Samopodobné řešení rovnice vedení tepla

Hledáme řešení, pro která

$$u(t,x) = \gamma u(\beta t, \alpha x) \quad \text{pro vhodná } \alpha, \beta, \gamma \in \mathbb{R}.$$

Z rovnice $u_t = \Delta u$ vidíme: $\beta = \alpha^2$

Zachování celkové energie: $\displaystyle\int_{\mathbb{R}^d} \gamma u(\beta t, \alpha x)\, dx = \left| \begin{matrix} s = \beta t \\ y = \alpha x \end{matrix} \right|$

$$= \int_{\mathbb{R}^d} \gamma \cdot \left(\frac{1}{\alpha}\right)^d u(\beta t, y)\, dy \Rightarrow \gamma = |\alpha|^d$$

Požadujeme $\quad u(t,x) = |\alpha|^d u(\alpha^2 t, \alpha x)$ pro $\alpha > 0,\ t > 0,\ x \in \mathbb{R}^d$.

nastavením $\alpha = \frac{1}{\sqrt{t}}$: $u(t,x) = \left(\frac{1}{\sqrt{t}}\right)^d u\left(1, \frac{x}{\sqrt{t}}\right)$

$\longrightarrow$ takto lze nalézt fundamentální řešení.

$$u(t,x) = \frac{1}{\left[4\pi\left(t+\frac{1}{2}\right)\right]^{d/2}} \exp\left(-\frac{|x|^2}{4\left(t+\frac{1}{2}\right)}\right) =: H\left(t+\tfrac{1}{2}, x\right)$$

fundamentální řešení

$$H(t,x) = \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{4t}\right)$$

$$H(\alpha^2 t, \alpha x)|\alpha|^d = \frac{\alpha^d}{(4\pi t)^{d/2} \cdot \alpha^{2 d/2}} \exp\left(-\frac{|\alpha x|^2}{4\alpha^2 t}\right) = H(t,x)$$

- H je samopodobné

$$u(\alpha^2 t, \alpha x)\alpha^d = \frac{\alpha^d}{\left[4\pi\left(\alpha^2 t+\frac{1}{2}\right)\right]^{d/2}} \exp\left(-\frac{|\alpha x|^2}{4\left(\alpha^2 t+\frac{1}{2}\right)}\right)$$

Vol $t = \frac{1}{2}$, $\alpha = \frac{1}{2}$: $u\left(\frac{1}{4}\cdot\frac{1}{2}, \frac{1}{2}\cdot 0\right)\left(\frac{1}{2}\right)^d = \dfrac{\left(\frac{1}{2}\right)^d}{(4\pi)^{d/2}\left(\frac{1}{8}+\frac{1}{2}\right)^{d/2}}$

$x = 0$

$$\neq u\left(\tfrac{1}{2}, 0\right) = \frac{1}{(4\pi)^{d/2}}$$

- u není samopodobné

K důsledku 3.2: Volme $u(t,x) = \mathcal{U}(t, x-\xi)$ pro jisté $\xi \in \mathbb{R}^d$.

Pak $u_0(x) = \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{|x-\xi|^2}{2}\right)$

$$H_1 u_0 = \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{|x-\xi|^2}{2}\right) \cdot \underbrace{\left(\frac{|x|^2 - |x-\xi|^2}{2}\right)}_{= (x,\xi) - \frac{|\xi|^2}{2}} \, dx$$

- Je U_j samopodobné?
- Odhad s podřešením?
- Lepší odhad, zachycující posun v ξ.

$$\| H(t,\cdot) - H(t+\tfrac{1}{2},\cdot)\|_1 = \int_{\mathbb{R}^d} \frac{1}{(4\pi)^{d/2}}\left[\frac{1}{t^{d/2}} \exp\left(-\frac{|x|^2}{4t}\right) - \frac{1}{(t+\frac{1}{2})^{d/2}} \exp\left(-\frac{|x|^2}{4t+2}\right)\right] dx$$

$$\leq \frac{1}{(4\pi)^{d/2}}\Bigg[\underbrace{\int_{\mathbb{R}^d}\left|\frac{1}{t^{d/2}}\left(\exp\left(-\frac{|x|^2}{4t}\right)-\exp\left(-\frac{|x|^2}{4t+2}\right)\right)\right| dx}_{I_1} + \underbrace{\int_{\mathbb{R}^d}\left|\left(\frac{1}{t^{d/2}}-\frac{1}{(t+1/2)^{d/2}}\right)\exp\left(-\frac{|x|^2}{4t+2}\right)\right| dx}_{I_2}$$

$$I_2 = (4\pi)^{d/2}\left[\left(\frac{t+\frac{1}{2}}{t}\right)^{d/2} - 1\right] \simeq (4\pi)^{d/2}\frac{d}{4t} \quad \text{pro } t \to +\infty$$

$$I_1 = \int_{\mathbb{R}^d}\left|\frac{1}{t^{d/2}}\exp\left(-\frac{|x|^2}{4t}\right)\right|\left(1-\exp\left(-|x|^2\underbrace{\left[\frac{1}{4t}-\frac{1}{4t+2}\right]}_{\frac{2}{4t(4t+2)}}\right)\right)\Bigg| dx$$

$$= \left| \begin{array}{l} y = \frac{x}{2\sqrt{t}} \\ dx = 2^d t^{d/2} dy \end{array}\right| = 2^d \int_{\mathbb{R}^d}\left| e^{-|y|^2}\left(1-\exp\left(-y^2\cdot\frac{2}{4t+2}\right)\right)\right| dy$$

$$= 2^d \int_{\mathbb{R}^d} e^{-|y|^2} \underbrace{\left|\frac{1-\exp\left(-y^2\frac{2}{4t+2}\right)}{-y^2\frac{2}{4t+2}}\right|}_{\xrightarrow{t\to+\infty} 1} \frac{2y^2}{4t+2}\, dy \quad \text{a dál nevím!}$$

Ale: $\left|1-\exp\left(-\frac{|x|^2}{2t(4t+2)}\right)\right| \leq \frac{|x|^2}{2t(4t+2)}$ podle Lagrange

Tedy $|I_1| \leq \int_{\mathbb{R}^d} \frac{1}{t^{d/2}}\exp\left(-\frac{|x|^2}{4t}\right)\cdot\frac{|x|^2}{2t(4t+2)}\, dx = \left|\begin{array}{l} y = \frac{x}{2\sqrt{t}} \\ dy = \frac{dx}{(4t)^{d/2}}\end{array}\right| =$

$$= \int_{\mathbb{R}^d} 2^d e^{-\frac{|y|^2}{2}} |y|^2 dy \frac{1}{2t+1} = K\frac{1}{2t+1}$$

Dohromady $\| H(t,\cdot) - H(t+\frac{1}{2},\cdot)\|_1 \leq K\frac{1}{t}$ pro j. K a ~~...~~ všech $t > 0$.

Důsledek 3.2:

a) ~~Ne~~ Pro každé u_0, ex. $K>0$ takové, že pro $t>1$ platí

$$\|u(t)-H(t)\|_1 \le \frac{K}{\sqrt{2t+1}}.$$

nebo

b) Pro každé u_0, a $\varepsilon>0$) ex. $t_0>0$ takové, že pro $t>t_0$ platí

$$\|u(t)-H(t)\|_1 \le \frac{\sqrt{8H_1(u_0)}+\varepsilon}{\sqrt{2t+1}}$$