x=1$, $y=1$, $z=1$, $\alpha=2$.

B is nonlinear similarly

C is linear: $F(r,N,\overset{\frac{\partial N}{\partial t}}{u},\overset{\frac{\partial N}{\partial r}}{v},\overset{\frac{\partial^2 N}{\partial r^2}}{w}) = u-w-\frac{1}{r}v+kN$ and $\forall\, r,N,u,v,w,N',u',v',w',\alpha,\beta\in\mathbb{R}$

$$F(r,\alpha N+\beta N',\alpha u+\beta u',\alpha v+\beta v',\alpha w+\beta w') = \alpha F(r,N,u,v,w)+\beta F(r,N',u',v',w')$$

Homework 2

1. Determine if the next ODE's are linear or separable.
$$x' + xt = e^x, \quad xx' + t^2 x = \sin(t).$$

2. Find a general solution to the following separable ODE's. In particular: 1) Apply general theory (existence, uniqueness, direction field method, find trivial solution), 2) Integrate the equation, 3) Express the explicit solutions together with its domain of definition, 4) Draw the complete picture of solutions. If you are not able to find exact roots of the solutions, try to estimate them. Course of the function may help you.
$$y' = \frac{x}{y^2\sqrt{1+x}}, \quad y' = \frac{x}{y\sqrt{1+x}}$$

3. Find a general solution to the following linear ODE's. Don't forget to write its domain of definition $(x^2+1)y' + xy - x = 0$.

4. Solve IVP $y' + 4y - e^{-x} = 0$, $y(0) = 4/3$.

Solution to homework 2

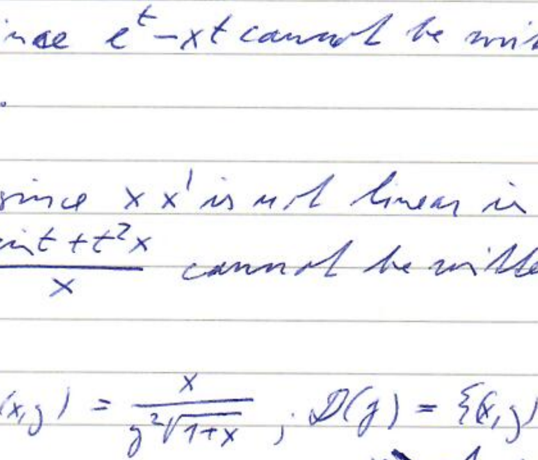
No.
Date 4.4

1) $x'+xt=e^t$ is linear, since $x'+xt$ is linear in x and x'.
with right hand side e^t.
It is not separable since e^t-xt cannot be written as a product $h(t)\cdot g(x)$.

$xx'+t^2x=\sin t$ is nonlinear since xx' is not linear in x and x'.
It is not separable since $\frac{\sin t+t^2x}{x}$ cannot be written as a product $h(t)g(x)$.

2) $y'=\frac{x}{y^2\sqrt{1+x}}$ general theory: $f(x,y)=\frac{x}{y^2\sqrt{1+x}}$, $\mathcal{D}(f)=\{(x,y)\in\mathbb{R}^2,\ x>-1,\ y\neq 0\}$
$f, \partial_y f$ are continuous on $\mathcal{D}(f)$ => on $\mathcal{D}(f)$ we have ex. and uniqueness

direction fields
& guess of solutions



(*)

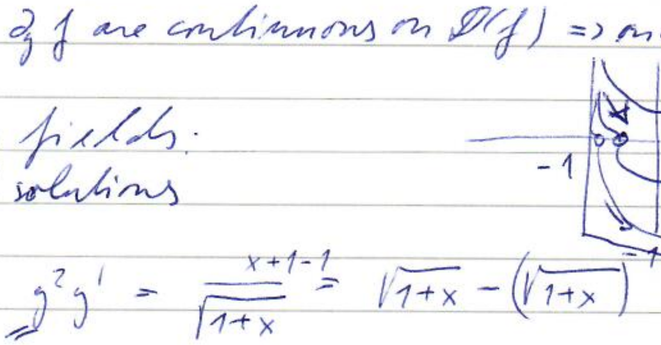
Integration: $y^2y'=\frac{x+1-1}{\sqrt{1+x}}=\sqrt{1+x}-\left(\sqrt{1+x}\right)^{-1}=\left(\frac{(1+x)^{3/2}}{\frac{3}{2}}-\frac{(1+x)^{1/2}}{1/2}+C\right)'$

$\left(\frac{y^3}{3}\right)'$

$$\Rightarrow\ y^3(x)=2(1+x)^{3/2}-6(1+x)^{1/2}+C$$

$y(x)=\sqrt[3]{2(1+x)^{3/2}-6(1+x)^{1/2}+C}$ for $C\in\mathbb{R}$ on set/intervals where $RHS\neq 0$

We solve $h(z)=2z^3-6z+C=0$, z stays for $(1+x)^{1/2}$
$h'(z)=6z^2-6$; $z=\pm 1$



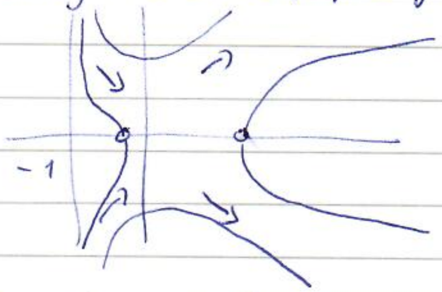
h has on $(0,+\infty)$ min in 1, $h(1)=C-4$

=> $C>4$ no root $\mathcal{D}(y)=(-1,+\infty)$
$C=4$, 1 root $\mathcal{D}(y)=(-1,0)$ and $(0,+\infty)$
$\neq 0$
$0<C<4$ 2 roots, one in $(-1,0)$ other ~~1 root in $(0,+\infty)$~~
in $(0,+\infty)$, we denote them r_1, r_2 then
$\mathcal{D}(y)=(-1,r_1),(r_1,r_2),(r_2,+\infty)$.

Solutions are plotted in (*)
~~$0\le$~~ $C<0$ 1 root in $(0,+\infty)$
$\mathcal{D}(y)=(-1,r_2)$ and $(r_2,+\infty)$

$y' = \dfrac{x}{y\sqrt{1+x}}$ general theory: same as before $f(x,y) = \dfrac{x}{y\sqrt{1+x}}$

direction fields:



(+)

Integration: $\underbrace{y\,y'}_{\left(\frac{y^2}{2}\right)'} = \dfrac{x}{\sqrt{1+x}} = \left(\frac{2}{3}(1+x)^{3/2} - 2(1+x)^{1/2} + C\right)'$

$y^2(x) = \frac{4}{3}(1+x)^{3/2} - 4(1+x)^{1/2} + C$ for any C on interval where is $RHS \geq 0$!

If $C > 4$ $\quad D(y) = (-1, +\infty)$

$C = 4$ $\quad = (-1, 0)$ and $(0, +\infty)$

$0 < C < 4$ $\quad (-1, r_1)$ and $(r_2, +\infty)$

$C \leq 0$ $\quad (r_2, +\infty)$

formula for sol: $y(x) = \pm\sqrt{\frac{4}{3}(1+x)^{3/2} - 4(1+x)^{1/2} + C}$.

Picture of sols is above (+)

3) $(x^2+1)y' + xy - x = 0$

~~Div~~ $y' + \frac{x}{x^2+1}y = \frac{x}{1+x^2}$ $\quad$ IF: $\exp\left(\int \frac{1}{2}\frac{2x}{x^2+1}\,dx\right) = \exp \frac{1}{2}\ln(1+x^2) = \sqrt{1+x^2}$

$(\sqrt{1+x^2}\,y)' = \sqrt{1+x^2}\,y' + y\frac{x}{\sqrt{1+x^2}} = \frac{2x}{\sqrt{1+x^2}} \cdot \frac{1}{2} = \left(\sqrt{1+x^2} + C\right)'$

$y(x) = 1 + \dfrac{C}{\sqrt{1+x^2}}$ for any $C \in \mathbb{R}$ on $\mathbb{R}$

$y' + 4y = e^{-x}$ $\quad$ IF: e^{4x}

$(e^{4x}y)' = e^{4x}y' + 4e^{4x}y = e^{3x} = \left(\frac{e^{3x}}{3} + C\right)'$

$y(x) = \frac{e^{-x}}{3} + C e^{-4x}$ for any $C \in \mathbb{R}$ on $\mathbb{R}$

IC $y(0) = 4/3$: $4/3 = y(0) = \frac{1}{3} + C \Rightarrow C = 1$

$\boxed{y(x) = \frac{e^{-x}}{3} + e^{-4x} \text{ on } \mathbb{R}}$

Homework 3

1. Solve the initial value problem

$$(1/x + 2y^2x)\,dx + (2yx^2 - \cos y)dy = 0 \;, \quad y(1) = \pi$$

2. A brine solution of salt flows at a constant rate of 6 L/min into a large tank that initially held 50 L of brine solution in which was dissolved 0.5 kg of salt. The solution inside the tank is kept well stirred and flows out of the tank at the same rate. If the concentration of salt in the brine entering the tank is 0.05 kg/L, determine the mass of salt in the tank after *t* min. When will the concentration of salt in the tank reach 0.03 kg/L?

3. The initial mass of a certain species of fish is 7 million tons. The mass of fish, if left alone, would increase at a rate proportional to the mass, with a proportionality constant of 2/yr. However, commercial fishing removes fish mass at a constant rate of 15 million tons per year. When will all the fish be gone? If the fishing rate is changed so that the mass of fish remains constant, what should that rate be?

4. It was noon on a cold December day in Tampa: 16°C. Detective Taylor arrived at the crime scene to find the sergeant leaning over the body. The sergeant said there were several suspects. If they knew the exact time of death, then they could narrow the list. Detective Taylor took out a thermometer and measured the temperature of the body: 34.5°C. He then left for lunch. Upon returning at 1:00 P.M., he found the body temperature to be 33.7°C. When did the murder occur? [*Hint:* Normal body temperature is 37°C.]

Homework 3 - ma 341

1)

$$\underbrace{\left(\frac{1}{x} + 2y^2x\right)}_{M(x,y)} dx + \underbrace{(2yx^2 - \cos y)}_{N(x,y)} dy = 0 \ ; \ y(1) = \pi$$

Is the equation exact?

$$\frac{\partial M}{\partial y}(x,y) = 4yx \quad = \quad \frac{\partial N}{\partial x}(x,y) = 4xy$$

Yes, it is! M

$$M = \frac{\partial F}{\partial x} \Rightarrow F(x,y) = \lg|x| + (xy)^2 + C(y)$$

$$N(x,y) = \frac{\partial F}{\partial y}(x,y) = \cancel{2x^2y} + C'(y) = \cancel{2yx^2} - \cos y$$

$$\Rightarrow C'(y) = -\cos y \Rightarrow C(y) = -\sin y + C$$

The unknown potential $F(x,y) = \lg|x| + (xy)^2 - \sin y$ ~~the~~

Implicit formula for solutions: $F(x,y) = C$. Insert i.c.

$(x = 1, y = \pi)$: $\pi^2 = C \Rightarrow C = \pi^2$

Solve $\lg|x| + (xy)^2 - \sin y = \pi^2$ for $y(x)$ to get solution.

2) First we construct equation:

initial state: 50 l incl. 0,5 kg of salt

inflow: rate 6L/min, concentration 0,05 kg/L

outflow: rate 6L/min, concentration $\frac{x(t)}{50}$

unknown: $x(t)$ .. amount of salt in tank at time t in kg, time in min

$$x'(t) = 0{,}3 - \frac{x(t)}{50}\cdot 6 \ ; \quad x'(t) + \frac{6}{50}x(t) = 0{,}3 \ , \ /\cdot e^{\frac{6}{50}t}$$

$$\left(e^{\frac{6}{50}t}x(t)\right)' = 0{,}3e^{\frac{6}{50}t} = \left(0{,}3\cdot e^{\frac{6}{50}t}\cdot\frac{50}{6} + c\right)'$$

$$x(t) = 2{,}5 + Ce^{-\frac{6}{50}t} \ ; \ \text{i.c. } x(0) = 2{,}5 + C = 0{,}5 \Rightarrow C = -2$$

The searched sol.: $x(t) = 2{,}5 - 2e^{-6/50\, t}$

When will the concentration = 0,03 ?

$$0{,}03 = \frac{x(t_0)}{50} = \frac{2{,}5}{50} - \frac{2e^{-\frac{6}{50}t_0}}{50}$$

$$1{,}5 = 2{,}5 - 2e^{-\frac{6}{50}t_0} \quad | \quad 2e^{-\frac{6}{50}t_0} = 1 \ , \ e^{-\frac{6}{50}t} = \frac{1}{2}$$

$$-\frac{6}{50}t_0 = -\lg 2 \qquad t_0 = \frac{50}{6}\cdot \lg 2 \cong 5{,}77 \text{ min.}$$

The concentration in tank will be 0,03 kg/L in $\frac{50}{6}\lg 2$ minutes.

3) Initial mass of fish $\quad 7\,Mt \qquad$ mass of fish at t $\quad m(t)$

Increase rate $\quad 2m(t)$

Decrease rate $\quad 15\,Mt/\text{years}$

$$m'(t) = 2m(t) - 15 \ ; \ m' - 2m = -15 \qquad \cdot\, e^{-2t}$$

$$m(0) = 7 \qquad (m(t)e^{-2t})' = -15e^{-2t} = \left(\frac{15}{2}e^{-2t} + C\right)'$$

$$m(t) = \frac{15}{2} + Ce^{+2t} \ ; \ m(0) = 7 = \frac{15}{2} + C \ \rightarrow C = -\frac{1}{2}$$

$$m(t) = \frac{15}{2} - \frac{1}{2}e^{+2t} \ ; \ 0 = 15 - e^{2t} \ ; \ 15 = e^{2t} \ ; \ 2t = \lg 15$$

$t = \frac{\lg 15}{2}$ .. The fish will be gone in $\frac{\lg 15}{2}$ years.

The required rate satisfies $2\cdot 7 - r = 0$ i.e. rate $= 14\,Mt/\text{year}$.

4) temp 16°C

body temp 34.5°C at 12:00 $\qquad t = 0$

33.7°C 1PM $\qquad t = 1$

37°C at time of murder

$T(t)$ temperature of body at time

The eqn: $T'(t) = -c(T(t) - 16)$

Solution: $T'(t) + cT(t) = +c \cdot 16 \qquad$ (i.f. e^{-ct}

$$\left(e^{+ct}T(t)\right)' = +c \cdot 16\, e^{+ct} = \left(D + 16\, e^{+ct}\right)'$$

$$T(t) = 16 + D e^{-ct}$$

$$34.5 = 16 + D \Rightarrow D = 18{,}5$$

$$T(1) = 33.7 = 16 + 18{,}5\, e^{-c}; \quad 17{,}7 = 18{,}5\, e^{-c}, \quad e^{-c} = \frac{17{,}7}{18{,}5}$$

$$-c = \lg \frac{17{,}7}{18{,}5}, \quad c = \lg \frac{18{,}5}{17{,}7}$$

$$T(t) = 16 + 18{,}5\, e^{-\lg\left(\frac{18{,}5}{17{,}7}\right)\cdot t} = 16 + 18{,}5 \left(\frac{17{,}7}{18{,}5}\right)^t$$

$$37 = 16 + 18{,}5\, e^{-\lg\left(\frac{18{,}5}{17{,}7}\right)\cdot t}, \quad \frac{21}{18{,}5} = e^{-\lg\left(\frac{18{,}5}{17{,}7}\right) t}$$

$$-\lg\left(\frac{18{,}5}{17{,}7}\right)\cdot t = \lg \frac{21}{18{,}5}, \quad t = -\frac{\lg\left(\frac{21}{18{,}5}\right)}{\lg\left(\frac{18{,}5}{17{,}7}\right)} \doteq \cancel{-4{,}5415}\ -2{,}87$$

$\Rightarrow$ The murder happened at time $t = -2{,}87$, i.e. at 9:08 a.m.

Homework 4

One way to define hyperbolic functions is by means of differential equations. Consider the eqation $y'' - y = 0$. The hyperbolic cosine, cosh, is defined as the solution of this equation subject to the initial values: $y(0) = 1$ and $y'(0) = 0$. The hyperbolic sine, sinh, is defined as the solution of this equation subject to the initial values: $y(0) = 0$ and $y'(0) = 1$.

1. Solve these initial value problems to derive explicit formulas for cosh and sinh. Also show that $\cosh' = \sinh$ and $\sinh' = \cosh$.
2. Prove that a general solution of the equation $y'' - y = 0$ is given by $y(t) = c_1 \cosh(t) + c_2 \sinh(t)$.
3. Suppose that a, b and c are given constants for which $ar^2 + br + c = 0$ has two distinct real roots. If the two roots are expressed in the form $\alpha - \beta$ and $\alpha + \beta$, show that a general solution of the equation $ay'' + by' + cy = 0$ is $y(t) = c_1 e^{\alpha t} \cosh(\beta t) + c_2 e^{\alpha t} \sinh(\beta t)$.
4. Use the result of previous part to solve the inital value problem: $y'' + y' - 6y = 0$, $y(0) = 2$, $y'(0) = -17/2$.

Homework 4 – ma 431

1) (*) $y'' - y = 0$; char. eqn: $\lambda^2 = 1$, fundamental system $\{e^t, e^{-t}\}$

• $y(0) = 1, y'(0) = 0$: $1 = Ae^t + Be^{-t}\big|_{t=0} = A+B$

$$0 = Ae^t - Be^{-t}\big|_{t=0} = A - B$$

$\Rightarrow 1 = 2A$; $A = 1/2$, $B = 1/2$

$\Rightarrow \cosh(t) = \frac{1}{2}(e^t + e^{-t})$

• $y(0) = 0, y'(0) = 1$: $\left.\begin{matrix} 0 = A+B \\ 1 = A-B \end{matrix}\right\} \Rightarrow 1 = 2A \Rightarrow A = 1/2$; $B = -1/2$

$\Rightarrow \sinh(t) = \frac{1}{2}(e^t - e^{-t})$

Clearly $\sinh' = \cosh$; $\cosh' = \sinh$

2) General sol of (*) is given by $y(t) = Ae^t + Be^{-t} =$

$= C\cosh t + D\sinh t$, for C, D satisfying

$C + D = 2A$

$C - D = 2B$, i.e., $C = A+B$, $D = A-B$.

So any solution can be written as $y(t) = C\cosh t + D\sinh t$

3) $\lambda_1 = \alpha + \beta$; $\lambda_2 = \alpha - \beta$ the general sol is

$y(t) = Ae^{(\alpha+\beta)t} + Be^{(\alpha-\beta)t} = e^{\alpha}(Ae^{\beta t} + Be^{-\beta t}) =$

$= e^{\alpha}\left((A+B)\cosh(\beta t) + (A-B)\sinh(\beta t)\right)$, see the previous computation in 2)

4) $\lambda^2 + \lambda - 6 = 0$; $\lambda_{1,2} = \frac{-1 \pm \sqrt{1+24}}{2} = -\frac{1}{2} \pm \frac{5}{2}$

$y(t) = e^{-\frac{t}{2}}\left(A\cosh\frac{5t}{2} + B\sinh\left(\frac{5t}{2}\right)\right)$; $y(0) = 2 \Rightarrow A = 2$

$y'(t) = -\frac{1}{2}y(t) + e^{-t/2}\left(A\frac{5}{2}\sinh\frac{5t}{2} + \frac{5}{2}B\cosh\left(\frac{5t}{2}\right)\right)$, $y'(0) = -\frac{17}{2} \Rightarrow B = \frac{1 - \frac{17}{2}}{\frac{5}{2}} = -3$

The solution is $y(t) = e^{-\frac{t}{2}}\left(2\cosh\frac{5t}{2} - 3\sinh\frac{5t}{2}\right)$

Homework 5

1. See [NSA12, 4.6, 17]. Find a general solution to the differential equation.

$$\frac{y''}{2} + 2y = \mathrm{tg}(2t) - \frac{1}{2}e^t.$$

2. See [NSA12, 4.9, 8]. A 20-kg mass is attached to a spring with stiffness 200 N/m. The damping constant for the system is 140 N-sec/m. If the mass is pulled 25 cm to the right of equilibrium and given an initial leftward velocity of 1 m/sec, when will it first return to its equilibrium position?

3. See [NSA12, 4.10, 5]. An undamped system is governed by

$$m\frac{\partial^2 y}{\partial t^2} + ky = F_0 \cos(\gamma t), \quad y(0) = 0, y'(0) = 0,$$

   where $\gamma \neq \omega := \sqrt{k/m}$.

   (a) Find the equation of motion of the system.

   (b) Use trigonometric identities to show that the solution can be written in the form

$$y(t) = \frac{2F_0}{m(\omega^2 - \gamma^2)} \sin(\frac{\omega+\gamma}{2}t) \sin(\frac{\omega-\gamma}{2}t).$$

   (c) When γ is near ω, then $\omega - \gamma$ is small, while $\omega + \gamma$ is relatively large compared with $\omega - \gamma$. Hence, $y(t)$ can be viewed as the product of a slowly varying sine function, $\sin((\omega - \gamma)t/2)$, and a rapidly varying sine function, $\sin((\omega + \gamma)t/2)$. The net effect is a sine function $y(t)$ with frequency $(\omega - \gamma)/4\pi$, which serves as the time-varying amplitude of a sine function with frequency $(\omega + \gamma)/4\pi$. This vibration phenom enon is referred to as beats and is used in tuning stringed instruments. This same phenomenon in electronics is called amplitude modulation. To illustrate this phenomenon, sketch the curve $y(t)$ for $F_0 = 32$, $m = 2$, $\omega = 9$ and $\gamma = 7$.

Reference

[NSA12] R.K. Nagle, E.B. Saff, and Snider A.D. *Fundamentals of Differential Equations and Boundary Value Problems.* Addison-Wesley, sixth edition, 2012.