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APPUNTI

LECTURE NOTES

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An Introduction to the Regularity Theory for Elliptic Systems, Harmonic Maps and Minimal Graphs

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An Introduction to the Regularity Theory for Elliptic Systems, Harmonic Maps and Minimal Graphs

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Preface to the first edition

Initially thought as lecture notes of a course given by the first author at the Scuola Normale Superiore in the academic year 2003-2004, this volume grew into the present form thanks to the constant enthusiasm of the second author.

Our aim here is to illustrate some of the relevant ideas in the theory of regularity of linear and nonlinear elliptic systems, looking in particular at the context and the specific situation in which they generate. Therefore this is not a reference volume: we always refrain from generalizations and extensions. For reasons of space we did not treat regularity questions in the linear and nonlinear Hodge theory, in Stokes and Navier-Stokes theory of fluids, in linear and nonlinear elasticity; other topics that should be treated, we are sure, were not treated because of our limited knowledge. Finally, we avoided to discuss more recent and technical contributions, in particular, we never entered regularity questions related to variational integrals or systems with general growth p.

In preparing this volume we particularly took advantage from the references [6] [37] [39] [52], from a series of unpublished notes by Giuseppe Modica, whom we want to thank particularly, from [98] and from the papers [109] [110] [111].

We would like to thank also Valentino Tosatti and Davide Vittone, who attended the course, made comments and remarks and read part of the manuscript.

Part of the work was carried out while the second author was a graduate student at Stanford, supported by a Stanford Graduate Fellowship.

Preface to the second edition

This second edition is a deeply revised version of the first edition, in which several typos were corrected, details to the proofs, exercises and examples were added, and new material was covered. In particular we added the recent results of T. Rivière [88] on the regularity of critical points of conformally invariant functionals in dimension 2 (especially 2-dimensional harmonic maps), and the partial regularity of stationary harmonic maps following the new approach of T. Rivière and M. Struwe [90], which avoids the use of the moving-frame technique of F. Hélein. This gave us the motivation to briefly discuss the limiting case $p = 1$ of the L^p-estimates for the Laplacian, introducing the Hardy space $\mathcal{H}^1$ and presenting the celebrated results of Wente [112] and of Coifman-Lions-Meyer-Semmes [22].

Part of the work was completed while the second author was visiting the Centro di Ricerca Matematica Ennio De Giorgi in Pisa, whose warm hospitality is gratefully acknowledged.

Chapter 1
Harmonic functions

We begin by illustrating some aspects of the classical model problem in the theory of elliptic regularity: the Dirichlet problem for the Laplace operator.

1.1 Introduction

From now on Ω will be a bounded, connected and open subset of $\mathbb{R}^n$.

Definition 1.1 *Given a function $u \in C^2(\Omega)$ we say that u is*

- *harmonic if* $\Delta u = 0$
- *subharmonic if* $\Delta u \geq 0$
- *superharmonic if* $\Delta u \leq 0$,

where

$$\Delta u(x) := \sum_{\alpha=1}^{n} D_\alpha^2 u(x), \qquad D_\alpha := \frac{\partial}{\partial x^\alpha}$$

is the Laplacian operator.

Exercise 1.2 Prove that if $f \in C^2(\mathbb{R})$ is convex and $u \in C^2(\Omega)$ is harmonic, then $f \circ u$ is subharmonic.

Throughout this chapter we shall study some important properties of harmonic functions and we shall be concerned with the problem of the existence of harmonic functions with prescribed boundary value, namely with the solution of the following *Dirichlet problem*:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

in $C^2(\Omega) \cap C^0(\overline{\Omega})$, for a given function $g \in C^0(\partial\Omega)$.

1.2 The variational method

The problem of finding a harmonic function with prescribed boundary value $g \in C^0(\partial\Omega)$ is tied, though not equivalent (see section 1.2.2), to the following one: find a minimizer u for the functional $\mathcal{D}$

$$\mathcal{D}(u) = \frac{1}{2}\int_\Omega |Du|^2 dx \tag{1.2}$$

in the class

$$\mathcal{A} = \{u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \ : \ u = g \text{ on } \partial\Omega\}.$$

The functional $\mathcal{D}$ is called *Dirichlet integral.*

In fact, formally, if a minimizer u exists, then *the first variation of the Dirichlet integral* vanishes:

$$\frac{d}{dt}\mathcal{D}(u + t\varphi)\Big|_{t=0} = 0$$

for all smooth compactly supported functions φ in Ω; an integration by parts then yields

$$\begin{aligned} 0 &= \frac{d}{dt}\mathcal{D}(u + t\varphi)\Big|_{t=0} \\ &= \int_\Omega \nabla u \cdot \nabla\varphi dx \\ &= -\int_\Omega \Delta u\varphi dx, \quad \forall\varphi \in C_0^\infty(\Omega), \end{aligned}$$

and by the arbitrariness of φ we conclude $\Delta u = 0$, which is the *Euler-Lagrange equation for the Dirichlet integral: minimizers of the Dirichlet integral are harmonic.*

This was stated as an equivalence by Dirichlet and used by Riemann in his geometric theory of functions.

Dirichlet's principle: A minimizer u of the Dirichlet integral in Ω with prescribed boundary value g always exists, is unique and is a harmonic function; it solves

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Conversely, any solution of (1.3) is a minimizer of the Dirichlet integral in the class of functions with boundary value g.

Dirichlet saw no need to prove this principle; however, as we shall see, in general Dirichlet's principle does not hold and, in the circumstances in which it holds, it is not trivial.

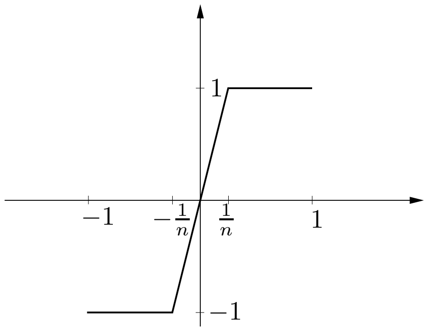

Figure 1.1: The function u_n as defined in (1.4)

1.2.1 Non-existence of minimizers of variational integrals

The following examples, the first being a classical example of Weierstrass, show that minimizers to a variational integral need not exist.

1. Consider the functional

$$\mathcal{F}(u) = \int_{-1}^{1} (x\dot{u})^2 dx$$

defined on the class of Lipschitz functions

$$\mathcal{A} = \{u \in \mathrm{Lip}([-1,1]) \ : \ u(-1) = -1, \ u(1) = 1\}.$$

The following sequence of functions in $\mathcal{A}$

$$u_n(x) := \begin{cases} -1 & \text{for } x \in [-1, -\frac{1}{n}] \\ 1 & \text{for } x \in [\frac{1}{n}, 1] \\ nx & \text{for } x \in [-\frac{1}{n}, \frac{1}{n}] \end{cases} \tag{1.4}$$

shows that $\inf_{\mathcal{A}} \mathcal{F} = 0$, but evidently $\mathcal{F}$ cannot attain the value 0 on $\mathcal{A}$.

2. Consider

$$\mathcal{F}(u) = \int_0^1 (1 + \dot{u}^2)^{\frac{1}{4}} dx,$$

defined on

$$\mathcal{A} = \{u \in \mathrm{Lip}([0,1]) \ : \ u(0) = 1, \ u(1) = 0\}.$$

The sequence of functions

$$u(x) = \begin{cases} 1 - nx & \text{for } x \in [0, \frac{1}{n}] \\ 0 & \text{for } x \in [\frac{1}{n}, 1] \end{cases}$$

shows that $\inf_{\mathcal{A}} \mathcal{F} = 1$. On the other hand, if $\mathcal{F}(u) = 1$, then u is constant, thus cannot belong to $\mathcal{A}$.

3. Consider the *area functional* defined on the unit ball $B_1 \subset \mathbb{R}^2$

$$\mathcal{F}(u) = \int_{B_1} \sqrt{1 + |Du|^2} dx,$$

defined on

$$\mathcal{A} = \{u \in \mathrm{Lip}(B_1) \ : \ u = 0 \text{ on } \partial B_1, \ u(0) = 1\}.$$

As $\mathcal{F}(u) \geq \pi$ for every $u \in \mathcal{A}$, the sequence of functions

$$u(x) = \begin{cases} 1 - n|x| & \text{for } |x| \in [0, \frac{1}{n}] \\ 0 & \text{for } |x| \in [\frac{1}{n}, 1] \end{cases}$$

shows that $\inf_{\mathcal{A}} \mathcal{F} = \pi$. On the other hand if $\mathcal{F}(u) = \pi$ for some $u \in \mathcal{A}$, then u is constant, thus cannot belong to $\mathcal{A}$.

1.2.2 Non-finiteness of the Dirichlet integral

We have seen that a minimizer of the Dirichlet integral is a harmonic function. In some sense the converse is not true: we exhibit a harmonic function with *infinite* Dirichlet integral.

The Laplacian in polar coordinates on $\mathbb{R}^2$ is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

and it is easily seen that $r^n \cos n\theta$ and $r^n \sin n\theta$ are harmonic functions. Now define on the unit ball $B_1 \subset \mathbb{R}^2$

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta).$$

Provided

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|) < \infty,$$

the series converges uniformly, while its derivatives converge uniformly on compact subsets of the ball, so that u belongs to $C^\infty(B_1) \cap C^0(\overline{B}_1)$ and is harmonic.

The Dirichlet integral of u is

$$\mathcal{D}(u) = \frac{1}{2}\int_0^{2\pi} d\theta \int_0^1 (|\partial_r u|^2 + \frac{1}{r^2}|\partial_\theta u|^2) r dr = \frac{\pi}{2}\sum_{n=1}^{\infty} n(a_n^2 + b_n^2).$$

Thus, if we choose $a_n = 0$ for all $n \geq 0$, $b_n = 0$ for all $n \geq 1$, with the exception of $b_{n!} = n^{-2}$, we obtain

$$u(r,\theta) = \sum_{n=1}^{\infty} r^{n!} n^{-2} \sin(n!\theta),$$

and we conclude that $u \in C^\infty(B_1) \cap C^0(\overline{B}_1)$, it is harmonic, yet

$$\mathcal{D}(u) = \frac{\pi}{2} \sum_{n=1}^{\infty} n^{-4} n! = \infty.$$

In fact, every function $v \in C^\infty(B_1) \cap C^0(\overline{B}_1)$ that agrees with the function u defined above on ∂B_1 has *infinite* Dirichlet integral.

1.3 Some properties of harmonic functions

Proposition 1.3 (Weak maximum principle) *If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is subharmonic, then*

$$\sup_\Omega u = \max_{\partial\Omega} u;$$

If u is superharmonic, then

$$\inf_\Omega u = \min_{\partial\Omega} u.$$

Proof. We prove the proposition for u subharmonic, since for a superharmonic u it is enough to consider $-u$. Suppose first that $\Delta u > 0$ in Ω. Were $x_0 \in \Omega$ such that $u(x_0) = \max_{\overline{\Omega}} u$, we would have $u_{x^i x^i}(x_0) \leq 0$ for every $1 \leq i \leq n$. Summing over i we would obtain $\Delta u(x_0) \leq 0$, contradiction.

For the general case $\Delta u \geq 0$ consider the function $v(x) = u(x) + \varepsilon |x|^2$. Then $\Delta v > 0$ and, by what we have just proved, $\sup_\Omega v = \max_{\partial\Omega} v$. On the other hand, as $\varepsilon \to 0$, we have $\sup_\Omega v \to \sup_\Omega u$ and $\max_{\partial\Omega} v \to \max_{\partial\Omega} u$. $\square$

Exercise 1.4 Similarly, prove the following generalization of Proposition 1.3: let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$\sum_{\alpha,\beta=1}^{n} A^{\alpha\beta} D_{\alpha\beta} u + \sum_{\alpha=1}^{n} b^\alpha D_\alpha u \geq 0,$$

where $A^{\alpha\beta}, b^\alpha \in C^0(\overline{\Omega})$ and $A^{\alpha\beta}$ is *elliptic*: $\sum_{\alpha,\beta=1}^n A^{\alpha\beta}\xi_\alpha \xi_\beta \geq \lambda |\xi|^2$, for some $\lambda > 0$ and every $\xi \in \mathbb{R}^n$. Then

$$\sup_\Omega u = \max_{\partial\Omega} u.$$

Remark 1.5 The continuity of the coefficients in Exercise 1.4 is necessary. Indeed Nadirashvili gave a counterexample to the maximum principle with $A^{\alpha\beta}$ elliptic and bounded, but discontinuous, see [82].

Proposition 1.6 (Comparison principle) *Let $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be such that u is subharmonic, v is superharmonic and $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in Ω.*

Proof. Since $u - v$ is subharmonic with $u - v \leq 0$ on $\partial\Omega$, from the weak maximum principle, Proposition 1.3, we get $u - v \leq 0$ in Ω. $\square$

Clearly

$$u \leq v + \max_{\partial\Omega} |u - v| \qquad \text{on } \partial\Omega,$$

consequently:

Corollary 1.7 (Maximum estimate) *Let u and v be two harmonic functions in Ω. Then*

$$\sup_{\Omega} |u - v| \leq \max_{\partial\Omega} |u - v|.$$

Corollary 1.8 (Uniqueness) *Two harmonic functions on Ω that agree on $\partial\Omega$ are equal.*

Proposition 1.9 (Mean value inequalities) *Suppose that $u \in C^2(\Omega)$ is subharmonic. Then for every ball $B_r(x) \Subset \Omega$*

$$u(x) \leq \fint_{\partial B_r(x)} u(y) d\mathcal{H}^{n-1}(y),^{1} \tag{1.5}$$

$$u(x) \leq \fint_{B_r(x)} u(y) dy. \tag{1.6}$$

If u is superharmonic, the reverse inequalities hold; consequently for u harmonic equalities are true.

[1]By $\fint_A f(x)dx$ we denote the average of f on A i.e., $\frac{1}{|A|}\int_A f(x)dx$. Similarly $\fint_A f d\mathcal{H}^{n-1} = \frac{1}{\mathcal{H}^{n-1}(A)} \int_A f d\mathcal{H}^{n-1}$.

Proof. Let u be subharmonic. From the divergence theorem, for each $\rho \in (0, r]$ we have

$$
\begin{aligned}
0 &\le \int_{B_\rho(x)} \Delta u(y) dy \\
&= \int_{\partial B_\rho(x)} \frac{\partial u}{\partial \nu}(y) d\mathcal{H}^{n-1}(y) \\
&= \int_{\partial B_1(0)} \frac{\partial u}{\partial \rho}(x + \rho y)\rho^{n-1} d\mathcal{H}^{n-1}(y) \\
&= \rho^{n-1} \frac{d}{d\rho} \int_{\partial B_1(0)} u(x + \rho y) d\mathcal{H}^{n-1}(y) \\
&= \rho^{n-1} \frac{d}{d\rho} \left(\frac{1}{\rho^{n-1}} \int_{\partial B_\rho(x)} u(y) d\mathcal{H}^{n-1}(y) \right) \\
&= n\omega_n \rho^{n-1} \frac{d}{d\rho} ⨍_{\partial B_\rho(x)} u(y) d\mathcal{H}^{n-1}(y),
\end{aligned}
\tag{1.7}
$$

where $\omega_n := |B_1|$. This implies that the last integral is non-decreasing and, since

$$
\lim_{\rho \to 0} ⨍_{\partial B_\rho(x)} u(y) d\mathcal{H}^{n-1}(y) = u(x),
$$

(1.5) follows. We leave the rest of the proof for the reader. □

Corollary 1.10 (Strong maximum principle) *If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is subharmonic (resp. superharmonic), then it cannot attain its maximum (resp. minimum) in Ω unless it is constant.*

Proof. Assume u is subharmonic and let $x_0 \in \Omega$ be such that $u(x_0) = \sup_\Omega u$. Then the set

$$
S := \{x \in \Omega : u(x) = u(x_0)\}
$$

is closed because u is continuous and is open thanks to (1.6). Since Ω is connected we have $S = \Omega$. □

Remark 1.11 If u is harmonic, the mean value inequality is also a direct consequence of the representation formula (1.11) below.

Exercise 1.12 Prove that if $u \in C^2(\Omega)$ satisfies one of the mean value properties, then it is correspondigly harmonic, subharmonic or superharmonic.

Exercise 1.13 Prove that if $u \in C^0(\Omega)$ satisfies the mean value equality

$$
u(x) = ⨍_{B_r(x)} u(y) dy, \quad \forall B_r(x) \subset \Omega
$$

then $u \in C^\infty(\Omega)$ and it is harmonic.

[Hint: Regularize u with a family $\varphi_\varepsilon = \rho_\varepsilon(|x|)$ of mollifiers with radial simmetry and use the mean value property to prove that $u * \rho_\varepsilon = u$ in any $\Omega_0 \Subset \Omega$ for ε small enough.]

Proposition 1.14 *Consider a sequence of harmonic functions u_j that converge locally uniformly in Ω to a function $u \in C^0(\Omega)$. Then u is harmonic.*

Proof. The mean value property is stable under uniform convergence, thus holds true for u, which is therefore harmonic thanks to Exercise 1.13. □

Remark 1.15 Being harmonic is preserved under the weaker hypothesis of weak L^p convergence, $1 \le p < \infty$, or even of the convergence is the sense of distributions. This follows at once from the so-called Weyl's lemma.

Lemma 1.16 (Weyl) *A function $u \in L^1_{\mathrm{loc}}(\Omega)$ is harmonic if and only if*

$$\int_\Omega u \Delta\varphi dx = 0, \quad \forall \varphi \in C^\infty_c(\Omega).$$

Proof. Consider a family of radial mollifiers ρ_ε, i.e. $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho(\varepsilon^{-1}x)$, where $\rho \in C^\infty(\mathbb{R}^n)$ is radially symmetric, $\mathrm{supp}(\rho) \subset B_1$ and $\int_{B_1} \rho(x)dx = 1$. Define $u_\varepsilon = u * \rho_\varepsilon$. Then, from the standard properties of convolution we find

$$\begin{aligned}\int_\Omega u_\varepsilon \Delta\varphi dx &= \int_\Omega u(\Delta\varphi * \rho_\varepsilon)dx \\ &= \int_\Omega u\Delta(\varphi * \rho_\varepsilon)dx \\ &= 0, \qquad \text{for every } \varphi \in C^\infty_c(\Omega_\varepsilon),\end{aligned}$$

where

$$\Omega_\varepsilon := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \varepsilon\}.$$

In particular $\Delta u_\varepsilon = 0$ on Ω_ε. Now fix $R > 0$ and let $0 < \varepsilon \le \frac{1}{2}R$. We have by Fubini's theorem

$$\begin{aligned}\int_{\Omega_\varepsilon} |u_\varepsilon(y)|dy &\le \int_{\Omega_\varepsilon} \frac{1}{\varepsilon^n}\int_\Omega \rho\left(\frac{|x-y|}{\varepsilon}\right)|u(x)|dxdy \\ &\le \int_\Omega |u(x)|dx.\end{aligned} \tag{1.8}$$

Here we may assume that $u \in L^1(\Omega)$, since being harmonic is a local property. By the mean value property applied with balls of radius $\frac{R}{2}$ and (1.8), we obtain that the u_ε are uniformly bounded in $\Omega_{R/2}$. They are also locally equicontinuous in Ω_R because for $x_0 \in \Omega_R$ and $x_1, x_2 \in B_{\frac{R}{2}}(x_0)$, still by the mean-value property,

$$\begin{aligned} |u_\varepsilon(x_1) - u_\varepsilon(x_2)| &\le \frac{2^n}{\omega_n R^n} \int_{B_{\frac{R}{2}}(x_1) \Delta B_{\frac{R}{2}}(x_2)} |u_\varepsilon(x)| dx \\ &\le \frac{2^n}{\omega_n R^n} \sup_{B_R(x_0)} |u_\varepsilon| \cdot \mathrm{meas}\left(B_{\frac{R}{2}}(x_2) \Delta B_{\frac{R}{2}}(x_1)\right), \end{aligned}$$

where

$$B_{\frac{R}{2}}(x_1) \Delta B_{\frac{R}{2}}(x_2) := \left(B_{\frac{R}{2}}(x_1) \backslash B_{\frac{R}{2}}(x_2)\right) \cup \left(B_{\frac{R}{2}}(x_2) \backslash B_{\frac{R}{2}}(x_1)\right).$$

By Ascoli-Arzelà's theorem (Theorem 2.3 below), we can extract a sequence u_{ε_k} which converges uniformly in Ω_R to a continuous function v as $k \to \infty$ and $\varepsilon_k \to 0$, which is harmonic thanks to Exercise 1.13. But $u = v$ almost everywhere in Ω_R by the properties of convolutions, hence u is harmonic in Ω_R. Letting $R \to 0$ we conclude. □

Proposition 1.17 *Given $u \in C^0(\Omega)$, the following facts are equivalent:*

(i) *For every ball $B_R(x) \Subset \Omega$ we have*

$$u(x) \le \fint_{\partial B_R(x)} u(y) d\mathcal{H}^{n-1}(y);$$

(ii) *for every ball $B_R(x) \Subset \Omega$ we have*

$$u(x) \le \fint_{B_R(x)} u(y) dy;$$

(iii) *for every $x \in \Omega$, $R_0 > 0$, there exist $R \in (0, R_0)$ such that $B_R(x) \Subset \Omega$ and*

$$u(x) \le \fint_{B_R(x)} u(y) dy; \tag{1.9}$$

(iv) *for each $h \in C^0(\Omega)$ harmonic in $\Omega' \Subset \Omega$ with $u \le h$ on $\partial\Omega'$, we have $u \le h$ in Ω';*

(v) $\int_\Omega u(x) \Delta\varphi(x) dx \ge 0$, $\forall \varphi \in C_c^\infty(\Omega)$, $\varphi \ge 0$.

Proof. Clearly (i) implies (ii) and (ii) implies (iii).
(iii)⇒(iv): Since h satisfies the mean value property the function $w := u - h$ satisfies

$$w(x) \le \fint_{B_R(x)} w(y) dy \quad \text{for all balls } B_R(x) \subset \Omega' \text{ s.t. (1.9) holds.}$$

Then

$$\sup_{\Omega'} w = \max_{\partial\Omega'} w \leq 0,$$

the first identity following exactly as in the proof of Corollary 1.10.
(iv)⇒(i): Let $B_R(x) \Subset \Omega$, and choose h harmonic in $B_R(x)$ and $h = u$ in $\Omega \backslash B_R(x)$. This can be done by Proposition 1.24 below. Then

$$u(x) \leq h(x) = \fint_{\partial B_R(x)} h d\mathcal{H}^{n-1} = \fint_{\partial B_R(x)} u d\mathcal{H}^{n-1}.$$

The equivalence of (v) to (ii) can be proved by mollifying u, compare Exercise 1.13. □

Often a continuous function satisfying one of the conditions in Proposition 1.17 is called *subharmonic*.

Exercise 1.18 Use Proposition 1.17 to prove the following:

1. A finite linear combination of harmonic functions is harmonic.
2. A positive finite linear combination of subharmonic (resp. superharmonic) functions is a subharmonic (resp. superharmonic) function.
3. The supremum (resp. infimum) of a finite number of subharmonic (resp. superharmonic) functions is a subharmonic (resp. superharmonic) function.

Theorem 1.19 (Harnack inequality) *Given a non-negative harmonic function $u \in C^2(\Omega)$, for every ball $B_{3r}(x_0) \Subset \Omega$ we have*

$$\sup_{B_r(x_0)} u \leq 3^n \inf_{B_r(x_0)} u.$$

Proof. By the mean value property, Proposition 1.9, and from $u \geq 0$ we get that for $y_1, y_2 \in B_r(x_0)$

$$\begin{aligned} u(y_1) &= \frac{1}{\omega_n r^n} \int_{B_r(y_1)} u dx \\ &\leq \frac{1}{\omega_n r^n} \int_{B_{2r}(x_0)} u dx \\ &= \frac{3^n}{\omega_n (3r)^n} \int_{B_{2r}(x_0)} u dx \\ &\leq \frac{3^n}{\omega_n (3r)^n} \int_{B_{3r}(y_2)} u dx \\ &= 3^n u(y_2). \end{aligned}$$

□

Theorem 1.20 (Liouville) *A bounded harmonic function $u : \mathbb{R}^n \to \mathbb{R}$ is constant.*

Proof. Define $m = \inf_{\mathbb{R}^n} u$. Then $u - m \geq 0$ and by Harnack's inequality, Theorem 1.19,

$$\sup_{B_R}(u-m) \leq 3^n \inf_{B_R}(u-m), \quad \forall R > 0.$$

Letting $R \to \infty$, the term on the right tends to 0 and we conclude that $\sup_{\mathbb{R}^n} u = m$. □

Proposition 1.21 *Let u be harmonic (hence smooth by Exercise 1.13) and bounded in $B_R(x_0)$. For $r < R$ we may find constants $c(k,n)$ such that*

$$\sup_{B_r(x_0)} |\nabla^k u| \leq \frac{c(k,n)}{(R-r)^k} \sup_{B_R(x_0)} |u|. \tag{1.10}$$

Exercise 1.22 Prove Proposition 1.21.
[Hint: First prove (1.10) for $k = 1$ using the mean-value identity (it might be easier to start with the case $r = R/2$ and then use a covering or a scaling argument). Then notice that each derivative of u is harmonic and use an inductive procedure.]

Proposition 1.23 *Let (u_k) be an equibounded sequence of harmonic functions in Ω, i.e. assume that $\sup_\Omega |u_k| \leq c$ for a constant c independent of k. Then up to extracting a subsequence $u_k \to u$ in $C^\ell_{\mathrm{loc}}(\Omega)$ for every ℓ, where u is a harmonic function on Ω.*

Proof. This follows easily from Proposition 1.21 and the Ascoli-Arzelà theorem (Theorem 2.3 below), with a simple covering argument. □

1.4 Existence in general bounded domains

Before dealing with the existence of harmonic functions is general domains we state a classical representation formula providing us with the solution of the Dirichlet problem (1.1) on a ball.

1.4.1 Solvability of the Dirichlet problem on balls: Poisson's formula

Proposition 1.24 (H.A. Schwarz or S.D. Poisson) *Let $a \in \mathbb{R}^n$, $r > 0$ and $g \in C^0(\partial B_r(a))$ be given and define the function u by*

$$u(x) := \begin{cases} \dfrac{r^2 - |x-a|^2}{n\omega_n r} \displaystyle\int_{\partial B_r(a)} \frac{g(y)}{|x-y|^n} d\mathcal{H}^{n-1}(y) & x \in B_r(a) \\ g(x) & x \in \partial B_r(a). \end{cases} \tag{1.11}$$

Then $u \in C^\infty(B_r(a)) \cap C^0(\overline{B_r(a)})$ and solves the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } B_r(a) \\ u = g & \text{on } \partial B_r(a) \end{cases}$$

Proof. We only sketch it. By direct computation we see that u is harmonic. For the continuity on the boundary assume, without loss of generality, that $a = 0$ and define

$$K(x,y) := \frac{r^2 - |x|^2}{n\omega_n r |x-y|^n}, \quad x \in B_r(0),\ y \in \partial B_r(0).$$

One can prove that

$$\int_{\partial B_r(0)} K(x,y) d\mathcal{H}^{n-1}(y) = 1, \quad \text{for every } x \in B_r(0).$$

Let $x_0 \in \partial B_r(0)$ and for any $\varepsilon > 0$ choose δ such that $|g(x) - g(x_0)| < \varepsilon$ if $x \in \partial B_r(0) \cap B_\delta(x_0)$. Then, for $x \in B_r(0) \cap B_{\delta/2}(x_0)$,

$$\begin{aligned} |u(x) - g(x_0)| &\leq \left| \int_{\partial B_r(0)} K(x,y)[g(y) - g(x_0)] d\mathcal{H}^{n-1}(y) \right| \\ &\leq \int_{\partial B_r(0) \cap B_\delta(x_0)} K(x,y)|g(y) - g(x_0)| d\mathcal{H}^{n-1}(y) \\ &\quad + \int_{\partial B_r(0) \setminus B_\delta(x_0)} K(x,y)|g(y) - g(x_0)| d\mathcal{H}^{n-1}(y) \\ &\leq \varepsilon + \frac{(r^2 - |x|^2) r^{n-2}}{\left(\frac{\delta}{2}\right)^n} 2 \sup_{\partial B_r(0)} |g|. \end{aligned}$$

Hence $|u(x) - g(x_0)| \to 0$ as $x \to x_0$. □

1.4.2 Perron's method

We now present a method for solving the Dirichlet problem (1.1).

Given an open bounded domain $\Omega \subset \mathbb{R}^n$ and $g \in C^0(\partial\Omega)$ define

$$S_- := \{u \in C^2(\Omega) \cap C^0(\overline{\Omega}) : \Delta u \geq 0 \text{ in } \Omega,\ u \leq g \text{ on } \partial\Omega\};$$
$$S_+ := \{u \in C^2(\Omega) \cap C^0(\overline{\Omega}) : \Delta u \leq 0 \text{ in } \Omega,\ u \geq g \text{ on } \partial\Omega\}.$$

These sets are non-empty, since g is bounded and constant functions are harmonic: $u \equiv \sup_\Omega g$ and $v \equiv \inf_\Omega g$ belong to S_+ and S_- respectively. We also observe that, by the comparison principle, $v \leq u$ for each $v \in S_-$ and $u \in S_+$. We define

$$u_*(x) = \sup_{u \in S_-} u(x), \quad u^*(x) = \inf_{u \in S_+} u(x).$$

and shall

1. prove that both u_* and u^* are harmonic;
2. find conditions on Ω in order to have $u_*, u^* \in C^0(\overline{\Omega})$ and $u_* = u^* = g$ on $\partial\Omega$.

This is referred to as *Perron's method.*

Step 1. It is enough to prove that u_* is harmonic in a generic ball $B \subset \Omega$. Fix $x_0 \in B$. By the definition of u_* we may find a sequence $v_j \in S_-$ such that $v_j(x_0) \to u_*(x_0)$. Define

$$v_j' := \max(v_1, \ldots, v_j) \in S_-,$$
$$v_j'' := P_B v_j',$$

where $P_B v_j'$ is obtained by (1.11) as the harmonic extention of v_j' on B matching v_j' on ∂B. Observe that by definition (v_j') is an increasing sequence and, by the maximum principle, (v_j'') is increasing as well. Since the sequence (v_j'') is equibounded and increasing it converges locally uniformly in B to a harmonic function h thanks to Proposition 1.23.

Observe that $h \leq u_*$ and $h(x_0) = u_*(x_0)$. We claim that $h = u_*$ in B. If $h(z) < u_*(z)$ for some $z \in B$, choose $w \in S_-$ such that $w(z) > h(z)$ and define $w_j = \max\{v_j'', w\}$. Also define w_j' and w_j'' as done before with v_j' and v_j''. Again we have that $w_j'' \to \tilde{h}$ for some harmonic function $\tilde{h}$. From the definition it is easy to prove that $v_j'' \leq w_j''$, thus $h \leq \tilde{h}$ and $h(x_0) = \tilde{h}(x_0)$. By the strong maximum principle, this implies $h = \tilde{h}$ on all of B. This is a contradiction because

$$\tilde{h}(z) = \lim w_j''(z) \geq w(z) > h(z) = \tilde{h}(z).$$

This proves that $h = u_*$ and then u_* is harmonic in B, hence in all of Ω since B was arbitrary. Clearly the same proof applies to u^*.

Step 2. The functions u^* and u_* need not achieve the boundary data g, and in general they don't.

Definition 1.25 *A point $x_0 \in \partial\Omega$ is called* regular *if for every $g \in C^0(\partial\Omega)$ and every $\varepsilon > 0$ there exist $v \in S_-$ and $w \in S^+$ such that $g(x_0) - v(x_0) \leq \varepsilon$ and $w(x_0) - g(x_0) \leq \varepsilon$.*

Exercise 1.26 The Dirichlet problem (1.1) has solution for every $g \in C^0(\partial\Omega)$ if and only if each point of $\partial\Omega$ is regular.
[Hint: Use Perron's method and prove that $u_* \in C^0(\overline{\Omega})$ and $u_* = g$ on $\partial\Omega$.]

Definition 1.27 *Given $x_0 \in \partial\Omega$, an* upper barrier *at x_0 is a superharmonic function $b \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that $b(x_0) = 0$ and $b > 0$ on $\overline{\Omega}\backslash\{x_0\}$. We say that b is a* lower barrier *if $-b$ is an upper barrier.*

Proposition 1.28 *Suppose that $x_0 \in \Omega$ admits upper and lower barriers. Then x_0 is a regular point.*

Proof. Define $M = \max_{\partial\Omega} |g|$ and, for each $\varepsilon > 0$, choose $\delta > 0$ such that for $x \in \Omega$ with $|x - x_0| < \delta$ we have $|g(x) - g(x_0)| < \varepsilon$. Let b be an upper barrier and choose $k > 0$ such that $kb(x) \geq 2M$ if $|x - x_0| \geq \delta$ (by compactness $\inf_{\overline{\Omega}\backslash B_\delta(x_0)} b > 0$). Then define

$$w(x) := g(x_0) + \varepsilon + kb(x);$$

$$v(x) := g(x_0) - \varepsilon - kb(x)$$

and observe that $w \in S_+$ and $v \in S_-$. Moreover $w(x_0) - g(x_0) = \varepsilon$ and $g(x_0) - v(x_0) = \varepsilon$. □

In the following proposition we see that, under suitable hypotheses on the geometry of Ω, the existence of barriers, and therefore of a solution to the Dirichlet problem, is guaranteed.

Proposition 1.29 *Suppose that for each $x_0 \in \partial\Omega$ there exists a ball $B_R(y)$ in the complement of Ω such that $\overline{B}_R(y) \cap \overline{\Omega} = \{x_0\}$ (see Figure 1.2). Then every point of $\partial\Omega$ is regular, hence the Dirichlet problem* (1.1) *is solvable on Ω for arbitrary continuous boundary data.*

Proof. For any $x_0 \in \partial\Omega$ and a ball $B_R(y)$ as in the statement of the proposition, consider the upper barrier $b(x) := R^{2-n} - |x - y|^{2-n}$ for $n > 2$ and $b(x) := \log \frac{|x-y|}{R}$ for $n = 2$, and the lower barrier $-b(x)$. One can easily verify that $\Delta b = 0$ in $\mathbb{R}^n \setminus \{y\}$. □

Exercise 1.30 The hypotesis of Proposition 1.29 is called *exterior sphere condition*. Show that convex domains and C^2 domains satisfy the exterior sphere condition.

Remark 1.31 The Perron method is non-constructive because it doesn't provide any way to find approximate solutions.

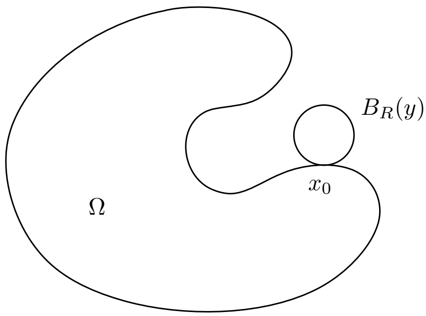

Figure 1.2: The exterior sphere condition.

1.4.3 Poincaré's method

We now present a different method of solving the Dirichlet problem (1.1).

Cover Ω with a sequence B_i of balls, i.e. choose balls $B_i \subset \Omega$, $i = 1, 2, 3, \dots$ such that $\Omega = \bigcup_{i=1}^{\infty} B_i$. Now define the sequence of integers

$$i_k = 1, 2, 1, 2, 3, 1, 2, 3, 4, \dots, 1, \dots, n, \dots$$

Given $g \in C^0(\overline{\Omega})$, define the sequence (u_k) by $u_1 := g$ and for $k > 1$

$$u_k(x) := \begin{cases} u_{k-1}(x) & \text{for } x \in \overline{\Omega} \setminus B_{i_k} \\ P_{i_k} u_{k-1}(x) & \text{for } x \in B_{i_k}, \end{cases}$$

where $P_{i_k} u_{k-1}$ is the harmonic extention on B_{i_k} of $u_{k-1}\big|_{\partial B_{i_k}}$, given by (1.11).

Proposition 1.32 *If each point of $\partial\Omega$ is regular, then u_k converges to the solution u of the Dirichlet problem* (1.1).

Proof. Suppose first $g \in C^0(\overline{\Omega})$ subharmonic, meaning that it satisfies the properties of Proposition 1.17. We can inductively prove that u_k is subharmonic and

$$g = u_1 \le u_2 \le \dots u_k \le \dots \le \sup_{\Omega} g.$$

Suppose indeed that u_k is subharmonic (this is true for $k = 1$ by assumption). Then by the comparison principle $u_{k+1} \ge u_k$, and it is not difficult to prove that u_{k+1} satisfies for instance (iii) or (iv) of Proposition 1.17, hence is subharmonic.

Since, for each i, u_k is harmonic in B_i for infinitely many k, increasing and uniformly bounded with respect to k, by Proposition 1.23 we see that its limit u is a harmonic functions in each ball B_i, hence in Ω. Using barriers it is not difficult to show that $u = g$ on the boundary.

Now suppose that g, not necessarily subharmonic, belongs to $C^2(\mathbb{R}^n)$ and $\Delta g \geq -\lambda$. Then $g_0(x) = g(x) + \frac{\lambda}{2n}|x|^2$ is subharmonic and we may solve the Dirichlet problem with boundary data g_0. We may also solve the Dirichlet problem with data $\frac{\lambda}{2n}|x|^2$ (that is subharmonic) and by linearity we may solve the Dirichlet problem with data g.

Finally, suppose $g \in C^0(\overline{\Omega})$, which we can think of as continuosly extended to $\mathbb{R}^n$, and regularize it by convolution. For each convoluted function $g_\varepsilon \in C^\infty(\overline{\Omega})$ we find a harmonic map u_ε with $u_\varepsilon = g_\varepsilon \to g$ uniformly on $\partial\Omega$. Then by the maximum principle, for any sequence $\varepsilon_k \to 0$ we have that (u_{ε_k}) is a Cauchy sequence in $C^0(\overline{\Omega})$, hence it uniformly converges to a harmonic function u which equals g on $\partial\Omega$. $\square$

Remark 1.33 The method of Poincaré decreases the Dirichlet integral:

$$\mathcal{D}(g) \geq \mathcal{D}(u_2) \geq \ldots \geq \mathcal{D}(u_k) \geq \ldots \geq \mathcal{D}(u).$$

Consequently if g has a $W^{1,2}$ extension i.e., an extension with finite Dirichlet integral, then the harmonic extension u lies in $W^{1,2}(\Omega)$ (for the definition of $W^{1,2}(\Omega)$ see Section 3.2 below).

On the other hand one can also have

$$\mathcal{D}(g) = \mathcal{D}(u_k) = \infty \quad \text{for every } k = 1, 2, \ldots,$$

compare section 1.2.2.

Remark 1.34 By Riemann's mapping theorem one can show that, if $\Omega \subset \mathbb{R}^2$ is the interior of a closed Jordan curve Γ, then all boundary points of Ω are regular. Lebesgue has instead exhibited a Jordan domain Ω in $\mathbb{R}^3$ (i.e. the interior of a homeomorphic image of S^2) where the problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$ cannot be solved for every $g \in C^0(\partial\Omega)$.

Chapter 2
Direct methods

In this chapter we shall study the existence of minimizers of variational integrals $\mathcal{F}$ defined on some space of functions A, say

$$\mathcal{F}(u) := \int_\Omega F(Du)dx, \qquad u \in A, \tag{2.1}$$

using the so-called *direct method*. This consists in introducing a possibly larger class $\overline{A} \supset A$ together with a topology that makes $\mathcal{F}$ *lower semicontinuous* and every (or at least one) minimizing sequence $\{u_j\}$ compact in $\overline{A}$, i.e. such that, modulo passing to a subsequence, $u_j \to \overline{u}$. Then $\overline{u}$ is a minimizer in $\overline{A}$, since

$$\mathcal{F}(\overline{u}) \leq \liminf_{j\to\infty} \mathcal{F}(u_j) = \inf_{u\in\overline{A}} \mathcal{F}(u).$$

Observe that the two conditions are in competition, since with a stronger topology it is easier to have semicontinuity, but more difficult to have compactness.

Examples of integrals of the form (2.1) are the following,[1]

1. $\mathcal{F}(u) := \int_\Omega |Du|^2 dx$
2. $\mathcal{F}(u) := \int_\Omega \sqrt{1+|Du|^2} dx$
3. $\mathcal{F}(u) := \int_\Omega e^{|Du|^2} dx$
4. $\mathcal{F}(u) := \int_\Omega |Du|^2 \log(1+|Du|^2) dx$
5. $\mathcal{F}(u) := \int_\Omega \left(\sum_{i=1}^{n-1} |D_i u|^2 + |D_n u|^k\right) dx, \quad k \geq 1$
6. $\mathcal{F}(u) := \int_\Omega (1+|Du|^k)^{\frac{1}{k}} dx, \quad k \geq 1.$

[1] These integrals are well defined on the space of Lipschitz functions $A = \mathrm{Lip}(\Omega)$ because, thanks to Rademacher's theorem, every Lipschitz function is differentiable almost everywhere and belongs to $W^{1,\infty}(\Omega)$. On the other hand, working with other spaces, such as Sobolev spaces, is often more suitable.

It turns out that in all these cases F is a convex function. This is a key property in the study of lower semicontinuity, and we shall assume it throughout this chapter.

2.1 Lower semicontinuity in classes of Lipschitz functions

By convexity of F we have for $u, v \in \mathrm{Lip}(\Omega)$

$$F(Dv(x)) \geq F(Du(x)) + F_{p_\alpha}(Du(x))(D_\alpha v(x) - D_\alpha u(x)), \text{ a.e. } x \in \Omega,$$

where F_{p_α} denotes the partial derivative of $F(p) = F(p_1, \ldots, p_n)$ with respect to the variable p_α, and here and in the following we use the convention of summing over repeated indexes. Consider a sequence $\{u_j\}$; for each u_j we have

$$\int_\Omega F(Du)dx \leq \int_\Omega F(Du_j)dx - \int_\Omega F_{p_\alpha}(Du)(D_\alpha u_j - D_\alpha u)dx. \qquad (2.2)$$

If we assume that F_{p_α} is continuous, then $F_{p_\alpha}(Du(x)) \in L^\infty(\Omega)$. Therefore if Du_j weakly converges to Du in $L^1(\Omega)$, the last integral vanishes and $\mathcal{F}(u) \leq \liminf \mathcal{F}(u_j)$, thus we have

Proposition 2.1 *A functional $\mathcal{F} : \mathrm{Lip}(\Omega) \to \mathbb{R}$ of the form*

$$\mathcal{F}(u) = \int_\Omega F(Du)dx$$

with F convex and F_p continuous is lower semicontinuous with respect to the weak-$W^{1,1}$ convergence.

Define the space

$$\mathrm{Lip}_k(\Omega) = \{u \in \mathrm{Lip}(\Omega) \ : \ |u|_1 \leq k\},$$

where $|u|_1$ is the Lipschitz seminorm:

$$|u|_1 := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x) - u(y)|}{|x - y|}.$$

Then we have

Proposition 2.2 *If F is convex and F_p continuous, then $\mathcal{F}$ is lower semicontinuous with respect to the uniform convergence of sequences with equibounded Lipschitz seminorm.*

Proof. If we approximate $F_p(Du)$ in $L^1(\Omega,\mathbb{R}^n)$ by smooth functions F_ε, the last term in (2.2) can be written as

$$\int_\Omega (F_{p_\alpha} - F_\varepsilon^\alpha)(D_\alpha u_j - D_\alpha u)dx + \int_\Omega F_\varepsilon^\alpha(D_\alpha u_j - D_\alpha u)dx.$$

Since $F_\varepsilon^\alpha \to F_{p_\alpha}(Du)$ in L^1, and the sequence (u_j) has equibounded gradient, taking ε small enough, the the first term can be made arbitrarily small. Integrating by parts the second term yields

$$\int_\Omega F_\varepsilon^\alpha(D_\alpha u_j - D_\alpha u)dx = -\int_\Omega D_\alpha F_\varepsilon^\alpha(u_j - u)dx,$$

which goes to zero as $u_j \to u$ uniformly. Lower semicontinuity follows from (2.2) letting $\varepsilon \to 0$. □

2.2 Existence of minimizers

2.2.1 Minimizers in $\mathrm{Lip}_k(\Omega)$

The reason for working in the classes $\mathrm{Lip}_k(\Omega)$ of equi-Lipschitz functions essentially lies in the compactness theorem of Ascoli and Arzelà.

Theorem 2.3 (Ascoli-Arzelà) *Given any equibounded and equicontinuous*[2] *sequence of functions $u_j : \Omega \to \mathbb{R}$, there exists a subsequence converging uniformly on compact subsets.*

Proposition 2.4 *Consider $g \in \mathrm{Lip}_k(\Omega)$. Then any variational integral $\mathcal{F}(u) = \int_\Omega F(Du)dx$ with F convex and F_p continuous has a minimizer in the class*

$$\mathcal{A}_k := \{u \in \mathrm{Lip}_k(\Omega) \ : \ u = g \text{ on } \partial\Omega\}.$$

Proof. Take a minimizing sequence $(u_j) \subset \mathcal{A}_k$. It is equibounded and equicontinuous hence, by Ascoli-Arzelà's theorem, we may extract a subsequence, still denoted by u_j, such that $u_j \to \overline{u} \in \mathrm{Lip}_k(\Omega)$ uniformly. Then Proposition 2.2 yields

$$\mathcal{F}(\overline{u}) \le \liminf_{j\to\infty} \mathcal{F}(u_j) = \inf_{u\in\mathcal{A}_k} \mathcal{F}(u).$$

□

[2] *Equibounded* means that there exists $K > 0$ such that $\sup_\Omega |u_j| \le K$ for every j; *equicontinuous* means that for every $x_0 \in \Omega$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|u_j(x) - u_j(x_0)| < \varepsilon, \quad \text{for } x \in \Omega \cap B_\delta(x_0), \text{ and for every } j.$$

The above proposition does not solve the problem of finding a minimizer among Lipschitz functions since it produces a function $\overline{u}$ minimizing in $\mathcal{A}_k$, but, in general, not in the class $\mathcal{A}$ of all Lipschitz functions with boundary value g. However $\overline{u}$ is a minimizer in $\mathcal{A}$, if a suitable *a priori estimate for its gradient* holds, as the following proposition shows.

Proposition 2.5 *Suppose that the minimizer $\overline{u}$ in $\mathcal{A}_k$ given by Proposition 2.4 satisfies $|\overline{u}|_1 < k$. Then u minimizes in*

$$\mathcal{A} := \{u \in \mathrm{Lip}(\Omega) \mid u = g \text{ on } \partial\Omega\}.$$

Proof. Take any $w \in \mathcal{A}$. Since $|\overline{u}|_1 < k$, we may choose $t \in (0,1)$ such that $tw + (1-t)\overline{u} \in \mathrm{Lip}_k(\Omega)$. Since $\overline{u}$ minimizes in $\mathcal{A}_k$ and F is convex, we have

$$\mathcal{F}(\overline{u}) \le \mathcal{F}(tw + (1-t)\overline{u}) \le t\mathcal{F}(w) + (1-t)\mathcal{F}(\overline{u}),$$

i.e., $\mathcal{F}(w) \ge \mathcal{F}(\overline{u})$. □

2.2.2 A priori gradient estimates

We now establish the a priori estimate required in Proposition 2.5, under suitable assumptions. This is achieved by comparison with suitable functions, called *barriers*, whose discussion is the aim of the following few paragraphs. We shall always assume F convex and F_p continuous.

Supersolutions and subsolutions

Definition 2.6 *Given the variational integral $\mathcal{F}$, we shall say that $u \in \mathrm{Lip}(\Omega)$ is a* supersolution *if*

$$\mathcal{F}(u + \varphi) \ge \mathcal{F}(u), \quad \forall \varphi \in \mathrm{Lip}(\Omega),\ \varphi \ge 0,\ \mathrm{spt}\,\varphi \Subset \Omega. \tag{2.3}$$

We shall say that v is a subsolution *if*

$$\mathcal{F}(v - \varphi) \ge \mathcal{F}(v), \quad \forall \varphi \in \mathrm{Lip}(\Omega),\ \varphi \ge 0,\ \mathrm{spt}\,\varphi \Subset \Omega. \tag{2.4}$$

If u is a supersolution, then we easily infer

$$\frac{d}{dt}\mathcal{F}(u + t\varphi)\Big|_{t=0^+} = \int_\Omega F_{p_\alpha}(Du) D_\alpha \varphi dx \ge 0, \quad \forall \varphi \ge 0,\ \mathrm{spt}\,\varphi \Subset \Omega,$$

or, in the sense of distributions, $\mathrm{div}(F_p(Du)) \le 0$. Similarly, a subsolution v satisfies $\mathrm{div}(F_p(Dv)) \ge 0$.

The comparison principle

Proposition 2.7 (Comparison principle) *Suppose that F is strictly convex. Then given a supersolution u and a subsolution v in* $\mathrm{Lip}(\Omega)$*, with $v \le u$ on $\partial\Omega$, we have $v \le u$ in Ω.*

Proof. Were the assertion false, the open set

$$K = \{x \in \Omega \mid v(x) > u(x)\}$$

would be non-empty. Consider now the functions

$$\widetilde{u}(x) := \begin{cases} u(x) & \text{if } x \in \Omega\backslash K \\ v(x) & \text{if } x \in K, \end{cases}$$

and

$$\widetilde{v}(x) := \begin{cases} v(x) & \text{if } x \in \Omega\backslash K \\ u(x) & \text{if } x \in K. \end{cases}$$

Then $\mathcal{F}(\widetilde{u}) \ge \mathcal{F}(u)$ and $\mathcal{F}(\widetilde{v}) \ge \mathcal{F}(v)$, hence

$$\int_\Omega F(Du)dx \le \int_\Omega F(D\widetilde{u})dx = \int_{\Omega\backslash K} F(Du)dx + \int_K F(Dv)dx,$$

whence

$$\int_K F(Du)dx \le \int_K F(Dv)dx.$$

Similarly

$$\int_\Omega F(Dv)dx \le \int_\Omega F(D\widetilde{v})dx = \int_{\Omega\backslash K} F(Dv)dx + \int_K F(Du)dx,$$

hence

$$\int_K F(Dv)dx \le \int_K F(Du)dx.$$

Then we infer

$$\int_K F(Du)dx = \int_K F(Dv)dx.$$

Now the strict convexity of F implies

$$\begin{aligned} \int_K F\left(\frac{Du+Dv}{2}\right)dx &< \frac{1}{2}\int_K F(Du)dx + \frac{1}{2}\int_K F(Dv)dx \\ &= \int_K F(Dv)dx. \end{aligned}$$

This is an absurd since replacing v in K with the smaller function $\frac{v+u}{2}$, decreases $\mathcal{F}$, contradicting the fact that v is a subsolution. $\square$

Exercise 2.8 Every constant function is both a supersolution and a subsolution. Moreover, if u is a supersolution (resp. subsolution), then $u + \lambda$ is a supersolution (resp. subsolution) for every constant $\lambda \in \mathbb{R}$.
[Hint: use (2.2) and integration by parts.]

Proposition 2.9 (Maximum principle) *Given a subsolution v and a supersolution u of $\mathcal{F}$ in* $\mathrm{Lip}(\Omega)$, *we have*

$$\sup_{\Omega}(v-u) \le \sup_{\partial\Omega}(v-u).$$

In particular if u is both a supersolution and a subsolution, then

$$\sup_{\Omega}|u| = \sup_{\partial\Omega}|u|$$

Proof. Since $u + \sup_{\partial\Omega}(v-u)$ is a supersolution by Exercise 2.8, and is not smaller than v on $\partial\Omega$, Proposition 2.7 yields

$$v \le u + \sup_{\partial\Omega}(v-u), \quad \text{in } \Omega.$$

□

Exercise 2.10 Show that the comparison principle holds true if we assume that u and v are respectively a supersolution and a subsolution in $\mathrm{Lip}_k(\Omega)$, which means that $u, v \in \mathrm{Lip}_k(\Omega)$, and in (2.3) and (2.4) we require $\varphi \in \mathrm{Lip}_k(\Omega)$.

Reduction to boundary estimates

It now comes the key estimate that allows us to infer global gradient estimates from boundary estimates. In fact the method we are presenting goes back to Haar and Radò, see [85]. In the Sixties of the last century the method was revisited by M. Miranda, P. Hartman and G. Stampacchia.

Proposition 2.11 (Haar-Radò) *Let $u \in \mathrm{Lip}(\Omega)$ be a minimizer of $\mathcal{F}$ in $\mathcal{A} = \{v \in \mathrm{Lip}(\Omega) : v = u \text{ on } \partial\Omega\}$. Then*

$$\sup_{x,y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|} = \sup_{x\in\Omega,\, y\in\partial\Omega} \frac{|u(x)-u(y)|}{|x-y|}. \tag{2.5}$$

Proof. For $x_1, x_2 \in \Omega$, $x_1 \ne x_2$, let $\tau = x_2 - x_1$. Define

$$u_\tau(x) := u(x+\tau), \quad \Omega_\tau := \{x \;:\; x+\tau \in \Omega\}.$$

Both u and u_τ are super and subsolutions in $\Omega \cap \Omega_\tau$, which is non-empty. By the comparison principle, Proposition 2.9, there exists $z \in \partial(\Omega \cap \Omega_\tau)$ such that

$$|u(x_1) - u(x_2)| = |u(x_1) - u_\tau(x_1)| \le |u(z) - u_\tau(z)| = |u(z) - u(z+\tau)|.$$

Now observe that $\partial(\Omega \cap \Omega_\tau) \subset (\partial\Omega \cup \partial\Omega_\tau)$ and this implies that at least one of the point z, $z+\tau$ belongs to $\partial\Omega$. Moreover, both z and $z+\tau$ belong to $\overline{\Omega}$. □

Boundary gradient estimates through the bounded slope condition

The *bounded slope condition* (BSC), essentially introduced by Haar, is defined as follows:

> A function $g \in \mathrm{Lip}(\partial\Omega)$ satisfies the bounded slope condition if there exists a constant $k > 0$ such that for every $x_0 \in \partial\Omega$ we may find two affine functions v and w with $|Dv| \le k$ and $|Dw| \le k$ such that:
>
> 1. $v(x_0) = w(x_0) = g(x_0)$
> 2. $v(x) \le g(x)$, $w(x) \ge g(x)$ for every $x \in \partial\Omega$.

Theorem 2.12 *Suppose that $g \in \mathrm{Lip}(\partial\Omega)$ satisfies the BSC with constant k. Then any variational integral $\mathcal{F}(u) = \int_\Omega F(Du)dx$ with F convex and F_p continuous attains a minimum in the class*

$$\mathcal{A} := \left\{u \in \mathrm{Lip}(\Omega)\ :\ u\big|_{\partial\Omega} = g\right\},$$

and such minimum belongs to $\mathrm{Lip}_k(\Omega)$.

Proof. By Proposition 2.4, there exists a minimizer u of $\mathcal{F}$ in

$$\mathcal{A}_{k+1} = \{v \in \mathrm{Lip}_{k+1}(\Omega) : v = g \text{ on } \partial\Omega\}.$$

Since the affine functions in the definition of the BSC are a supersolution and a subsolution, the comparison principle implies that $|Du| \le k$ on $\partial\Omega$ and, by Proposition 2.11, $|u|_1 \le k < k+1$. We conclude with Proposition 2.5. □

Remark 2.13 The BSC is a pretty strong condition: for instance, it can be true only if Ω is convex. On the other hand, notice that the above result holds for a wide class of functionals.

2.2.3 Constructing barriers: the distance function

Since the *BSC* is very restrictive, we will discuss other conditions on a domain Ω and a function $g \in \mathrm{Lip}(\partial\Omega)$ which allow to construct barriers and minimize a given variational integral $\mathcal{F}(u) = \int_\Omega F(Du)dx$, with F convex and F_p continuous.

Definition 2.14 *Given a boundary datum $g \in \mathrm{Lip}(\partial\Omega)$, an upper barrier at $x_0 \in \partial\Omega$ is a supersolution $b_+ \in \mathrm{Lip}(\Omega)$ of $\mathcal{F}$ such that $b_+(x_0) = g(x_0)$ and $b_+ \ge g$ on $\partial\Omega$. Lower barriers are defined analogously.*

Suppose that Ω is of class C^k, $k \geq 1$; then there exist an interior tubular neighborhood N of $\partial\Omega$,

$$N = \{x \in \Omega \ : \ \mathrm{dist}(x, \partial\Omega) < \varepsilon\},$$

where the corresponging projection $p : N \to \partial\Omega$ is of class C^{k-1}. Let $d(x) := \mathrm{dist}(x, \partial\Omega)$ be the distance function from $\partial\Omega$; then for every $x \in N$,

$$\nabla d(x) = \nu(p(x)),$$

where $\nu(x)$ is the interior unit normal to $\partial\Omega$. Since ν and p are of class C^{k-1}, we have that $d \in C^k(N) \cap C^0(\overline{N})$.

For $x \in N$, denote by $H(x)$ the mean curvature (compare Section 11.1.3) at x of the hypersurface in $\mathbb{R}^n$

$$\Gamma_{d(x)} := \{y \in \Omega \ : \ d(y) = d(x)\}.$$

Then it can be proved that:

$$(n-1)H(x) = -\Delta d \geq (n-1)H(p(x)).$$

Given a boundary datum g, that we assume of class C^2 in a neighborhood of Ω, one may try to construct Lipschitz barriers of the form

$$b_+(x) := g(x) + h_+(d(x)), \qquad b_-(x) := g(x) + h_-(d(x)), \tag{2.6}$$

with $h_+ : [0, \varepsilon) \to \mathbb{R}$ increasing, differentiable at 0, *independent of* u, and $h_+(0) = 0$, such that $b_+(x)$ is a supersolution of $\mathcal{F}$, and similarly $h_- : [0, \varepsilon) \to \mathbb{R}$ decreasing, differentiable at 0, *independent of* u, and $h_-(0) = 0$, such that $b_-(x)$ is a subsolution of $\mathcal{F}$. Though in general impossible, this can be done if we assume additional structural conditions on Ω and F, compare e.g. [52], [96]. For instance, still assuming that $F = F(|p|)$, indicate with $F_{p_\alpha p_\beta} := \frac{\partial^2 F}{\partial p_\alpha \partial p_\beta}$ the Hessian of F. Assume that F is strictly convex and C^2, so that $F_{p_i p_j} = F_{p_j p_i}$ and

$$\lambda(p)|\xi|^2 \leq F_{p_\alpha p_\beta}(p)\xi_\alpha \xi_\beta \leq \Lambda(p)|\xi|^2,$$

for positive functions $0 < \lambda(p) \leq \Lambda(p)$. Define the *Bernstein function*

$$\mathcal{E}(p) := F_{p_\alpha p_\beta}(p) p_\alpha p_\beta.$$

Then in the following cases the construction of barriers of the form (2.6) is possible.

(i) $\limsup\limits_{|p|\to\infty} \frac{|p|\Lambda(p)}{\mathcal{E}(p)} < \infty$

(ii) a. $\limsup\limits_{|p|\to\infty} \frac{\Lambda(p)}{\mathcal{E}(p)} < \infty$ and

b. the mean curvature of $\partial\Omega$ is non-negative.

For instance (i) is trivially verified if F_{pp} is uniformly elliptic, i.e. $\lambda(p) \geq \gamma\Lambda(p)$ for every $p \in \mathbb{R}^n$, and some $\gamma > 0$. Uniform ellipticity, however, is not necessary because $F(p) = e^{|p|^2}$, which is not uniformly elliptic, satisfies (i).

Exercise 2.15 The area functional, $\mathcal{F}(u) = \int_\Omega \sqrt{1+|Du|^2}dx$, which is elliptic, but not uniformly, satisfies (ii)a, but not (i).[3]

By the comparison principle the existence of such Lipschitz barriers yields the a priori estimate for the gradient on the boundary (compare also Proposition 11.41, where we shall also prove the existence of barriers in the case of the area functional). In particular:

Theorem 2.16 *Consider*

$$\mathcal{F}(u) = \int_\Omega F(Du)dx$$

with F convex and of class C^2. If F satisfies (i), or if F and Ω satisfy (ii) above, then $\mathcal{F}$ has a minimizer in

$$\mathcal{A} = \{u \in \mathrm{Lip}(\Omega) : u = g \text{ on } \partial\Omega\}$$

for every $g \in \mathrm{Lip}(\partial\Omega)$.

2.3 Non-existence of minimizers

Condition (ii)a in the last section does not guarantee the existence of barriers without the assumption (ii)b. We shall now see an explicit example.

2.3.1 An example of Bernstein

We shall prove that the area functional

$$\mathcal{F}(u) := \int_\Omega \sqrt{1+|Du|^2}dx,$$

which satisfies (ii)a of the previous section, need not have a minimizer if (ii)b is not met. This will be made more general in the next section.

[3] Given a function $u \in \mathrm{Lip}(\Omega)$, $\Omega \subset \mathbb{R}^n$, it can be shown using the area formula that

$$\mathcal{F}(u) := \int_\Omega \sqrt{1+|Du|^2}dx = \mathcal{H}^n(\mathrm{graph}(u)),$$

where $\mathrm{graph}(u) := \{(x, u(x)) : x \in \Omega\} \subset \mathbb{R}^{n+1}$.

For fixed $0 < \rho < R$, consider the domain

$$\Omega = \{x \in \mathbb{R}^n \ : \ \rho < |x| < R\}.$$

Define the boundary value g by

$$g(x) := \begin{cases} m & \text{if } |x| = \rho \\ 0 & \text{if } |x| = R \end{cases}$$

Exercise 2.17 Suppose that u is a minimizer for the area functional with the above boundary condition. Then u is radial, i.e. $u = u(r)$.
[Hint: the function

$$\overline{u}(r) := \frac{1}{2\pi r} \int u(r,\theta) d\theta$$

satisfies $\mathcal{A}(\overline{u}) < \mathcal{A}(u)$ if $u \neq \overline{u}$, by the strict convexity of $F(p) = \sqrt{1+|p|^2}$ and Jensen's inequality.]

By Exercise 2.17, a minimizer with boundary value g must be radial. Then the area can be computed as

$$\mathcal{F}(u) = 2\pi \int_\rho^R r\sqrt{1+u_r^2} dr.$$

The corresponding Euler-Lagrange equation is the ordinary differential equation

$$\frac{r u_r(r)}{\sqrt{1+u_r(r)^2}} = -c, \tag{2.7}$$

where c is a constant depending on $m = u(\rho)$. The unique solution to (2.7) with $u = 0$ on $\partial B_R(0)$ is

$$u(r) = c \log\left(\frac{R + \sqrt{R^2 - c^2}}{r + \sqrt{r^2 - c^2}}\right). \tag{2.8}$$

In particular $c \leq \rho$ and

$$\begin{aligned} \sup_{0 \leq c \leq \rho} u(\rho) &= \sup_{0 \leq c \leq \rho} c \log\left(\frac{R + \sqrt{R^2 - c^2}}{\rho + \sqrt{\rho^2 - c^2}}\right) \\ &= \rho \log\left(\frac{R + \sqrt{R^2 - \rho^2}}{\rho}\right) \\ &=: c(\rho, R), \end{aligned}$$

that forces m to be less than $c(\rho, R)$.

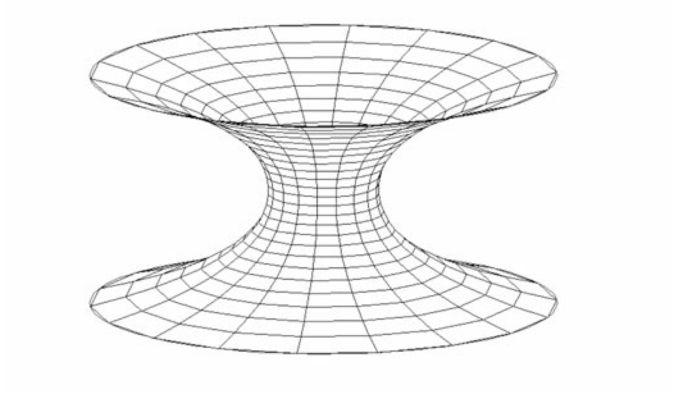

Figure 2.1: A piece of catenoid which cannot be expressed as graph of a function.

Remark 2.18 For

$$m = \rho \log \left(\frac{R + \sqrt{R^2 - \rho^2}}{\rho} \right),$$

which is the highest value for which the Dirichlet problem is solvable, the solution u is not smooth up to the boundary since

$$\lim_{r \to \rho^+} |u_r(r)| = +\infty.$$

Remark 2.19 Observing that

$$\cosh^{-1}(r) = \log(r + \sqrt{r^2 - 1}),$$

we see that the graph of the solution given by (2.8) is the revolution surface obtained by rotating a catenoid. For

$$m > \rho \log \frac{R + \sqrt{R^2 - \rho^2}}{\rho}$$

a catenoid matching the boundary conditions is no longer expressible as the graph of a function, see Figure2.1.

2.3.2 Sharpness of the mean curvature condition

We now show that, at least in the case of the area functional

$$\mathcal{F}(u) = \int_\Omega \sqrt{1 + |Du|^2} dx,$$

condition (ii)b is sharp.

Theorem 2.20 *Let $x_0 \in \partial\Omega$ be such that $H(x_0) < 0$, where it is assumed that Ω is a C^2 domain. Then for every $\varepsilon > 0$ there exists $g \in \mathrm{Lip}(\partial\Omega)$ with $\max_{\partial\Omega} |g| < \varepsilon$ such that the Dirichlet problem for the area functional cannot be solved with boundary data g, i.e. the area functional $\mathcal{F}$ has no minimizer in $\mathcal{A} = \{u \in \mathrm{Lip}(\Omega) : u = g \text{ on } \partial\Omega\}$.*

This follows from Lemma 2.22 below by choosing $\varepsilon > 0$, consequently fixing Γ and finally imposing $g = 0$ on $\partial\Omega\backslash\Gamma$ and $g(x_0) > \frac{\varepsilon}{2}$. In the proof of Lemma 2.22 we will need the following lemma.

Lemma 2.21 *Let $u \in \mathrm{Lip}(\Omega)$ be a subsolution and a supersolution of $\mathcal{F}$ and let $v \in C^1(\Omega) \cap C^0(\overline{\Omega})$ be a supersolution of $\mathcal{F}$. Let A be open in $\overline{\Omega}$ and set $\partial^1\Omega := \partial\Omega \cap A$. Assume that*

1. *$u \le v$ on $\partial^0\Omega := \partial\Omega\backslash\partial^1\Omega$,*
2. $\displaystyle \liminf_{t\to 0^+} \inf_{A\cap\Gamma_t} \frac{\partial v}{\partial \nu} > |u|_1 := \sup_{x,y\in\Omega,\ x\neq y} \frac{|u(x)-u(y)|}{|x-y|},$

where

$$\Gamma_t := \{x \in \Omega : d(x, \partial\Omega) = t\},$$

and ν is the interior unit normal to Γ_t. Then $u \le v$ in $\overline{\Omega}$.

Proof. It is enough to prove the claim for $w = v + \varepsilon$ instead of v, and let $\varepsilon \to 0$. By the comparison principle it suffices to show that

$$u \le w \quad \text{on } \partial^0\Omega.$$

If not, there exists $t > 0$ as small as we want such that

$$\gamma_t := \sup_{A\cap\Gamma_t} (u - w) > 0$$

and

$$u - w \le 0 \quad \text{on } \Gamma_t\backslash A.$$

In $\Omega_t := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > t\}$ we have $u \le w + \gamma_t$ since $u \le w + \gamma_t$ on $\partial\Omega_t$, and by the maximum principle (Proposition 2.9) there exists $x_0 \in \Gamma_t \cap A$ with $u(x_0) = w(x_0) + \gamma_t$ and thus

$$\frac{\partial}{\partial \nu}(u - v)(x_0) = \frac{\partial}{\partial \nu}(u - w)(x_0) \ge 0.$$

Since $t > 0$ can be chose arbitrarily small, we found a contradiction to hypothesis 2. $\square$

Lemma 2.22 *For every $\varepsilon > 0$ there exists a neighborhood Γ of x_0 in $\partial\Omega$ such that if u minimizes $\mathcal{F}$ in the class $\mathcal{A}$, then*

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega\setminus\Gamma} |g| + \frac{\varepsilon}{2}.$$

Proof. We may assume $x_0 = 0$ and choose $R > 0$ such that $H(x) < 0$ for $x \in B_R(0) \cap \overline{\Omega}$. Remember that $H(x)$ is the mean-curvature of the hypersurface $\Gamma_{d(x)} \subset \Omega$. Define

$$v(x) = b + \psi(|x|) \quad \text{for } x \in \overline{\Omega} \setminus B_R(0),$$

where

$$\psi(r) := -R\cosh^{-1}\left(\frac{r}{R}\right) = -R\log\left(\frac{r + \sqrt{r^2 - R^2}}{R}\right),$$

and

$$b := \sup_{\partial\Omega\setminus B_R(0)} |g| + R\cosh^{-1}\frac{\operatorname{diam}\Omega}{R}.$$

By the above, we know that

$$\frac{\partial v}{\partial \nu} = +\infty \quad \text{on } \partial B_R(x) \cap \Omega.$$

Also, $u \leq v$ on $\partial\Omega\setminus B_R(0)$ and, by Lemma 2.21 applied to the domain $\Omega \setminus B_R(0)$, we infer $u \leq v$ in $\overline{\Omega}\setminus B_R(0)$.

We now work in $\Omega \cap B_R(0)$, where we define

$$w(x) := a\left(\sqrt{R} - \sqrt{d(x)}\right) + b.$$

Using that $H(x) > 0$ for $x \in \overline{\Omega} \cap \overline{B}_R$, and $-\Delta d(x) = (n-1)H(x)$, we can compute for $a > 0$ large enough, more precisely

$$a \geq \Big(-2(n-1)\inf_{\Omega\cap B_R(0)} H\Big)^{-1},$$

we compute

$$\operatorname{div}(F_p(w)) = D_\alpha \frac{D_\alpha w}{\sqrt{1+|Dw|^2}} \leq 0 \quad \text{weakly},$$

i.e. w is a supersolution. Moreover we have

$$w \geq u \quad \text{on } \partial B_R(0) \cap \Omega \quad \text{and} \quad \frac{\partial w}{\partial \nu} = +\infty \quad \text{on } \partial\Omega \cap B_R(0),$$

hence by Lemma 2.21 applied to the domain $\Omega \cap B_R(0)$ we have

$$u(x) \leq w(x) \leq b + a\sqrt{R}, \quad \text{for } x \in \Omega \cap B_R(0).$$

In conclusion

$$|u(x)| \le \sup_{\partial\Omega\setminus B_R(0)} |g| + R\cosh^{-1}\left(\frac{\operatorname{diam}\Omega}{R}\right) + a\sqrt{R}. \tag{2.9}$$

For $R = R(\varepsilon)$ small enough and choosing $\Gamma := \partial\Omega \cap B_R(0)$ we get the conclusion. □

Remark 2.23 Any minimizer of $\mathcal{F}$ in $\mathcal{A} = \{u \in \mathrm{Lip}(\Omega) : u = 0 \text{ on } \partial\Omega\}$ actually belongs to $C^\infty(\Omega)$. Hence minimizing in $\mathcal{A}$ is equivalent to minimizing in $\widetilde{\mathcal{A}} = \{u \in C^\infty(\Omega) \cap \mathrm{Lip}(\Omega) : u = 0 \text{ on } \partial\Omega\}$. One might wonder whether under the assumption of Theorem 2.20 minimizers of $\mathcal{F}$ can be found in the larger class $\mathcal{A}^* = \{u \in C^\infty(\Omega) \cap C^0(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$. This is not the case. For the proof, which is slightly more technical but based on the same ideas of Theorem 2.20, we refer to [6].

2.4 Finiteness of the area of graphs with zero mean curvature

We would like to stress one more difference between the Dirichlet problem for the Laplacian and the minimal surface equation. As we have seen in Section 1.2.2, a $C^2(\Omega) \cap C^0(\overline{\Omega})$ solution to the Dirichlet problem for the Laplace equation need not have finite Dirichlet energy. In the area problem things go differently. Let us first notice that the Euler-Lagrange equation of the area functional

$$\mathcal{F}(u) = \int_\Omega \sqrt{1 + |Du|^2}\, dx$$

is

$$\sum_{i=1}^n D_i \frac{D_i u}{\sqrt{1 + |Du|^2}} = 0. \tag{2.10}$$

One might wonder whether it is possible to find a solution u to (2.10) with $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ (as usual Ω is bounded) and $\mathcal{F}(u) = \infty$. As we now see (at least if we assume Ω of class C^1 for simplicity), this is not the case.

Proposition 2.24 *Suppose that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution to the minimal surface equation* (2.10)[4] *Then the area of the graph of u is finite, i.e.*

$$\mathcal{F}(u) := \int_\Omega \sqrt{1 + |Du|^2}\, dx < \infty.$$

[4] Minimizers of the area functional (with prescribed boundary data) satisfy (2.10), but by convexity of the area functional, every Lipschitz solution u of (2.10) with $\mathcal{F}(u) < \infty$ is in fact the only minimizer of the area relative to its boundary value.

Proposition 2.24 will be a consequence of the following lemma.

Lemma 2.25 *For every* $u \in \mathrm{Lip}(\Omega)$ *which minimizes the area in*

$$\{w \in \mathrm{Lip}(\Omega) : w = u \ \text{on} \ \partial\Omega\}$$

and every $v \in C^1(\overline{\Omega})$ *we have*

$$\mathcal{F}(u) \le \mathcal{F}(v) + \int_{\partial\Omega} |u - v| d\mathcal{H}^{n-1}. \tag{2.11}$$

Proof. Choose a sequence of smooth domains $\Omega_\varepsilon \subset \Omega$ with

$$\Omega_\varepsilon \uparrow \Omega, \quad \text{and} \quad \mathcal{H}^{n-1}(\partial\Omega_\varepsilon) \to \mathcal{H}^{n-1}(\partial\Omega) \quad \text{as } \varepsilon \to 0,$$

and choose functions $\eta_\varepsilon \in C_c^\infty(\Omega)$ with $0 \le \eta_\varepsilon \le 1$, $\eta_\varepsilon \equiv 1$ on Ω_ε, roughly

$$\Omega_\varepsilon := \big\{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \varepsilon\big\}, \quad \eta_\varepsilon(x) \sim \frac{1}{\varepsilon}\,\mathrm{dist}(x, \partial\Omega), \ x \in \Omega\backslash\Omega_\varepsilon.$$

The claim then follows easily from

$$\mathcal{F}(u) \le \mathcal{F}(\eta_\varepsilon v + (1 - \eta_\varepsilon)u) \to \mathcal{F}(v) + \int_{\partial\Omega} |u - v| d\mathcal{H}^{n-1} \quad \text{as } \varepsilon \to 0.$$

□

Proof of Proposition 2.24. We apply Lemma 2.25 in the domain

$$\Omega_\varepsilon := \big\{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \varepsilon\big\}.$$

As already noticed, since

$$\mathcal{F}(u, \Omega_\varepsilon) := \int_{\Omega_\varepsilon} \sqrt{1 + |Du|^2} dx < \infty,$$

we have that u is the only minimizer of $\mathcal{F}(\cdot, \Omega_\varepsilon)$ in

$$\{w \in \mathrm{Lip}(\Omega_\varepsilon) : v = u \text{ on } \partial\Omega_\varepsilon\},$$

compare Theorem 11.29. Then, by Lemma 2.25 with $v = 0$ we infer

$$\begin{aligned}\mathcal{F}(u, \Omega_\varepsilon) &\le \mathcal{F}(v, \Omega_\varepsilon) + \int_{\partial\Omega_\varepsilon} |u| d\mathcal{H}^{n-1} \\ &\le \mathcal{H}^n(\Omega_\varepsilon) + \sup_\Omega |u|\, \mathcal{H}^{n-1}(\partial\Omega_\varepsilon).\end{aligned}$$

Letting $\varepsilon \to 0$ we conclude

$$\mathcal{F}(u, \Omega) \le \mathcal{H}^n(\Omega) + \sup_\Omega |u|\, \mathcal{H}^{n-1}(\partial\Omega) < \infty.$$

□

Exercise 2.26 Construct a function $u \in C^2(B_1(0)) \cap C^0(\overline{B_1(0)})$ with

$$\int_{B_1(0)} \sqrt{1 + |Du|^2} dx = \infty.$$

2.5 The relaxed area functional in BV

In this section we discuss (giving the main ideas and omitting many details) how to use variational methods to find minimizers of the area functional with prescribed boundary value (in a suitable relaxed sense) even on domains Ω not satisfying condition (ii)b, i.e. when the mean curvature of $\partial\Omega$ is negative at some points.

Given a Lipschitz or smooth function in Ω, the area of its graph is given by

$$\mathcal{F}(u) = \int_\Omega \sqrt{1+|Du|^2}dx.$$

If u is merely continuous, we define its relaxed area, according to Lebesgue, as

$$\mathcal{F}(u) = \inf\left\{ \liminf_{k\to\infty} \mathcal{F}(u_k) \;\middle|\; u_k \to u \text{ uniformly, } u_k \in C^1(\overline{\Omega}) \right\}.$$

Exercise 2.27 Prove that the relaxed area functional is lower semicontinuous with respect to the uniform convergence.
[Hint: The area functional $\mathcal{F}$ for Lipschitz functions is lower semicontinuous.]

Exercise 2.28 The relaxed area functional agrees with the standard area functional on Lipschitz functions.

In order to understand which functions have finite relaxed area, we extend the above definition to L^1, replacing the uniform convergence with the L^1 convergence: for each $u \in L^1(\Omega)$

$$\mathcal{F}(u) = \inf\left\{ \liminf_{k\to\infty} \mathcal{F}(u_k) \;:\; u_k \to u \text{ in } L^1, \;\; u_k \in C^1(\overline{\Omega}) \right\}.$$

Functions of bounded variation

Definition 2.29 *An $L^1(\Omega)$ function is said to be of* bounded variation *when its partial derivatives in the sense of distributions are signed measures with finite total variation. The subspace of $L^1(\Omega)$ consisting of such functions is called $BV(\Omega)$.*

Equivalently, $BV(\Omega)$ is the space of $L^1(\Omega)$ functions such that

$$\mathcal{F}(u) < +\infty, \tag{2.12}$$

where

$$\mathcal{F}(u) := \sup\left\{ \int_\Omega \left(u\sum_{i=1}^n D_i g_i + g_{n+1} \right) dx \;:\; g \in C^1_c(\Omega, \mathbb{R}^{n+1}),\; |g| \le 1 \right\}. \tag{2.13}$$

Exercise 2.30 Prove the latter claim: a function $u \in L^1(\Omega)$ belongs to $BV(\Omega)$ if and only if it satisfies (2.12).
[Hint: To show that (2.12) implies that u has bounded variation use Riesz's representation theorem]

It turns out that the relaxed area agrees with the quantity in (2.13), often denoted by $\int_\Omega \sqrt{1+|Du|^2}$,[5] which is the *total variation* of the vector measure $(-Du, \mathcal{L}^n)$. BV-functions are exactly the functions having graphs of finite area.

In particular, given any $u \in BV(\Omega)$ or any $u \in C^0(\overline{\Omega})$ with finite area, there exists a sequence $u_k \in C^\infty(\Omega)$ ($C^\infty(\overline{\Omega})$ provided, of course $\partial\Omega$ is smooth) such that

$$u_k \to u \text{ in } L^1 \text{ (or uniformly) and } \mathcal{F}(u_k) \to \mathcal{F}(u).$$

We shall not prove this, see e.g. [6] [49] [51].

2.5.1 BV minimizers for the area functional

We now want to use direct methods to prove existence of minimal graphs with prescribed boundary. The natural space to work with is $BV(\Omega)$. Since a function $u \in L^1(\Omega)$ is defined up to a set of zero measure, we cannot naïvely make sense of the boundary datum $u\big|_{\partial\Omega}$. On the other hand for $u \in BV(\Omega)$, its *trace* on $\partial\Omega$ is well defined. This follows from the theorem below, whose proof can be found in [51]:

Proposition 2.31 (Trace) *Let $\Omega \subset \mathbb{R}^n$ be bounded domain with Lipschitz boundary. Then there exists a unique continuous linear operator*

$$\mathrm{Trace} : BV(\Omega) \to L^1(\partial\Omega)$$

such that for $u \in C^\infty(\overline{\Omega})$ we have $\mathrm{Trace}\, u = u\big|_{\partial\Omega}$. Moreover the map Trace *is* surjective.

We can now define the class of BV functions with boundary value $g \in L^1(\partial\Omega)$:

$$\mathcal{A} := \big\{u \in BV(\Omega) : \mathrm{Trace}\, u = g\big\},$$

and look for a minimizer of the area functional in $\mathcal{A}$.

This problem is not in general solvable and the reason lies essentially in the boundary behaviour of minimizing sequences, as we shall see. Remember that direct methods are based on semicontinuity and compactness. For BV functions and the area functional we have both:

Theorem 2.32 (Compactness) *The immersion $BV(\Omega) \hookrightarrow L^1(\Omega)$ is compact.*

[5] Notice the absence of "dx", to emphasize that $\sqrt{1+|Du|^2}$ is in general not absolutely continuous when $u \in BV(\Omega)$.

Proof. We only remark that if $\Omega = Q$ is a cube, then the proof is exactly the same as in Theorem 3.18 in the next chapter. $\square$

Theorem 2.33 (Semicontinuity) *Let $u_j \to u$ in $L^1(\Omega)$, where $u_j \in BV(\Omega)$. Then $u \in BV(\Omega)$ and*

$$\int_\Omega \sqrt{1+|Du|^2} \leq \liminf_{j\to+\infty} \sqrt{1+|Du_j|^2}.$$

Consequently, from any minimizing sequence $u_j \in \mathcal{A}$, we can extract a subsequence converging in $L^1(\Omega)$ to a function $u \in BV(\Omega)$, with

$$\int_\Omega \sqrt{1+|Du|^2} \leq \int_\Omega \sqrt{1+|Dv|^2}, \qquad \text{for every } v \in \mathcal{A}.$$

But it is false in general that $\operatorname{Trace} u = g$.

This leads us to relax the problem further. We allow for functions u which do not attain the value g at the boundary, and we modify the area functional so that the area spanned to connect u to g on $\partial\Omega$ is taken into account. We obtain the functional on $BV(\Omega)$

$$\mathcal{J}(u) := \int_\Omega \sqrt{1+|Du|^2} + \int_{\partial\Omega} |u-g|d\mathcal{H}^{n-1}. \tag{2.14}$$

Theorem 2.34 *Assume that $\partial\Omega$ is Lipschitz continuous. Then for any boundary data $g \in L^1(\partial\Omega)$, there exists a function $u \in BV(\Omega)$ which minimizes the area functional $\mathcal{J}$ in (2.14) among all functions in $BV(\Omega)$.*

Proof. Instead of minimizing $\mathcal{J}$ we consider a ball $B_R(0)$ such that $\overline{\Omega} \subset B_R(0)$, and extend g to a function in $W^{1,1}(B_R(0)\backslash\Omega)$. This can be done since the trace operator

$$\text{Trace} : W^{1,1}(\Omega \setminus B_R(0)) \to L^1(\partial\Omega \cup \partial B_R(0))$$

is surjective. Now, for every $v \in BV(\Omega)$, denote by v_g the function

$$v_g(x) := \begin{cases} v(x) & \text{if } x \in \Omega \\ g(x) & \text{if } x \in B_R(0)\backslash\Omega. \end{cases}$$

Then $v_g \in BV(B_R(0))$, and in fact $|Dv_g|(\partial\Omega) = \int_{\partial\Omega} |v-g|d\mathcal{H}^{n-1}$, whence

$$\begin{aligned} \int_{B_R(0)} \sqrt{1+|Dv_g|^2} &= \int_\Omega \sqrt{1+|Dv|^2} + \int_{B_R(0)\backslash\overline{\Omega}} \sqrt{1+|Dg|^2}dx \\ &\quad + \int_{\partial\Omega} |v-g|d\mathcal{H}^{n-1} \\ &= \mathcal{J}(v,\Omega) + \int_{B_R(0)\backslash\overline{\Omega}} \sqrt{1+|Dg|^2}dx. \end{aligned}$$

Therefore our original problem reduces to minimizing $\int_{B_R(0)} \sqrt{1+|Dv|^2}$ among all the functions in $v \in BV(B_R(0))$ such that $v = g$ in $B_R(0)\backslash\overline{\Omega}$. Since this last condition is preserved under convergence in $L^1(\Omega)$, we may consider a minimizing sequence, bounded in $BV(\Omega)$ by the Poincaré inequality, see Proposition 2.35 below. and apply Theorems 2.32 and 2.33 to conclude. □

In the proof of Theorem 2.34 we used the following version of Poincaré's inequality:

Proposition 2.35 *For any $f \in BV(\Omega)$ with* $\mathrm{Trace}\, f = 0$, *we have*

$$\int_\Omega |f|dx \le C(\Omega)\int_\Omega |Df|. \tag{2.15}$$

For $f \in C_c^1(\Omega)$, (2.15) shall be proven in Proposition 3.10 in the next chapter. The general case follows at once from the following approximation property:

Proposition 2.36 *Given $f \in BV(\Omega)$ with* $\mathrm{Trace}\, f = 0$, *there exists a sequence of functions $f_n \in C_c^1(\Omega)$ with*

$$f_n \to f \ \text{in } L^1(\Omega), \qquad \int_\Omega |Df_n|dx \to \int_\Omega |Df|.$$

We shall see in Chapter 11 that a minimizer in Theorem 2.34 is smooth in Ω. Regularity up to the boundary is in general false: u may not even attain the boundary data g, as Theorem 2.20 implies. On the other hand, if the mean curvature of $\partial\Omega$ is non-negative, we have the following result of M. Miranda [75].

Theorem 2.37 *Assume that $\partial\Omega$ is of class C^2 and has non-negative mean curvature at x_0. Furthermore, assume that g is continuous at x_0, and let u be a minimizer of the relaxed area functional $\mathcal{J}$ in* (2.14). *Then*

$$\lim_{x\to x_0} u(x) = g(x_0).$$

Finally we state the following uniqueness theorem, compare [6] [52]:

Theorem 2.38 *Let $\Omega \subset \mathbb{R}^n$ be bounded with Lipschitz continuous boundary, and assume $g \in C^0(\partial\Omega)$. Then the functional $\mathcal{J}$ in* (2.14) *has exactly one minimizer in $BV(\Omega)$.*

Chapter 3
Hilbert space methods

Let us recall a few simple facts concerning the geometry of Hilbert spaces, see e.g. [47]. We will use them to solve the Dirichlet problem for the Laplace equation (1.1) or more general linear equations and systems.

3.1 The Dirichlet principle

The abstract Dirichlet's principle

Given a Hilbert space H with inner product $(\ ,\ )$ and norm $\|\ \|$, and $L \in H^*$, its dual, define

$$\mathcal{F}(u) := \frac{1}{2}\|u\|^2 - L(u). \tag{3.1}$$

Then we have

1. $\mathcal{F}$ achieves a unique minimum $\overline{u}$ in H and every minimizing sequence converges to $\overline{u}$;

2. $\overline{u}$ is the unique solution of

$$(\varphi, \overline{u}) = L(\varphi) \quad \forall \varphi \in H.$$

   Moreover $\|\overline{u}\| = \|L\|_{H^*}$, where

$$\|L\|_{H^*} := \sup_{\substack{u \in H \\ \|u\|_H = 1}} |Lu|.$$

The theorem of Riesz

As a consequence of the Dirichlet principle we have:

1. For each $L \in H^*$ there exists a unique u_L such that $L(\cdot) = (\cdot, u_L)$; indeed this is equivalent to the Dirichlet principle and the minimizer $\overline{u}$ of (3.1) is u_L.

2. $L \to u_L$ is a continuous bijective application from H^* to H, an isometry which identifies H and H^*.

The projection theorem

Given a closed subspace V of the Hilbert space H, we have

1. for every $f \in H$ there exists a unique $u_f \in V$ such that
$$\|f - u_f\| = \inf_{v \in V} \|f - v\|;$$
2. for such a projection u_f of f we have that $(f - u_f|\varphi) = 0$ for all $\varphi \in V$.

The projection theorem is equivalent to the Dirichlet principle.

Exercise 3.1 Prove the previous statements.
[Hint: To prove the existence of a minimizer in the abstract Dirichlet principle, first use $|L(v)| \le \|L\|_{H^*}\|v\|$ to prove that

$$\mathcal{F}(v) \ge -\frac{1}{2}\|L\|^2_{H^*}, \quad \forall v \in H;$$

then use the parallelogram identity to prove that

$$\frac{1}{4}\|u - v\|^2 = \mathcal{F}(u) + \mathcal{F}(v) - 2\mathcal{F}\Big(\frac{u+v}{2}\Big),$$

so that if (u_n) is a minimizing sequence, i.e. if $\mathcal{F}(u_n) \to \inf_{v \in H} \mathcal{F}(v) > -\infty$ as $n \to \infty$, then (u_n) is a Cauchy sequence and converges to the unique minimizer of $\mathcal{F}$.]

Bilinear symmetric forms

Suppose $\mathcal{B}$ is a symmetric, continuous and coercive bilinear form on H, where continuous and coercive respectively mean that there exist $\Lambda, \lambda > 0$ such that

$$|\mathcal{B}(u, v)| \le \Lambda\|u\|\|v\|, \quad \mathcal{B}(u, u) \ge \lambda\|u\|^2, \quad \text{for all } u, v \in H.$$

Then $\mathcal{B}$ is a scalar product equivalent to the original $(\cdot, \cdot)$ and the Dirichlet principle applies, giving the following theorem.

Theorem 3.2 *The functional*

$$\mathcal{F}(u) = \frac{1}{2}\mathcal{B}(u, u) - L(u)$$

has a unique minimizer $\overline{u}$. Moreover $\overline{u}$ satisfies $\mathcal{B}(\overline{u}, v) = L(v)$ for each $v \in H$.

The Lax-Milgram theorem

In the Fifties of last century it was proved that the symmetry condition on $\mathcal{B}$ used previously is not necessary; Theorem 3.2 without the symmetry assumption is known as *Lax-Milgram's theorem*. In fact, fix $u \in H$ and $L(v) := \mathcal{B}(u,v)$. By Riesz theorem, L is uniquely represented by a vector which we call Tu:

$$\mathcal{B}(u,v) = (Tu, v).$$

Observe that T is linear and continuous and define the *symmetric*, continuous and coercive bilinear form

$$\widetilde{\mathcal{B}}(u,v) := (T^*u, T^*v).$$

Here T^* is the adjoint of T, defined by

$$(Tu, v) = (u, T^*v), \quad \text{for all } u, v \in H.$$

Minimize

$$\frac{1}{2}\widetilde{\mathcal{B}}(u,u) - L(u),$$

finding $u_L \in H$ such that for $\varphi \in H$

$$L(\varphi) = \widetilde{\mathcal{B}}(u_L, \varphi) = (T^*u_L, T^*\varphi) = (TT^*u_L, \varphi) = \mathcal{B}(T^*u_L, \varphi).$$

Thus L may be represented also by $\mathcal{B}$, or $v := T^*u_L$ solves

$$\mathcal{B}(v,\varphi) = L(\varphi) \qquad \forall \varphi \in H.$$

3.2 Sobolev spaces

Sobolev spaces play an important role in the theory of elliptic equations. For this reason we collect here a few basic definitions and facts.

3.2.1 Strong and weak derivatives

Let $\Omega \subset \mathbb{R}^n$ and $1 \le p < \infty$. We say that a function $u \in L^p(\Omega)$ has *strong derivatives* $v_1, \dots, v_n$ in L^p if there exists a sequence of functions $\{u_k\} \subset C^1(\Omega) \cap L^p(\Omega)$ such that

$$u_k \to u, \qquad D_i u_k \to v_i \qquad \text{in } L^p(\Omega), \quad i = 1, \dots, n.$$

It is easily seen that if the strong derivatives exist they are uniquely determined by u. They are denoted by $D_i u$, since they agree with the classical derivatives if u is smooth.

Definition 3.3 *The class of functions $u \in L^p(\Omega)$ that possess strong derivatives in L^p is denoted by $H^{1,p}(\Omega)$.*

$H^{1,p}(\Omega)$ is a linear space, actually, it is a Banach space with the natural norm

$$\|u\|^p_{H^{1,p}(\Omega)} := \int_\Omega |u|^p dx + \int_\Omega |Du|^p dx.$$

The closure of $C^\infty_c(\Omega)$ in $H^{1,p}(\Omega)$ is denoted by $H^{1,p}_0(\Omega)$.

We say that $u \in L^p(\Omega)$ has *weak derivatives* $v_1, \dots, v_n$ in L^p if for all $i = 1, \dots, n$

$$\int_\Omega u D_i \varphi dx = -\int_\Omega v_i \varphi dx \qquad \forall \varphi \in C^\infty_c(\Omega).$$

It is easily seen that again weak derivatives are uniquely determined by u, if they exist, and that strong derivatives are also weak derivatives.

Definition 3.4 *The class of functions $u \in L^p(\Omega)$ that possess weak derivatives in L^p is denoted by $W^{1,p}(\Omega)$.*

Exercise 3.5 Prove that $H^{1,p}(\Omega) \subset W^{1,p}(\Omega)$.

The following property is often used.

Definition 3.6 *We say that an open set $\Omega \subset \mathbb{R}^n$ has the extension property if for $1 \le p < \infty$ and for any open set $\widetilde{\Omega} \Supset \Omega$ and every function $u \in W^{1,p}(\Omega)$ there exists $\widetilde{u} \in W^{1,p}(\widetilde{\Omega})$ with $\|\widetilde{u}\|_{W^{1,p}(\widetilde{\Omega})} \le c(\Omega, \widetilde{\Omega}) \|u\|_{W^{1,p}(\Omega)}$. This is true for instance if Ω is star-shaped or C^1 or even just Lipschitz.*

Exercise 3.7 Show that the set $\Omega := ([-1,1] \times [-1,1]) \setminus (\{0\} \times [0,1])$ does not have the extension property.

Assume that Ω has the extension property and $u \in W^{1,p}(\Omega)$. Then, by mollifying $\widetilde{u}$ we find a sequence of smooth functions

$$\{u_k\} \subset C^\infty(\Omega) \cap L^p(\Omega),$$

and actually in $C^\infty(\overline{\Omega})$, converging in the $H^{1,p}$-norm to u. For a general open set Ω, stepping down the parameter of mollification when approaching $\partial\Omega$, one can show the following theorem, of N. G. Meyers and J. Serrin [72], known as the $H = W$ theorem.

Theorem 3.8 *Let Ω be an open set. Then $H^{1,p}(\Omega) = W^{1,p}(\Omega)$.*

Remark 3.9 The definitions of $H^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$ also extend to the case $p = \infty$, but we have $H^{1,\infty}(\Omega) \neq W^{1,\infty}(\Omega)$.

3.2.2 Poincaré inequalities

The well-known Poincaré inequalities show that in many cases the L^p-norm of the derivative of a Sobolev function u controls the L^p-norm of u itself.

Proposition 3.10 *For every $u \in W_0^{1,p}(\Omega)$, $1 \le p < +\infty$ we have*

$$\int_\Omega |u|^p dx \le (\operatorname{diam}\Omega)^p \int_\Omega |Du|^p dx.$$

Proof. Write $x = (x_1, x_2, \ldots, x_n) = (x_1, \widetilde{x})$ and suppose $u \in C_c^1(\Omega)$; up to a translation we can assume that

$$\Omega \subset [-a, a] \times \mathbb{R}^{n-1}, \quad a := \frac{\operatorname{diam}(\Omega)}{2}.$$

Set $u = 0$ outside Ω. Then by Jensen's inequality

$$\begin{aligned} |u(x)|^p &= \left| \int_{-a}^{x_1} Du(t, \widetilde{x}) dt \right|^p \\ &\le (2a)^p \left| \fint_{-a}^{a} Du(t, \widetilde{x}) dt \right|^p \\ &\le (2a)^{p-1} \int_{-a}^{a} |Du(t, \widetilde{x})|^p dt. \end{aligned}$$

Integrating with respect to $\tilde{x}$ and x_1 yields

$$\begin{aligned} \int_\Omega |u|^p dx &\le (2a)^{p-1} \int_\Omega dx \int_{-a}^{a} |Du(t, \widetilde{x})|^p dt \\ &= (2a)^{p-1} \int_{-a}^{a} dx_1 \int_\Omega |Du|^p dx \\ &= (2a)^p \int_\Omega |Du|^p dx. \end{aligned}$$

The claim in the general case follows by density of $C_c^1(\Omega)$ in $W_0^{1,p}(\Omega)$. □

Exercise 3.11 On the Banach space $W_0^{1,p}(\Omega)$, Ω bounded and $1 \le p < \infty$, the standard $H^{1,p}$ norm is equivalent to

$$\|u\|^p_{W_0^{1,p}} := \int_\Omega |Du|^p dx.$$

Proposition 3.12 *There is a constant $c = c(n, p)$ such that, if $\Omega \subset \mathbb{R}^n$ is a convex set of diameter ℓ and $u \in W^{1,p}(\Omega)$, then*

$$\int_\Omega |u - u_\Omega|^p dx \le c\ell^p \int_\Omega |Du|^p dx, \tag{3.2}$$

where $u_\Omega = \frac{1}{|\Omega|} \int_\Omega u$.

Proof. Since smooth functions are dense in $W^{1,p}(\Omega)$, it is enough to prove (3.2) for $u \in C^\infty(\overline{\Omega})$. By Jensen's inequality

$$\begin{aligned}\int_\Omega |u - u_\Omega|^p dx &= \int_\Omega \left| u(x) - \fint_\Omega u(y)dy \right|^p dx \\ &\leq \int_\Omega \fint_\Omega |u(x) - u(y)|^p dy dx.\end{aligned}$$

Noticing that

$$u(x) - u(y) = \sum_{i=1}^n \int_{y_i}^{x_i} \frac{\partial u}{\partial x_i}(x_1, \ldots, x_{i-1}, \xi, y_{i+1}, \ldots, y_n) d\xi,$$

integrating over $\Omega \times \Omega$ and using Jensen's inequality and Fubini's theorem we find

$$\int_\Omega \int_\Omega |u(x) - u(y)|^p dy dx \leq c(n,p)|\Omega|\ell^p \int_\Omega |Du|^p dx. \tag{3.3}$$

□

Remark 3.13 The above Proposition still holds for non-convex sets with the extension property if we replace the constant $c(n,p)\ell^p$ in (3.2) by a more general constant $c(p,\Omega)$, which can be very large even for domains of diameter 1. A proof can be given by choosing a ball B containing Ω and extending any $u \in W^{1,p}(\Omega)$ to a function $\widetilde{u} \in W^{1,p}(B)$ (the cost of this extension can be large, depending on Ω), then applying (3.2) to $\widetilde{u}$. A different proof will be given using compactness, see Proposition 3.21.

Exercise 3.14 Prove the claims in Remark 3.13. For instance, consider for $\mu > 0$ the domain

$$\Omega_\mu = B_1(\xi_-) \cup ([-2,2] \times [-\mu,\mu]) \cup B_1(\xi_+) \subset \mathbb{R}^2, \quad \xi_\pm = (\pm 2, 0).$$

Show that if (3.2) holds on Ω_μ with a constant $c(\Omega_\mu)$, then necessarily $c(\Omega_\mu) \to \infty$ as $\mu \to 0$.
[Hint: Choose $u = \pm 1$ on $B_1(\xi_\pm)$.]

Proposition 3.15 *There is a constant $c = c(n,p)$ such that, if $\Omega \subset \mathbb{R}^n$ is a convex set of diameter ℓ and $u \in W^{1,p}(\Omega)$, with $u \equiv 0$ in Ω_0 for some measurable set $\Omega_0 \subset \Omega$ with $|\Omega_0| > 0$ then*

$$\int_\Omega |u|^p dx \leq c\ell^p \frac{|\Omega|}{|\Omega_0|} \int_\Omega |Du|^p dx, \tag{3.4}$$

Proof. By Jensen's inequality

$$\begin{aligned}\int_\Omega |u|^p dx &= \int_\Omega \left| u(x) - \fint_{\Omega_0} u(y)dy \right|^p dx \\ &\leq \fint_\Omega \int_{\Omega_0} |u(x) - u(y)|^p dy dx.\end{aligned}$$

Then with (3.3) (which was proven for u smooth, but holds for every $u \in W^{1,p}(\Omega)$ by density) we conclude

$$\begin{aligned}\int_\Omega |u|^p dx &\leq \frac{1}{|\Omega_0|} \int_\Omega \int_{\Omega_0} |u(x) - u(y)|^p dy dx \\ &\leq \frac{1}{|\Omega_0|} \int_\Omega \int_\Omega |u(x) - u(y)|^p dy dx \\ &\leq c(n,p)\ell^p \frac{|\Omega|}{|\Omega_0|} \int_\Omega |Du|^p dx.\end{aligned}$$

□

Exercise 3.16 Show that in dimension 2 and higher one cannot in general replace $c\ell^p \frac{|\Omega|}{|\Omega_0|}$ by a constant independent of Ω_0.
[Hint. Consider a function $u \in W^{1,p}(B_1(0))$ with $B_1(0) \subset \mathbb{R}^n$, $1 \leq p < n$ and

$$u = 0 \text{ in } B_\varepsilon(0), \quad u = 1 \text{ in } B_1(0) \setminus B_{2\varepsilon}(0), \quad |\nabla u| \leq \frac{2}{\varepsilon},$$

and let $\varepsilon \to 0$.]

Remark 3.17 Using the same idea of Remark 3.13, also Proposition 3.15 can be extended to non-convex domains enjoying the extension property, replacing the constant $c\ell^p \frac{|\Omega|}{|\Omega_0|}$ with a more general constant $\frac{c(p,\Omega)}{|\Omega_0|}$.

3.2.3 Rellich's theorem

Theorem 3.18 *Suppose that Ω is a bounded domain with the extension property (for instance a star-shaped domain, or a domain with Lipschitz boundary). Then, for $1 \leq p < +\infty$, the following immersion*

$$W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$$

is compact.

Proof. We first show that the immersion $W^{1,p}(Q) \hookrightarrow L^p(Q)$ is compact, where Q is a cube of side ℓ. Let $\{u_k\} \subset W^{1,p}(Q)$ with $\|u_k\|_{W^{1,p}} \leq M$. Fix $\varepsilon > 0$ and let $Q_1, \dots, Q_s$ be a subdivision of Q in cubes with disjoint interiors and side σ, $\sigma < \varepsilon$. Of course

$$|(u_k)_{Q_j}| := \left| \fint_{Q_j} u_k(x)dx \right| \leq \frac{c}{\sigma^n}.$$

Consider the finite family G of simple functions

$$g(x) = n_1 \varepsilon \chi_{Q_1}(x) + \ldots + n_s \varepsilon \chi_{Q_s}(x),$$

where $n_1, \ldots, n_s$ are integers in $(-N, N)$ with $N > \frac{c}{\varepsilon \sigma^n}$ and χ_{Q_j} is the characteristic function of Q_j. We now show that each u_k has L^p-distance from one of the g in G not greater than $c_2 \varepsilon$ for some $c_2(\ell, n, p)$: this concludes the proof since then G is a finite $c_2\varepsilon$-net.

Define

$$u_k^*(x) := \sum_{j=1}^{s} (u_k)_{Q_j} \chi_{Q_j}(x).$$

Poincaré inequality (3.2) yields

$$\begin{aligned}
\int_Q |u_k - u_k^*|^p dx &\le \sum_{j=1}^{s} \int_{Q_j} |u_k - u_k^*|^p dx \\
&\le c(n,p)\sigma^p \sum_{j=1}^{s} \int_{Q_j} |Du_k|^p \\
&\le cM^p \sigma^p.
\end{aligned}$$

On the other hand, there is $g \in G$ such that for all $x \in Q$

$$|g(x) - u_k^*(x)| < \varepsilon,$$

hence

$$\|u_k - g\|_{L^p} \le \|u_k - u_k^*\|_{L^p} + \|u_k^* - g\|_{L^p} \le c_1 \sigma + \ell^n \varepsilon \le c_2 \varepsilon.$$

In order to complete the proof, use the extension property to extend (with uniform bounds on the norms) every function in $W^{1,p}(\Omega)$ to a function in $W^{1,p}(Q)$ for a cube $Q \supset\supset \Omega$ and then apply the previous part of the proof. $\square$

Remark 3.19 If the extension property does not hold we still have the compactness of the embedding

$$W_0^{1,p}(\Omega) \to L^p(\Omega),$$

since for $u \in W_0^{1,p}(\Omega)$ the function $\tilde{u}$ defined by setting $\tilde{u} = 0$ on $Q \setminus \Omega$ (again $\Omega \Subset Q$ for a fixed cube Q) and $\tilde{u} = u$ in Ω belongs to $W^{1,p}(Q)$.

Remark 3.20 Some assumption on the regularity of Ω is necessary, as the following counterexample shows. Define a domain Ω as in figure 3.1, with the squares Q_n of side length $\frac{1}{n^2}$ and the connecting aisles A_n of length $\frac{1}{n^2}$ and width $\frac{1}{n^4}$.

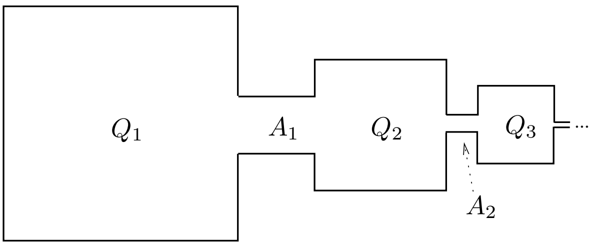

Figure 3.1: Counterexample to Rellich's theorem. The domain is bounded, but its boundary is not continuous.

Next consider the sequence of functions u_n defined by:

$$u_n := \begin{cases} n^4 & \text{on } Q_n \\ 0 & \text{on } Q_j,\ j \neq n; \\ 0 & \text{on } A_j,\ j \neq n, n-1. \end{cases}$$

On A_{n-1} and A_n set u_n to be the only affine function such that u_n is continuous on Ω. Then $\|u_n\|_{L^1(\Omega)} \geq 1$, while $\|u_n\|_{W^{1,1}(\Omega)}$ is uniformly bounded. Since $u_n \rightharpoonup 0$ in L^1, had a subsequence limit in L^1, the limit would be zero, in contrast with $\|u_n\|_{L^1(\Omega)} \geq 1$.

The following useful version of Poincaré's inequality has essentially been proven in Proposition 3.12 and Remark 3.13, but we shall give a simple alternative proof based on Rellich's theorem.

Proposition 3.21 *For every bounded and connected domain Ω with the extension property there is a constant $c = c(n, p, \Omega)$ such that for each $u \in W^{1,p}(\Omega)$ we have*

$$\int_\Omega |u - u_\Omega|^p dx \leq c \int_\Omega |Du|^p dx,$$

where as usual $u_\Omega := \fint_\Omega u dx$. When Ω is a ball of radius r or a cube of side length r, then we can take $c(n, p, \Omega) = c(n) r^p$.

Proof. Were the assertion false, we could find a sequence u_j with

$$\int_\Omega |Du_j|^p dx \to 0, \qquad (u_j)_\Omega = 0, \qquad \int_\Omega |u|^p dx = 1.$$

By Rellich's and Banach-Alaoglu's theorems we may find a subsequence (u_{n_k}) such that

$$u_{n_k} \xrightarrow{L^p} u, \quad u_{n_k} \overset{W^{1,p}}{\rightharpoonup} u.$$

In particular $Du = 0$, i.e. u is constant, $\|u\|_{L^p(\Omega)} = 1$ and $u_\Omega = 0$, which is clearly impossible.

The last claim of the proposition follows by scaling. $\square$

3.2.4 The chain rule in Sobolev spaces

The following properties of Sobolev functions are often used. As usual we will consider Ω bounded.

Proposition 3.22 *Let $f \in C^1(\mathbb{R})$ with $f' \in L^\infty(\mathbb{R})$ and $u \in W^{1,p}(\Omega)$ for some $p \in [1, \infty]$. Then $f \circ u \in W^{1,p}(\Omega)$ and*

$$D(f \circ u) = f'(u)Du.$$

Proof. It clearly suffices to prove the proposition for $p = 1$, since $u \in W^{1,p}(\Omega)$ implies $u \in W^{1,1}(\Omega)$, hence (by the case $p = 1$) the weak derivative of $f \circ u$ is $f'(u)Du$ which clearly belongs to L^p, hence $f \circ u \in W^{1,p}(\Omega)$. Hence let us assume $p = 1$. Since $|f(t)| \leq C(1 + |t|)$ for $t \in \mathbb{R}$, if easily follows that $f \circ u \in L^1(\Omega)$. Choose a sequence $(u_k) \subset C^\infty(\Omega)$ with $u_k \to u$ in $W^{1,1}_{\text{loc}}(\Omega)$. Then by the classical chain rule we have for every $\varphi \in C_c^\infty(\Omega)$

$$\int_\Omega (f \circ u_k) D\varphi dx = -\int_\Omega D(f \circ u_k)\varphi dx = -\int_\Omega f'(u_k) Du_k \varphi dx. \tag{3.5}$$

Since

$$|f \circ u_k(x) - f \circ u(x)| \leq \sup_{t \in \mathbb{R}} |f'(t)||u_k(x) - u(x)|, \quad x \in \Omega,$$

we have $f \circ u_k \to f \circ u$ in $L^1_{\text{loc}}(\Omega)$, hence

$$\lim_{k \to \infty} \int_\Omega (f \circ u_k) D\varphi dx = \int_\Omega (f \circ u) D\varphi dx.$$

Moreover, up to extracting a subsequence $u_k \to u$ a.e. in Ω, hence also $f'(u_k) \to f'(u)$ a.e. in Ω, and by the dominated convergene theorem

$$\begin{aligned}
\int_\Omega |f'(u_k)Du_k\varphi - f'(u)Du\varphi| dx &\leq \int_\Omega |f'(u_k)Du_k - f'(u_k)Du||\varphi| dx \\
&\quad + \int_\Omega |f'(u_k)Du - f'(u)Du||\varphi| dx \\
&\leq \sup_{\mathbb{R}} |f'| \int_\Omega |Du_k - Du||\varphi| dx \\
&\quad + \int_\Omega |f'(u_k) - f'(u)||Du||\varphi| dx \\
&\to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

Going back to (3.5) we see that $f'(u)Du \in L^1(\Omega)$ is the weak derivative of $f \circ u$ and we conclude. $\square$

Proposition 3.23 *Let $u \in W^{1,p}(\Omega)$, $1 \le p \le \infty$. Then $u^+, u^-, |u| \in W^{1,p}(\Omega)$, where*

$$u^+ := \max\{u, 0\}, \quad u^- := \min\{u, 0\}.$$

Moreover

$$Du^+(x) = \begin{cases} Du(x) & \text{if } u(x) > 0 \\ 0 & \text{if } u(x) \le 0, \end{cases}$$

$$Du^-(x) = \begin{cases} 0 & \text{if } u(x) \ge 0 \\ Du(x) & \text{if } u(x) < 0, \end{cases}$$

$$D|u|(x) = \begin{cases} Du(x) & \text{if } u(x) > 0 \\ 0 & \text{if } u(x) = 0 \\ -Du(x) & \text{if } u(x) < 0. \end{cases}$$

Finally given any $t \in \mathbb{R}$, $Du = 0$ almost everywhere on $\{x \in \Omega : u(x) = t\}$.

Proof. As in the proof of Proposition 3.22, it suffices to consider the case $p = 1$. We first deal with u^+. For any $\varepsilon > 0$ set

$$f_\varepsilon(t) = \begin{cases} \sqrt{t^2 + \varepsilon^2} - \varepsilon & \text{if } t > 0 \\ 0 & \text{if } t \le 0. \end{cases}$$

Notice that $|f'| \le 1$. Then by Proposition 3.22 we have $f_\varepsilon \circ u \in W^{1,1}(\Omega)$ and

$$\int_\Omega (f_\varepsilon \circ u) D\varphi dx = -\int_{\{x\in\Omega : u(x)>0\}} \frac{uDu}{\sqrt{u^2+\varepsilon^2}} \varphi dx, \quad \forall \varphi \in C_c^\infty(\Omega).$$

Taking the limit as $\varepsilon \to 0$ and using the dominated convergence theorem we infer

$$\int_\Omega u^+ D\varphi dx = -\int_{\{x\in\Omega : u(x)>0\}} Du\varphi dx, \quad \forall \varphi \in C_c^\infty(\Omega),$$

hence

$$u^+ \in W^{1,1}(\Omega) \quad \text{and} \quad Du^+ = Du\chi_{\{x\in\Omega : u(x)>0\}}.$$

Since $u^- = -(-u)^+$ and $|u| = u^+ - u^-$, also the claims about u^- and $|u|$ easily follow.

In order to prove the last claim, assume without loss of generality that $t = 0$, and simply observe that $u = u^+ + u^-$, hence $Du = Du^+ + Du^-$, and both Du^+ and Du^- vanish a.e. on $\{x \in \Omega : u(x) = 0\}$. □

Proposition 3.24 *Let $f \in C^0(\mathbb{R})$ be piecewise C^1, i.e. there are points $t_1, \dots, t_\ell$ such that $f \in C^1((-\infty, t_1])$, $f \in C^1([t_1, t_2])$, etc... Assume also that $f' \in L^\infty(\mathbb{R})$. Then for every $u \in W^{1,p}(\Omega)$, $1 \le p \le \infty$, we have $f \circ u \in W^{1,p}(\Omega)$ and*

$$D(f\circ u)(x) = \begin{cases} f'(u(x))Du(x) & \text{if } u(x) \notin \{t_1, \dots, t_\ell\} \\ 0 & \text{if } u(x) \in \{t_1, \dots, t_\ell\}. \end{cases}$$

Proof. Working by induction we can assume $\ell = 1$ and there is no loss of generality in assuming that $t_1 = 0$. We can find $f_1, f_2 \in C^1(\mathbb{R})$ with $f_1', f_2' \in L^\infty$ and $f_1(t) = f(t)$ for $t \geq 0$, $f_2(t) = f(t)$ for $t \leq 0$. Then

$$f \circ u = f_1 \circ u^+ + f_2 \circ u^-,$$

and the claim follows from Propositions 3.22 and 3.23. □

Corollary 3.25 *Given $u \in W^{1,p}(\Omega)$, $1 \leq p \leq \infty$ and $k \in \mathbb{R}$ we have $(u-k)^+ \in W^{1,p}(\Omega)$ and*

$$D(u-k)^+(x) = \begin{cases} Du(x) & \text{if } u(x) > k \\ 0 & \text{if } u(x) \leq k. \end{cases}$$

Proof. Apply Proposition 3.24 with

$$f(t) = \begin{cases} t-k & \text{if } t \geq k \\ 0 & \text{if } t \leq k. \end{cases}$$

□

3.2.5 The Sobolev embedding theorem

For later use, we recall without proof (see for instance [2] and compare Theorem 7.29)

Theorem 3.26 (Sobolev-Morrey) *Assume that Ω has the extension property (Definition 3.6) and let $p \in [1, +\infty)$, $k \geq 1$. Then*

1. *if $kp < n$, we have a continuous immersion*

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \forall q \in [p, q^*], \quad q^* := \frac{np}{n-kp}, \tag{3.6}$$

which is also compact for $q \in [p, q^*)$ if Ω is bounded; moreover

$$\|u\|_{L^{p^*}} \leq c(p,k,\Omega)\|u\|_{W^{k,p}}, \quad \text{for every } u \in W^{k,p}(\Omega); \tag{3.7}$$

2. *if $kp = n$ we have a continuous (actually compact if Ω is bounded) immersion*

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \forall q \in [p, +\infty), \tag{3.8}$$

and

$$\|u\|_{L^q} \leq c(p,q,k,\Omega)\|u\|_{W^{k,p}}, \quad \text{for every } u \in W^{k,p}(\Omega);$$

3. *if $kp > n + pr$ for some $r \in \mathbb{N}$, we have a continuous (actually compact if Ω is bounded) immersion*

$$W^{k,p}(\Omega) \hookrightarrow C^r(\overline{\Omega}), \tag{3.9}$$

and

$$\|u\|_{C^r} \leq c(p,\Omega)\|u\|_{W^{k,p}}, \quad \textit{for every } u \in W^{1,p}(\Omega).$$

Imbeddings (3.6) and (3.8) are essentially due to Sobolev [101], [102], (Kondracov for the compactness) while imbedding (3.9) is due to Morrey [76].

3.2.6 The Sobolev-Poincaré inequality

Mixing the Sobolev inequality (3.7) and the Poincaré inequality of Proposition 3.21 one obtains

Proposition 3.27 *For every bounded and connected domain Ω with the extension property there is a constant $c = c(p,\Omega)$ such that for every $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$, we have*

$$\left(\int_\Omega |u - u_\Omega|^{p^*} dx\right)^{\frac{1}{p^*}} \leq c \left(\int_\Omega |Du|^p dx\right)^{\frac{1}{p}},$$

where $u_\Omega := \fint_\Omega u dx$.

Proof. Applying (3.7) to $u - u_\Omega$, and then Proposition 3.21 we estimate

$$\begin{aligned}\left(\int_\Omega |u - u_\Omega|^{p^*} dx\right)^{\frac{1}{p^*}} &\leq c \left(\int_\Omega |Du|^p dx\right)^{\frac{1}{p}} + c \left(\int_\Omega |u - u_\Omega|^p dx\right)^{\frac{1}{p}} \\ &\leq c_1 \left(\int_\Omega |Du|^p dx\right)^{\frac{1}{p}}.\end{aligned}$$

□

Remark 3.28 Using Propositions 3.10 or 3.15 instead of Proposition 3.21, one can state similar versions of the Sobolev-Poincaré inequality for functions in $W^{1,p}_0$, or for functions vanishing on subsets of positive measure.

3.3 Elliptic equations: existence of weak solutions

We discuss here the solvability of Dirichlet and Neumann boundary value problems for linear elliptic equations in Sobolev spaces as consequence of Lax-Milgram's theorem and in fact of the abstract Dirichlet principle. In the next section we shall deal with linear systems.

3.3.1 Dirichlet boundary condition

As usual $\Omega \subset \mathbb{R}^n$ is a bounded open domain. It is understood that we sum over repeated indices.

Theorem 3.29 *Let $A^{\alpha\beta} \in L^\infty(\Omega)$ be elliptic and bounded, that is for some $\lambda, \Lambda > 0$*

$$\lambda|\xi|^2 \leq A^{\alpha\beta}(x)\xi_\alpha\xi_\beta \leq \Lambda|\xi|^2, \quad \forall x \in \Omega. \tag{3.10}$$

Then, for each $g \in W^{1,2}(\Omega)$ and $f, f^\alpha \in L^2(\Omega)$, $\alpha = 1, \ldots, n$ there exists one and only one weak solution *$u \in W^{1,2}(\Omega)$ to the Dirichlet problem*

$$\begin{cases} -D_\beta(A^{\alpha\beta}D_\alpha u) = f_0 - D_\alpha f^\alpha & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{3.11}$$

meaning that $u - g \in W^{1,2}_0(\Omega)$ and

$$\int_\Omega A^{\alpha\beta}D_\alpha u D_\beta\varphi dx = \int_\Omega \big(f_0\varphi + f^\alpha D_\alpha\varphi\big)dx$$

for all $\varphi \in W^{1,2}_0(\Omega)$ or, equivalently, for all $\varphi \in C^\infty_c(\Omega)$.

If in addition $A^{\alpha\beta} = A^{\beta\alpha}$, then the solution u is the unique minimizer of the functional

$$\mathcal{F}(v) = \frac{1}{2}\int_\Omega A^{\alpha\beta}D_\alpha v D_\beta v dx - \int_\Omega f_0 v dx - \int_\Omega f^\alpha D_\alpha v dx \tag{3.12}$$

in the class

$$\mathcal{A} = \{v \in W^{1,2}(\Omega) : v - g \in W^{1,2}_0(\Omega)\}.$$

Proof. Step 1. Define on the Hilbert space $H := W^{1,2}_0(\Omega)$ the bilinear form

$$\mathcal{B}(v, w) := \int_\Omega A^{\alpha\beta}D_\alpha v D_\beta w dx,$$

which is coercive thanks the Poincaré inequality and the ellipticity of $A^{\alpha\beta}$. Set $\widetilde{u} = u - g$, so that the initial problem is reduced to finding $\widetilde{u} \in H$ such that for every $v \in W^{1,2}_0(\Omega)$

$$\int_\Omega A^{\alpha\beta}D_\alpha\widetilde{u}D_\beta v dx = \int_\Omega f_0 v dx + \int_\Omega [A^{\alpha\beta}D_\alpha g + f^\beta]D_\beta v dx =: L(v).$$

Notice that $L \in H^*$, see also Remark 3.30. By Lax-Milgram's theorem there is exactly one $\widetilde{u}$ such that

$$\mathcal{B}(\widetilde{u}, v) = L(v), \quad \forall v \in W^{1,2}_0(\Omega),$$

thus $u = \widetilde{u} + g$ is the unique solution to (3.11).

Step 2. If $A^{\alpha\beta}$ is symmetric, the second derivative of $\mathcal{F}$ is

$$D^2\mathcal{F}_u(v,w) = \int_\Omega A^{\alpha\beta} D_\alpha v D_\beta w dx,$$

so that $\mathcal{F}$ is convex on $W^{1,2}$ and strictly convex on $\mathcal{A}$. Thus a critical point of $\mathcal{F}$ in $\mathcal{A}$ is the only minimizer (if it exists) of $\mathcal{F}$ in $\mathcal{A}$. On the other hand, the Euler-Lagrange equation of $\mathcal{F}$ is

$$\int_\Omega A^{\alpha\beta} D_\alpha u D_\beta \varphi dx - \int_\Omega f_0 \varphi dx - \int_\Omega f^\alpha D_\alpha \varphi dx = 0,$$

that is (3.11). This implies that a solution of (3.11) is the unique of $\mathcal{F}$ in $\mathcal{A}$. $\square$

Remark 3.30 The right-hand-side of (3.11) represents a generic element of the dual space $W_0^{1,2}(\Omega)^*$, since every continuous linear functional $L : W_0^{1,2}(\Omega) \to \mathbb{R}$ is of the form

$$L(\varphi) := \int_\Omega f_0 \varphi dx + \int_\Omega f^\alpha D_\alpha \varphi dx,$$

for some $f_0, f^\alpha \in L^2(\Omega)$, $\alpha = 1, \ldots, n$.

3.3.2 Neumann boundary condition

The Dirichlet boundary condition makes the functional $\mathcal{F}$ in (3.12) coercive on the class $\mathcal{A}$. Slightly modifying $\mathcal{F}$, we make it coercive on all of $W^{1,2}(\Omega)$; consequently a Neumann boundary condition *naturally* arises.

Theorem 3.31 *Let $A^{\alpha\beta} \in L^\infty(\Omega)$ be elliptic and bounded as in* (3.10). *Then for every $\gamma > 0$, $f_0, f^\alpha \in L^2(\Omega)$, $\alpha = 1, \ldots, n$, there exists a unique* weak solution *to*

$$\begin{cases} -D_\beta(A^{\alpha\beta} D_\alpha u) + \gamma u = f_0 - D_\alpha f^\alpha & \text{in } \Omega \\ A^{\alpha\beta} D_\alpha u \nu_\beta = f^\beta \nu_\beta & \text{on } \partial\Omega, \end{cases} \tag{3.13}$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the exterior unit normal to $\partial\Omega$, if Ω is smooth enough, see Remark 3.32.

If $A^{\alpha\beta} = A^{\beta\alpha}$, then such a solution is the unique minimizer in $W^{1,2}(\Omega)$ of

$$\mathcal{F}(v) = \frac{1}{2}\int_\Omega A^{\alpha\beta} D_\alpha v D_\beta v dx + \frac{\gamma}{2}\int_\Omega v^2 dx - \int_\Omega f_0 v dx - \int_\Omega f^\alpha D_\alpha v dx. \tag{3.14}$$

Remark 3.32 Unless we provide some smoothness assumption, (3.13) means, by definition,

$$\int_\Omega A^{\alpha\beta} D_\alpha u D_\beta \varphi dx + \gamma \int_\Omega u\varphi dx = \int_\Omega f_0 \varphi dx + \int_\Omega f^\alpha D_\alpha \varphi dx, \quad \text{for every } \varphi \in W^{1,2}(\Omega). \tag{3.15}$$

Observe that the test function φ need not vanish on $\partial\Omega$, compare the proof of Theorem 3.31.

Proof of Theorem 3.31. The bilinear form

$$\mathcal{B}(v,w) := \int_\Omega A^{\alpha\beta} D_\alpha v D_\beta w dx + \gamma \int_\Omega vw dx$$

is coercive, being $\mathcal{B}(u,u) \geq \min\{\lambda, \gamma\} \cdot \|u\|^2_{W^{1,2}}$. Set

$$L(v) := \int_\Omega f_0 v dx + \int_\Omega f^\alpha D_\alpha v dx.$$

By Lax-Milgram theorem applied to the Hilbert space $H = W^{1,2}(\Omega)$, a solution to the equation

$$\mathcal{B}(u,\varphi) = L(\varphi), \quad \text{for every } \varphi \in W^{1,2}(\Omega),$$

i.e. equation (3.15), exists and is unique. Such a solution is a minimizer if $A^{\alpha\beta}$ is symmetric, as in Theorem 3.29. $\square$

To obtain the Neumann boundary condition in (3.13) at least formally we integrate by parts in (3.15) and get

$$\int_\Omega \Big[- D_\beta(A^{\alpha\beta} D_\alpha u) + \gamma u - f_0 + D_\alpha f^\alpha \Big] \varphi dx + \int_{\partial\Omega} \Big[A^{\alpha\beta} D_\alpha u \nu_\beta - f^\alpha \nu_\alpha \Big] \varphi d\mathcal{H}^{n-1} = 0,$$

for every $\varphi \in W^{1,2}(\Omega)$. When $\varphi \in W^{1,2}_0(\Omega)$, the second term on the left hand side vanishes, giving

$$-D_\beta(A^{\alpha\beta} D_\alpha u) + \gamma u = f_0 - D_\alpha f^\alpha$$

in the sense of distributions (or pointwise if u, f_0 and f^α are regular enough), therefore when φ is generic we infer

$$\int_{\partial\Omega} \Big[A^{\alpha\beta} D_\alpha u \nu_\beta - f^\alpha \nu_\alpha \Big] \varphi d\mathcal{H}^{n-1} = 0, \quad \forall \varphi \in W^{1,2}(\Omega),$$

that yields the boundary condition in (3.13).

Also notice that geometrically this boundary condition means that on $\partial\Omega$ the vector field $A^{\alpha\beta} D_\alpha u - f^\alpha$ is tangent to $\partial\Omega$ (when the objects involved are regular enough).

Exercise 3.33 Making the above argument precise, show that the boundary condition in (3.13) holds pointwise if Ω, $A^{\alpha\beta}$, f_0, f^α and u are regular enough.

Exercise 3.34 When $A^{\alpha\beta} = \delta^{\alpha\beta}$, that is when the higher order part of equation (3.13) is the Laplacian, the boundary condition becomes $\frac{\partial u}{\partial \nu} = f^\alpha \nu_\alpha$.

Remark 3.35 More generally, consider a variational integral

$$\mathcal{F}(u) := \int_\Omega F(x, u, Du)dx,$$

with $\partial\Omega$ and F of class C^1. The Euler-Lagrange equation of $\mathcal{F}$,

$$\frac{d}{dt}\mathcal{F}(u + t\varphi)\Big|_{t=0} = 0$$

is

$$\int_\Omega \big\{F_{p_\alpha}(x, u, Du)D_\alpha\varphi + F_u(x, u, Du)\varphi\big\}dx = 0, \quad \forall\varphi \in C^\infty(\overline{\Omega}).$$

The natural boundary condition arising from minimizing $\mathcal{F}$ in $W^{1,2}(\Omega)$ in this case is

$$\nu_\alpha F_{p_\alpha}(x, u, Du) = 0 \text{ on } \partial\Omega.$$

Something similar holds for systems, as the reader can verify.

3.4 Elliptic systems: existence of weak solutions

Let us now discuss *systems* of linear equations.

3.4.1 The Legendre and Legendre-Hadamard ellipticity conditions

Definition 3.36 *A matrix of coefficients* $\big(A_{ij}^{\alpha\beta}\big)_{1\le i,j\le m}^{1\le\alpha,\beta\le n}$ *is said to satisfy*

1. *the* very strong ellipticity *condition, or the* Legendre *condition, if there is a* $\lambda > 0$ *such that*

$$A_{ij}^{\alpha\beta}\xi_\alpha^i\xi_\beta^j \ge \lambda|\xi|^2, \quad \forall\xi \in \mathbb{R}^{m\times n}; \tag{3.16}$$

2. *the* strong ellipticity condition, *or the* Legendre-Hadamard *condition, if there is a* $\lambda > 0$ *such that*

$$A_{ij}^{\alpha\beta}\xi_\alpha\xi_\beta\eta^i\eta^j \ge \lambda|\xi|^2|\eta|^2, \quad \forall\xi \in \mathbb{R}^n, \forall\eta \in \mathbb{R}^m. \tag{3.17}$$

Remark 3.37 The Legendre condition implies the Legendre-Hadamard condition: just insert $\xi^i_\alpha := \xi_\alpha \eta^i$ in (3.16). The converse is trivially true in case $m = 1$ or $n = 1$, but is false in general as the following example shows.

Example 3.38 Let $n = m = 2$ and define for some $\lambda > 0$

$$A^{12}_{12} = A^{21}_{21} = 1, \quad A^{21}_{12} = A^{12}_{21} = -1,$$

$$A^{11}_{11} = A^{11}_{22} = A^{22}_{11} = A^{22}_{22} = \lambda,$$

so that

$$A^{\alpha\beta}_{ij} \xi^i_\alpha \xi^j_\beta = \det(\xi) + \lambda |\xi|^2.$$

Since $\det(\xi_\alpha \eta^i) = 0$ for every choice of vectors $\xi, \eta \in \mathbb{R}^2$, we have

$$A^{\alpha\beta}_{ij} \xi_\alpha \xi_\beta \eta^i \eta^j = \lambda |\xi|^2 |\eta|^2, \quad \forall \xi \in \mathbb{R}^n, \forall \eta \in \mathbb{R}^m,$$

and

$$A^{\alpha\beta}_{ij} \xi^i_\alpha \xi^j_\beta = \det(\xi^i_\alpha) + \lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^{m\times n}.$$

This shows that for every $\lambda > 0$ the Legendre-Hadamard condition is satisfied, while for $\lambda \leq 1/2$ the Legendre condition is not (choose e.g. $\xi^1_1 = 1$, $\xi^2_2 = -1$, $\xi^1_2 = \xi^2_1 = 0$).

3.4.2 Boundary value problems for very strongly elliptic systems

Theorems analogous to 3.29 and 3.31 hold true trivially for very strongly elliptic systems.

Theorem 3.39 *Let $A^{\alpha\beta}_{ij} \in L^\infty(\Omega)$ be bounded and satisfy the Legendre condition, that is for some $\lambda > 0$ (3.16) holds. Then, for each $g \in W^{1,2}(\Omega, \mathbb{R}^m)$ and $f_i, f^\alpha_i \in L^2(\Omega)$, $i = 1, \dots, m$, $\alpha = 1, \dots, n$, there exists one and only one weak solution $u \in W^{1,2}(\Omega, \mathbb{R}^m)$ to the Dirichlet problem*

$$\begin{cases} -D_\beta(A^{\alpha\beta}_{ij} D_\alpha u^j) = f_i - D_\alpha f^\alpha_i & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.18}$$

As before we have to interpret (3.18) *in the weak sense as follows: the first equation means*

$$\int_\Omega A^{\alpha\beta}_{ij} D_\alpha u^j D_\beta \varphi^i dx = \int_\Omega (f_i \varphi^i + f^\alpha_i D_\alpha \varphi^i) dx, \quad \textit{for all } \varphi \in W^{1,2}_0(\Omega, \mathbb{R}^m)$$

and the boundary condition simply corresponds to

$$u - g \in W^{1,2}_0(\Omega, \mathbb{R}^m).$$

If $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$, then the solution u is the unique minimizer of the functional

$$\mathcal{F}(v) = \frac{1}{2}\int_\Omega A_{ij}^{\alpha\beta} D_\alpha v^i D_\beta v^j dx - \int_\Omega f_i v^i dx - \int_\Omega f_i^\alpha D_\alpha v^i dx \tag{3.19}$$

in the class

$$\mathcal{A} = \{v \in W^{1,2}(\Omega, \mathbb{R}^m) : v - g \in W_0^{1,2}(\Omega, \mathbb{R}^m)\}.$$

Theorem 3.40 *Let $A_{ij}^{\alpha\beta}$ be very strongly elliptic and bounded, $f_i, f_i^\alpha \in L^2(\Omega)$ and $\gamma > 0$. Then there exists a unique solution to*

$$\begin{cases} -D_\beta(A_{ij}^{\alpha\beta} D_\alpha u^j) + \gamma u^i = f_i - D_\alpha f_i^\alpha & \text{in } \Omega \\ A_{ij}^{\alpha\beta} D_\beta u^j \nu_\alpha = f_i^\alpha \nu_\alpha & \text{on } \partial\Omega. \end{cases} \tag{3.20}$$

As in Theorem 3.31, without further regularity (3.15) *means*

$$\int_\Omega A_{ij}^{\alpha\beta} D_\alpha u^j D_\beta \varphi^i dx + \gamma \int_\Omega u^i \varphi^i dx = \int_\Omega f_i \varphi^i dx + \int_\Omega f_i^\alpha D_\alpha \varphi^i dx, \quad \text{for every } \varphi \in W^{1,2}(\Omega, \mathbb{R}^m). \tag{3.21}$$

If $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$, this solution is the unique minimizer of

$$\mathcal{F}(v) = \frac{1}{2}\int_\Omega A_{ij}^{\alpha\beta} D_\alpha v^i D_\beta v^j dx + \frac{\gamma}{2}\int_\Omega |v|^2 dx - \int_\Omega f_i v^i dx - \int_\Omega f_i^\alpha D_\alpha v^i dx \tag{3.22}$$

in $W^{1,2}(\Omega, \mathbb{R}^m)$.

Proof of Theorems 3.39 and 3.40. The strong ellipticity condition gives the coercivity on $W_0^{1,2}(\Omega, \mathbb{R}^m)$ of the bilinear form

$$\mathcal{B}(v, w) := \int_\Omega A_{ij}^{\alpha\beta} D_\alpha v^i D_\beta w^j dx,$$

and the coercivity on $W^{1,2}(\Omega, \mathbb{R}^m)$ of

$$\mathcal{B}(v, w) := \int_\Omega A_{ij}^{\alpha\beta} D_\alpha v^i D_\beta w^j dx + \gamma \int_\Omega v^i u^i dx.$$

Then we can repeat the proofs of Theorems 3.29 and 3.31. □

3.4.3 Strongly elliptic systems: Gårding's inequality

If the coefficients $A_{ij}^{\alpha\beta}$ satisfy only the Legendre-Hadamard condition (3.17), in general the bilinear form

$$\mathcal{B}(u, v) := \int_\Omega A_{ij}^{\alpha\beta} D_\alpha u^i D_\beta v^j dx$$

is not coercive on $W_0^{1,2}(\Omega,\mathbb{R}^m)$: we shall see that, under suitable hypothesis, $\mathcal{B}$ is weakly coercive, according to the following definition.

Definition 3.41 *A bilinear form $\mathcal{B}$ on $W_0^{1,2}(\Omega,\mathbb{R}^m)$ is said to be* weakly coercive *if there exist $\lambda_0>0$ and $\lambda_1\geq 0$ such that*

$$\mathcal{B}(u,u)\geq \lambda_0\int_\Omega |Du|^2dx-\lambda_1\int_\Omega |u|^2dx. \tag{3.23}$$

Theorem 3.42 (Gårding's inequality) *Assume that $A_{ij}^{\alpha\beta}$ are uniformly continuous on Ω and that they satisfy the Legendre-Hadamard condition* (3.17) *for some $\lambda>0$ independent of $x\in\Omega$. Then the bilinear form on $W_0^{1,2}(\Omega,\mathbb{R}^m)$ defined by*

$$\mathcal{B}(u,v):=\int_\Omega A_{ij}^{\alpha\beta}D_\alpha u^i D_\beta v^j\,dx$$

is weakly coercive. If $A_{ij}^{\alpha\beta}$ are constant then $\mathcal{B}$ is in fact coercive.

Proof. The idea is to use the Fourier tranform to decouple the terms $D_\alpha u^i$ and $D_\beta u^j$ and then apply the Legendre-Hadamard condition. Recall that for a given function $f\in L^2(\Omega)$ the Fourier transform of f is

$$\widehat{f}(x):=\int_{\mathbb{R}^n} f(y)e^{-2\pi i x\cdot y}dy.$$

The Fourier transform satisfies

$$\widehat{D_\alpha f}(x)=2\pi i x_\alpha \widehat{f} \tag{3.24}$$

and the Parseval identities

$$(\widehat{f},\widehat{g})_{L^2}=(f,g)_{L^2},\quad \|\widehat{f}\|_{L^2}=\|f\|_{L^2}, \tag{3.25}$$

where, since $\widehat{f}$ and $\widehat{g}$ take values into $\mathbb{C}$, we define

$$(\widehat{f},\widehat{g})_{L^2}=\int_{\mathbb{R}^n}\widehat{f}(\xi)\overline{\widehat{g}}(\xi)d\xi,$$

where $\overline{\widehat{g}}$ is the complex conjugate of $\widehat{g}$.
Step 1. Assume $A_{ij}^{\alpha\beta}$ are constant. For $u\in W_0^{1,2}(\Omega,\mathbb{R}^m)$, that we think of as being extended to zero outside Ω, we bound with (3.24), (3.25) and the Legendre-Hadamard condition

$$\begin{aligned}
\mathcal{B}(u,u) &= A_{ij}^{\alpha\beta}\int_{\mathbb{R}^n}\widehat{D_\alpha u^i}\,\overline{\widehat{D_\beta u^j}}d\xi=(2\pi)^2A_{ij}^{\alpha\beta}\int_{\mathbb{R}^n}\xi_\alpha\xi_\beta\widehat{u}^i\overline{\widehat{u}^j}d\xi\\
&\geq (2\pi)^2\lambda\int_{\mathbb{R}^n}|\xi|^2|\widehat{u}|^2d\xi=(2\pi)^2\lambda\delta^{\alpha\beta}\delta_{ij}\int_{\mathbb{R}^n}\xi_\alpha\xi_\beta\widehat{u}^i\overline{\widehat{u}^j}d\xi\\
&= \lambda\delta^{\alpha\beta}\delta_{ij}\int_{\mathbb{R}^n}\widehat{D_\alpha u^i}\,\overline{\widehat{D_\beta u^j}}d\xi=\lambda\int_{\mathbb{R}^n}\left|\widehat{Du}\right|^2d\xi\\
&= \lambda\int_{\mathbb{R}^n}|Du|^2dx.
\end{aligned} \tag{3.26}$$

Step 2. Let us now drop the assumption that A is constant and take $u \in W_0^{1,2}(\Omega, \mathbb{R}^m)$ supported in $B_r(x_0)$ for some $x_0 \in \Omega$ and r small. Then, by step 1 we get

$$\begin{aligned}\mathcal{B}(u,u) &= A_{ij}^{\alpha\beta}(x_0)\int_\Omega D_\alpha u^i D_\beta u^j\,dx \\ &\quad + \int_{B_r(x_0)} \left[A_{ij}^{\alpha\beta}(x) - A_{ij}^{\alpha\beta}(x_0)\right] D_\alpha u^i D_\beta u^j\,dx \\ &\geq \lambda \int_\Omega |Du|^2 dx - \omega(r)\int_\Omega |Du|^2 dx,\end{aligned}$$

where

$$\omega(r) := \sup_{x,y\in\Omega,\ |x-y|\leq r}\ \max_{\alpha,\beta,i,j} |A_{ij}^{\alpha\beta}(x) - A_{ij}^{\alpha\beta}(y)|$$

is the modulus of continuity of $A_{ij}^{\alpha\beta}$.

Step 3. Now choose $r > 0$ such that $\lambda_0^* := \lambda - \omega(r) > 0$ and cover $\overline{\Omega}$ with finitely many balls $B_r(x_k)$, $k = 1, \dots, N$. Fix a partition of the unity φ_k^2 subordinated to the covering $\{B_r(x_k)\}$, i.e. non-negative smooth functions φ_k with

$$\text{support}(\varphi_k) \subset B_r(x_k), \qquad \sum_{k=1}^N \varphi_k^2 = 1 \text{ on } \overline{\Omega},$$

so that for $u \in W_0^{1,2}(\Omega, \mathbb{R}^m)$ (again extended to 0 outside Ω)

$$u = \sum_{k=1}^N (u\varphi_k^2), \qquad \text{support}(u\varphi_k^2) \subset B_r(x_k).$$

Then

$$\begin{aligned}\mathcal{B}(u,u) &= \int_\Omega A_{ij}^{\alpha\beta} \sum_{k=1}^N \varphi_k^2 D_\alpha u^i D_\beta u^j\,dx \\ &= \sum_{k=1}^N \int_\Omega A_{ij}^{\alpha\beta} D_\alpha(\varphi_k u^i) D_\beta(\varphi_k u^j) dx \\ &\quad - \sum_{k=1}^N \int_\Omega A_{ij}^{\alpha\beta} u^i u^j D_\alpha\varphi_k D_\beta\varphi_k dx \\ &\quad - \sum_{k=1}^N \int_\Omega A_{ij}^{\alpha\beta} \varphi_k u^j D_\alpha u^i D_\beta \varphi_k dx \\ &\quad - \sum_{k=1}^N \int_\Omega A_{ij}^{\alpha\beta} \varphi_k u^i D_\alpha \varphi_k D_\beta u^j\,dx.\end{aligned} \tag{3.27}$$

By Step 2

$$\sum_{k=1}^{N}\int_\Omega A_{ij}^{\alpha\beta} D_\alpha(\varphi_k u^i)_\beta(\varphi_k u^j)dx \geq \lambda_0^* \sum_{k=1}^{N}\int_\Omega |D(\varphi_k u)|^2 dx$$

$$= \lambda_0^* \sum_{k=1}^{N}\int_\Omega \Big[\varphi_k^2|Du|^2 + |u|^2|D\varphi_k|^2 + 2\varphi_k u^i D_\alpha\varphi_k D_\alpha u^i\Big]dx.$$

Applying Young's inequality $2ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon}$ to the last term ($a = |Du|$, $b = \varphi_k|u||D\varphi_k|$) we get

$$\sum_{k=1}^{N}\int_\Omega A_{ij}^{\alpha\beta} D_\alpha(\varphi_k u^i)D_\beta(\varphi_k u^j)dx \geq (\lambda_0^* - \varepsilon)\int_\Omega |Du|^2 dx - c_\varepsilon \int_\Omega |u|^2 dx.$$

The last three terms in (3.27) are estimated as follows:

$$\left|\int_\Omega A_{ij}^{\alpha\beta} u^i u^j D_\alpha\varphi_k D_\beta\varphi_k dx\right| \leq c \sup_\Omega |A| \int_\Omega |u|^2 dx,$$

and using Young's inequality as before

$$\left|\int_\Omega A_{ij}^{\alpha\beta}\varphi_k u^j D_\alpha u^i D_\beta\varphi_k dx\right| \leq \varepsilon\int_\Omega |Du|^2 dx + c\frac{\sup|A|^2}{\varepsilon}\int_\Omega |u|^2 dx.$$

Going back to (3.27) and choosing $\varepsilon < \frac{\lambda_0^*}{3}$ we conclude that $\mathcal{B}$ is weakly coercive with $\lambda_0 = \lambda_0^* - 3\varepsilon$ and $\lambda_1 = \lambda_1(\Omega, \omega, \varepsilon)$ in (3.23). □

Exercise 3.43 Show that, under the assumptions of Theorem 3.42, the bilinear form

$$\mathcal{B}(u,v) := \int_\Omega A_{ij}^{\alpha\beta} D_\alpha u^i D_\beta v^j dx + \int_\Omega b_{ij}^\alpha D_\alpha u^i v^j dx + \int_\Omega c_{ij} u^i v^j dx$$

is weakly coercive in $W_0^{1,2}(\Omega, \mathbb{R}^m)$ provided, for instance, $b_{ij}^\alpha, c_{ij} \in L^\infty(\Omega)$.

Corollary 3.44 *Let $A_{ij}^{\alpha\beta}$ be as in Theorem 3.42. Then, for any $g \in W^{1,2}(\Omega, \mathbb{R}^m)$ and $f_i, f_i^\alpha \in L^2(\Omega)$, $\alpha = 1, \dots, n$, $i = 1, \dots, m$, there exists a unique weak solution $u \in W^{1,2}(\Omega, \mathbb{R}^m)$ to the Dirichlet problem*

$$\begin{cases} -D_\beta(A_{ij}^{\alpha\beta} D_\alpha u^j) + \gamma u^j = f_i - D_\alpha f_i^\alpha & \text{in } \Omega \\ u = g & \text{in } \partial\Omega \end{cases} \tag{3.28}$$

for γ sufficiently large.

Proof. Write the equation in (3.28) for $\widetilde{u} := u - g$. Since the bilinear form

$$\mathcal{B}(u,v) := \int_\Omega A_{ij}^{\alpha\beta} D_\alpha u^i D_\beta v^j + \gamma \int_\Omega u \cdot v$$

is coercive for γ large thanks to Theorem 3.42, the conclusion follows as in Theorem 3.29. □

Fix γ as in Corollary 3.44. The linear map that to $(g, f_i, f_i^\alpha)_{i=1,\dots,m}^{\alpha=1,\dots,n}$ associates the solution $u \in W^{1,2}(\Omega, \mathbb{R}^m) \subset L^2(\Omega, \mathbb{R}^m)$ to (3.28) is compact, therefore we may conclude from the theory of compact operators that (3.28) is uniquely solvable for all g, f_i and f_i^α except for at most a discrete countable set of values of γ, the eigenvalues, which lie in $(-\infty, \lambda_0)$, for a suitable $\lambda_0 < \infty$.

For later use we state a simplified but useful version of Theorem 3.42, corresponding to the estimate in (3.26).

Proposition 3.45 *Let the coefficients $A_{ij}^{\alpha\beta}$ be constant and satisfy the Legendre-Hadamard condition* (3.17) *for some $\lambda > 0$. Then*

$$\mathcal{B}(u,u) := \int_\Omega A_{ij}^{\alpha\beta} D_\alpha u^i D_\beta u^j \, dx \geq \lambda \int_\Omega |Du|^2 dx$$

for all $u \in W_0^{1,2}(\Omega, \mathbb{R}^m)$.

Corollary 3.46 *Let $A_{ij}^{\alpha\beta}$ be constant and satisfy the Legendre-Hadamard condition* (3.17)*. Then, for every $g \in W^{1,2}(\Omega, \mathbb{R}^m)$ and $f_i, f_i^\alpha \in L^2(\Omega)$, $\alpha = 1, \dots, n$, $i = 1, \dots, m$, there exists one and only one weak solution $u \in W^{1,2}(\Omega, \mathbb{R}^m)$ to the Dirichlet problem*

$$\begin{cases} -D_\beta(A_{ij}^{\alpha\beta} D_\alpha u^j) = f_i - D_\alpha f_i^\alpha & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Chapter 4
L^2-regularity: the Caccioppoli inequality

In this chapter we discuss regularity in terms of square summability of the derivatives of weak solutions to a linear elliptic system

$$-D_\alpha(A_{ij}^{\alpha\beta}D_\beta u^j) = f_i - D_\alpha f_i^\alpha$$

in dependence of the regularity of the coefficients and boundary data, i.e., we deal with the *energy estimates* for the derivatives of solutions. The basic tool we use is the *Caccioppoli inequality*, sometimes also called *reverse Poincaré inequality*, which enables us to give *a priori* estimates of the L^2-norm of the derivatives of a solution u in terms of the L^2-norm of u.

4.1 The simplest case: harmonic functions

Theorem 4.1 (Caccioppoli inequality) *Let $u \in W^{1,2}(\Omega)$ be a weak solution of $\Delta u = 0$, that is*

$$\int_\Omega D_\alpha u D_\alpha \varphi dx = 0, \quad \forall \varphi \in W_0^{1,2}(\Omega). \tag{4.1}$$

Then for each $x_0 \in \Omega$, $0 < \rho < R \leq \operatorname{dist}(x_0, \partial\Omega)$ we have

$$\int_{B_\rho(x_0)} |Du|^2 dx \leq \frac{c}{(R-\rho)^2} \int_{B_R(x_0)\setminus B_\rho(x_0)} |u-\lambda|^2 dx, \quad \forall \lambda \in \mathbb{R} \tag{4.2}$$

for some universal constant c.

Proof. Define a "cut-off" function $\eta \in C_c^\infty(\Omega)$ such that

1. $0 \leq \eta \leq 1$;
2. $\eta \equiv 1$ on $B_\rho(x_0)$ and $\eta \equiv 0$ on $B_R(x_0) \setminus B_\rho(x_0)$
3. $|D\eta| \leq \frac{2}{R-\rho}$.

Choosing as test function $\varphi := (u-\lambda)\eta^2$ in (4.1) we get

$$\int_\Omega |Du|^2\eta^2 dx + \int_\Omega D_\alpha u(u-\lambda)2\eta D_\alpha \eta dx = 0,$$

hence using Hölder's inequality

$$\begin{aligned}\int_{B_R(x_0)} |Du|^2\eta^2 dx \le & \int_{B_R(x_0)} |Du||u-\lambda|2\eta|D\eta| dx \\ \le & \left(\int_{B_R(x_0)} |Du|^2\eta^2 dx\right)^{\frac{1}{2}} \left(\int_{B_R(x_0)} 4|u-\lambda|^2|D\eta|^2 dx\right)^{\frac{1}{2}}.\end{aligned}$$

Dividing by

$$\left(\int_{B_R(x_0)} |Du|^2\eta^2 dx\right)^{\frac{1}{2}},$$

squaring and taking into account the properties of η yield:

$$\begin{aligned}\int_{B_\rho(x_0)} |Du|^2 dx \le & \int_{B_R(x_0)} |Du|^2\eta^2 dx \\ \le & \frac{16}{(R-\rho)^2}\int_{B_R(x_0)\setminus B_\rho(x_0)} |u-\lambda|^2 dx.\end{aligned}$$

□

Exercise 4.2 (Higher order estimates) For $k>0$ and any $B_R(x_0)\subset\Omega$ there is a constant $c(k,R)$ such that, whenever u is a *smooth* harmonic function, then

$$\int_{B_{\frac{R}{2}}(x_0)} |D^k u|^2 dx \le c(k,R)\int_{B_R(x_0)} |u|^2 dx. \tag{4.3}$$

[Hint: Prove that all partial derivatives of u are harmonic functions and apply repeatedly Theorem 4.1 on suitable annuli.]

Exercise 4.3 (Smoothness of harmonic functions) Using inequality (4.3) prove that a harmonic function $u\in W^{1,2}(\Omega)$ belongs to $W^{k,2}_{\mathrm{loc}}(\Omega)$ for all k, consequently it is smooth.

[Hint: Consider the convoluted functions $u_\varepsilon := u*\rho_\varepsilon$ for some mollifying symmetric kernel ρ. Show that u_ε is harmonic and use the derivative estimates together with Rellich's and Sobolev's embedding theorems to conclude that, up to subsequences, $u_\varepsilon\to u$ in C^k for each $k\ge 0$.]

4.2 Caccioppoli's inequality for elliptic systems

Theorem 4.4 *Let $u \in W^{1,2}(\Omega, \mathbb{R}^m)$ be a weak solution of*

$$-D_\alpha(A^{\alpha\beta}_{ij} D_\beta u^j) = f_i - D_\alpha F^\alpha_i \tag{4.4}$$

with $f_i, F^\alpha_i \in L^2(\Omega)$, and assume that one of the following conditions holds:

1. $A^{\alpha\beta}_{ij} \in L^\infty(\Omega)$ *and satisfy the Legendre ellipticity condition* (3.16)*;*
2. $A^{\alpha\beta}_{ij} \equiv$ *const and satisfy the Legendre-Hadamard condition* (3.17)*;*
3. $A^{\alpha\beta}_{ij} \in C^0(\overline{\Omega})$ *satisfy the Legendre-Hadamard condition.*

Then for any ball $B_R(x_0) \subset \Omega$ (with $R < R_0$ small enough under condition 3) and $0 < \rho < R$ the following Caccioppoli inequality *holds:*

$$\int_{B_\rho(x_0)} |Du|^2 dx \leq c\left\{\frac{1}{(R-\rho)^2}\int_{B_R(x_0)\setminus B_\rho(x_0)} |u-\xi|^2 dx \right.$$
$$\left. + R^2 \int_{B_R(x_0)} |f|^2 dx + \int_{B_R(x_0)} |F|^2 dx\right\}, \tag{4.5}$$

for every vector $\xi \in \mathbb{R}^m$, where under conditions 1 or 2

$$c = c(\lambda, \Lambda), \quad \Lambda := \sup |A|.$$

Under condition 3 the constant c also depends on the modulus of continuity of $A^{\alpha\beta}_{ij}$ and R_0.

Proof. We give the simple proof when hypothesis 1 is satisfied. The other cases can be treated in a similar way using Gårding's inequality; the details are left for the reader.

First assume $f_i = 0$. Define a cut-off function η as in the proof of Theorem 4.1 and choose as test function $(u - \xi)\eta^2$ into (4.4). From the Legendre condition we obtain

$$\begin{aligned}
\lambda \int_{B_R(x_0)} \eta^2 |Du|^2 dx &\leq \int_{B_R(x_0)} \eta^2 A^{\alpha\beta}_{ij} D_\alpha u^j D_\beta u^i dx \\
&= - \int_{B_R(x_0)} 2\eta(u^i - \xi^i) A^{\alpha\beta}_{ij} D_\alpha u^j D_\beta \eta dx \\
&\quad + \int_{B_R(x_0)} \eta^2 F^\alpha_i D_\alpha u^i dx + \int_{B_R(x_0)} 2\eta F^\alpha_i (u^i - \xi^i) D_\alpha \eta dx \\
&=: (I) + (II) + (III).
\end{aligned}$$

Now we can bound with Young's inequality $2ab \le \varepsilon a^2 + \frac{b^2}{\varepsilon}$ and the properties of η

$$
\begin{aligned}
(I) &\le \varepsilon\Lambda \int_{B_R(x_0)} \eta^2 |Du|^2 dx + \frac{4\Lambda}{\varepsilon(R-\rho)^2} \int_{B_R(x_0)\setminus B_\rho(x_0)} |u-\xi|^2 dx,\\
(II) &\le \varepsilon\Lambda \int_{B_R(x_0)} \eta^2 |Du|^2 dx + \frac{1}{4\varepsilon\Lambda} \int_{B_R(x_0)} |F|^2 dx\\
(III) &\le \frac{4}{(R-\rho)^2} \int_{B_R(x_0)\setminus B_\rho(x_0)} |u-\xi|^2 dx + \int_{B_R(x_0)} |F|^2 dx.
\end{aligned}
$$

Choosing $\varepsilon = \frac{\lambda}{4\Lambda}$, using that

$$
\int_{B_\rho(x_0)} |Du|^2 dx \le \int_{B_R(x_0)} \eta^2 |Du|^2 dx
$$

and simplifying yield the result.

When $f_i \neq 0$ define

$$
\tilde{F}_i^1(x) = \int_{-\infty}^{x_1} f_i(t, x_2, \ldots, x_n) \chi_{B_R(x_0)}(t, x_2, \ldots, x_n) dt,
$$

and prove using Jensens inequality that

$$
\int_{B_R(x_0)} (\tilde{F}_i^1)^2 dx \le cR^2 \int_{B_R(x_0)} f_i^2 dx.
$$

Then the term $f_i = D_1 \tilde{F}_i^1$ can be added to term F_i^1. □

Exercise 4.5 Prove that in Theorem 4.4 we can replace the assumption $f_i \in L^2(\Omega)$ with the weaker assumption $f_i \in L^{2_*}(\Omega)$, $2_* := \frac{2n}{n+2}$, and in (4.5) we can replace the term

$$
R^2 \int_{B_R(x_0)} |f|^2 dx \quad \text{with} \quad \left(\int_{B_R(x_0)} |f|^{2_*} dx \right)^{\frac{2}{2_*}}.
$$

4.3 The difference quotient method

In order to prove L^2-estimates for the derivatives of a solution u we show that the difference quotients of u (a sort of discrete derivative) satisfy an elliptic system; by Caccioppoli's inequality we get L^2-estimates on the derivatives of the difference quotients $D\tau_{h,s}u$ and apply Proposition 4.8 to conlude the existence of the s-derivative $D_s Du$ with a suitable estimate in L^2. The procedure can be used inductively to obtain higher order differentiability.

Definition 4.6 (Difference quotient) *Given a function $u : \Omega \to \mathbb{R}^m$, an integer $s \in \{1, \dots, n\}$ and $h > 0$ we define the* difference quotient

$$\tau_{h,s}u(x) := \frac{u(x + he_s) - u(x)}{h}, \quad \forall x \in \Omega_{s,h} := \{x \in \Omega : x + he_s \in \Omega\},$$

where $e_s \in \mathbb{R}^n$ is the unit vector $(0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the s-th position.

Exercise 4.7 If $u \in W^{1,p}(\Omega)$, then $\tau_{s,h}u \in W^{1,p}(\Omega_{s,h})$ and $\tau_{s,h}Du = D\tau_{s,h}u$. Moreover for u or v compactly supported in Ω and h small enough we have

$$\int_\Omega u\, \tau_{h,s} v dx = -\int_\Omega \tau_{-h,s} u\, v dx, \tag{4.6}$$

and Leibniz's rule holds

$$\tau_{h,s}(uv)(x) = u(x + he_s)\tau_{h,s}v(x) + \tau_{h,s}u(x)v(x). \tag{4.7}$$

Proposition 4.8 *Let $1 < p < +\infty$ and $\Omega_0 \Subset \Omega$. Then*

(i) *There is a constant $c = c(n)$ such that, for every $u \in W^{1,p}(\Omega)$ and $s = 1, \dots, n$, we have*

$$\|\tau_{h,s}u\|_{L^p(\Omega_0)} \le c\|Du\|_{L^p(\Omega)}, \quad |h| < \frac{\mathrm{dist}(\Omega_0, \partial\Omega)}{2}. \tag{4.8}$$

(ii) *If $u \in L^p(\Omega)$ and there exists $L \ge 0$ such that, for every $h <$ $\mathrm{dist}(\Omega_0, \partial\Omega)$, $s = 1, \dots, n$, we have*

$$\|\tau_{h,s}u\|_{L^p(\Omega_0)} \le L, \tag{4.9}$$

then $u \in W^{1,p}(\Omega_0)$, $\|Du\|_{L^p(\Omega_0)} \le L$ and $\tau_{h,s}u \to D_s u$ in $L^p(\Omega_0)$ as $h \to 0$.

Proof. (i) Assume first that $u \in C^\infty(\Omega)$. By the fundamental theorem of calculus

$$u(x + he_s) - u(x) = \int_0^h \frac{\partial}{\partial x^s} u(x + \xi e_s) d\xi,$$

whence

$$\tau_{h,s}u(x) = \fint_0^h \frac{\partial}{\partial x^s} u(x + \xi e_s) d\xi.$$

Therefore, by Jensen's inequality and Fubini's theorem we get

$$\begin{aligned}
\|\tau_{h,s}u\|^p_{L^p(\Omega_0)} &= \int_{\Omega_0} \left| \fint_0^h \frac{\partial}{\partial x^s} u(x + \xi e_s) d\xi \right|^p dx \\
&\le \int_{\Omega_0} \left(\fint_0^h |Du(x + \xi e_s)|^p d\xi \right) dx \\
&\le \fint_0^h \left(\int_\Omega |Du(x)|^p dx \right) d\xi = \|Du\|^p_{L^p(\Omega)}.
\end{aligned}$$

For a general $u \in W^{1,p}(\Omega)$ we approximate it $W^{1,p}(\Omega)$ with a sequence of smooth functions and notice that (4.8) is stable under convergence in $W^{1,p}(\Omega)$.

(ii) We have that $L^p(\Omega)$ is reflexive and $\tau_{h,s}u$ is bounded in $L^p(\Omega_0)$ uniformly with respect to h. According to Banach-Alaoglu's theorem, the unit ball of a reflexive Banach space is sequentially weakly compact. Therefore we may extract a weakly converging subsequence:

$$\tau_{h_k,s} \rightharpoonup g \quad \text{in } L^p(\Omega_0)$$

and $g = D_s u$ in the sense of distributions because $\forall \varphi \in C_c^\infty(\Omega_0)$ we have

$$\begin{aligned}\int_{\Omega_0} g\varphi dx &= \lim_{h_k \to 0} \int_{\Omega_0} \tau_{h_k,s} u\varphi dx \\ &= -\lim_{h_k \to 0} \int_{\Omega_0} u\tau_{-h_k,s}\varphi dx \\ &= -\int_{\Omega_0} uD_s\varphi dx.\end{aligned}$$

Thus $D_s u \in L^p(\Omega_0)$. To prove that the convergence $\tau_{h,s}u \to D_s u$ is strong in $L^p(\Omega_0)$, take any $w \in C^\infty(\Omega)$; then

$$\tau_{h,s}u - D_s u = \tau_{h,s}(u-w) + \tau_{h,s}w - D_s w + D_s(w-u),$$

and by part 1

$$\|\tau_{h,s}u - D_s u\|_{L^p(\Omega_0)} \le \|\tau_{h,s}w - D_s w\|_{L^p(\Omega_0)} + c\|D_s(w-u)\|_{L^p(\Omega)},$$

where $c = c(n,p)$. Since $C^\infty(\Omega)$ is dense in $W^{1,p}(\Omega)$, the second term on the right-hand side can be made arbitrarily small, while the first term goes to 0 as $h \to 0$ since $\tau_{h,s}w \to D_s w$ uniformly on compact sets. □

4.3.1 Interior L^2-estimates

The following theorem is a direct consequence of the Caccioppoli inequality and Proposition 4.8.

Theorem 4.9 *Let $u \in W^{1,2}(\Omega, \mathbb{R}^m)$ be a weak solution of*

$$-D_\alpha(A_{ij}^{\alpha\beta} D_\beta u^j) = f_i - D_\alpha F_i^\alpha \tag{4.10}$$

with $f_i \in L^2(\Omega)$, $F_i^\alpha \in W^{1,2}(\Omega)$. Assume that $A_{ij}^{\alpha\beta} \in \mathrm{Lip}(\Omega)$ satisfy the Legendre-Hadamard condition. Then $u \in W^{2,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$, and for any relatively compact subset Ω_0 of Ω we have

$$\|D^2 u\|_{L^2(\Omega_0)} \le c\big(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} + \|DF\|_{L^2(\Omega)}\big), \tag{4.11}$$

c being a constant depending on Ω_0, Ω and the ellipticity and Lipschitz constants of A.

Proof. Remember that (4.10) means

$$\int_\Omega A_{ij}^{\alpha\beta} D_\beta u^j D_\beta \varphi^i dx = \int_\Omega f_i \varphi^i dx + \int_\Omega F_i^\alpha D_\alpha \varphi^i dx, \quad \forall \varphi \in C_c^\infty(\Omega, \mathbb{R}^m). \tag{4.12}$$

Assume that $f_i = 0$ (for the general case see the following exercise), choose a test function φ and, for h small, insert $\varphi(x - he_s)$ in (4.12) to obtain

$$\int_\Omega A_{ij}^{\alpha\beta}(x + he_s) D_\beta u^j(x + he_s) D_\beta \varphi^i(x) dx = \int_\Omega F_i^\alpha(x + he_s) D_\alpha \varphi^i(x) dx. \tag{4.13}$$

Subtract (4.12) form (4.13) to obtain

$$\begin{aligned} \int_\Omega A_{ij}^{\alpha\beta}(x + he_s)\tau_{h,s}(D_\beta u^j) D_\alpha \varphi^i dx + \int_\Omega \tau_{h,s} A_{ij}^{\alpha\beta} D_\beta u^j D_\alpha \varphi^i dx& \\ = \int_\Omega \tau_{h,s} F_i^\alpha D_\alpha \varphi^i dx.& \end{aligned} \tag{4.14}$$

Remember that $\tau_{h,s}(Du) = D(\tau_{h,s}u)$ and apply Caccioppoli inequality (4.5) to $\tau_{h,s}u$ in (4.14): for any $B_{4R}(x_0) \subset \Omega$

$$\begin{aligned} \int_{B_R(x_0)} |\tau_{h,s} Du|^2 dx \le & \frac{c}{R^2} \int_{B_{2R}(x_0)} |\tau_{h,s} u|^2 dx \\ & + c \int_{B_{2R}(x_0)} |\tau_{h,s} A|^2 |Du|^2 dx \\ & + c \int_{B_{2R}(x_0)} |\tau_{h,s} F|^2 dx. \end{aligned}$$

As $h \to 0$ all the terms on the right-hand side remain bounded thanks to the first part of Proposition 4.8 and A being Lipschitz; thus the second part of the same proposition implies that $Du \in W^{1,2}(B_R(x_0))$; taking the limit as $h \to 0$ and applying Caccioppoli's inequality again we bound

$$\begin{aligned} \int_{B_R(x_0)} |D^2 u|^2 dx \le & \frac{c}{R^2} \int_{B_{2R}(x_0)} |Du|^2 dx + cL^2 \int_{B_{2R}(x_0)} |Du|^2 dx \\ & + c \int_{B_{2R}(x_0)} |DF|^2 dx \\ \le & c_1(R, L) \int_{B_{4R}(x_0)} |u|^2 dx + \int_{B_{2R}(x_0)} |DF|^2 dx, \end{aligned}$$

where L is the Lipschitz constant of A. By a covering argument we get (4.11). □

Exercise 4.10 Complete the proof of Theorem 4.9 by dropping the assumption that $f = 0$.

[Hint. Choose as test function $\varphi := \tau_{-h,s}(\eta^2\tau_{h,s}u)$ with the usual cut-off function η. We obtain

$$\int_\Omega A_{ij}^{\alpha\beta} D_\beta u^j D_\alpha \varphi^i dx \geq c\int_\Omega |D(\eta\tau_{h,s}u)^2| dx - c_1\int_\Omega (\eta^2|Du|^2 + |D\eta|^2|\tau_{h,s}u|^2)dx.$$

Rearranging one gets

$$\begin{aligned} c\int_{B_\rho} |D(\eta\tau_{h,s}u)|^2 dx &\leq c_1\int_\Omega (\eta^2|Du|^2 + |D\eta|^2|\tau_{h,s}u|^2)dx \\ &\quad + \int_\Omega |f||\varphi|dx + \int_\Omega |F||D\varphi|dx. \end{aligned} \tag{4.15}$$

The last term is quite easy to estimate and for the term involving f

$$\begin{aligned} \int_\Omega |f||\tau_{-h,s}(\eta^2\tau_{h,s}u)|dx &\leq \varepsilon\int_\omega |D(\eta^2\tau_{h,s}u)|^2 dx + \frac{1}{\varepsilon}\int_\Omega |f|^2 dx \\ &\leq \varepsilon\int_\Omega |D(\eta\tau_{h,s}u)|^2 dx + \varepsilon\int_\Omega |D\eta|^2|\tau_{h,s}u|^2 dx \\ &\quad + \frac{1}{\varepsilon}\int_\Omega |f|^2 dx. \end{aligned} \tag{4.16}$$

Insert (4.16) into (4.15) to obtain an L^2-estimate of $\tau_{h,s}Du$ and use Proposition 4.8 to get the result.]

By induction Theorem 4.9 generalizes to the following regularity result.

Theorem 4.11 *Assume that $u \in W^{1,2}(\Omega, \mathbb{R}^m)$ is a weak solution of the system*

$$-D_\alpha(A_{ij}^{\alpha\beta} D_\beta u^j) = f_i - D_\alpha F_i^\alpha,$$

where $A_{ij}^{\alpha\beta}$ satisfies the Legendre-Hadamard condition and for some integer $k \geq 0$

1. $A_{ij}^{\alpha\beta} \in C^{k,1}(\Omega)$, *i.e.* $D^k A_{ij}^{\alpha\beta} \in \mathrm{Lip}(\Omega)$,
2. $f_i \in W^{k,2}(\Omega)$,
3. $F_i^\alpha \in W^{k+1,2}(\Omega)$.

Then $u \in W^{k+2,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ and for every $\Omega_0 \Subset \Omega$ there is a constant c depending on k, Ω, Ω_0 and $\|A\|_{C^{k,1}}$ such that

$$\|D^{k+2}u\|_{L^2(\Omega_0)} \leq c(\|u\|_{L^2(\Omega)} + \|f\|_{W^{k,2}(\Omega)} + \|DF\|_{W^{k,2}(\Omega)}).$$

Proof. For $k = 0$ this reduces to Theorem 4.9. Now we assume the theorem true for some $k - 1 \geq 0$ and prove it for k.

For a given $\psi \in C_c^\infty(\Omega)$ consider the test function $\varphi := D_s\psi$, $1 \leq s \leq n$. Integration by parts yields

$$\int_\Omega D_s(A^{\alpha\beta} D_\beta u^j) D_\alpha \psi^i dx = \int_\Omega D_s f_i \psi^i dx + \int_\Omega D_s F_i^\alpha D_\alpha \psi^i dx,$$

that becomes

$$\int_\Omega A^{\alpha\beta}_{ij} D_\beta(D_s u^j) D_\alpha \psi^i dx = \int_\Omega D_s f_i \psi^i dx$$
$$+ \int_\Omega \big[- D_s A^{\alpha\beta}_{ij} D_\beta u^j + D_s F^\alpha_i \big] D_\alpha \psi^i dx.$$

Now given $\Omega_0 \Subset \Omega$ choose $\tilde\Omega$ with $\Omega_0 \Subset \tilde\Omega \Subset \Omega$. We have $\widetilde{f_i} := D_s f_i \in W^{k-1,2}(\Omega)$ and $\widetilde{F}^\alpha_i := -D_s A^{\alpha\beta}_{ij} D_\beta u^j + D_s F^\alpha_i \in W^{k,2}(\tilde\Omega)$ so that by inductive hypothesis we have $D_s u \in W^{k+1,2}(\Omega_0)$ for every s, i.e. $u \in W^{k+2,2}(\Omega_0)$ with the claimed estimate easily following. □

Corollary 4.12 *Let u be a weak solution of the elliptic system*

$$-D_\alpha(A^{\alpha\beta}_{ij} D_\beta u^j) = f_i - D_\alpha F^\alpha_i$$

with $A^{\alpha\beta}_{ij}, f_i, F^\alpha_i \in C^\infty(\Omega)$. Then $u \in C^\infty(\Omega, \mathbb{R}^m)$.

Proof. By Theorem 4.11 $u \in W^{k,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ for every $k \geq 0$ and the result follows at once from the Sobolev embedding theorem. □

4.3.2 Boundary regularity

The solution to an elliptic system with prescribed boundary data g is regular up to the boundary according to the regularity of $\partial\Omega$ and g.

Definition 4.13 *A domain Ω is said to be* of class C^k *if for every point $x_0 \in \partial\Omega$ there exist a neighborhood U of x_0 in $\overline{\Omega}$ and C^k-diffeomorphism*

$$\gamma : \overline{B}^+ \to \overline{U},$$

where B^+ is the half-ball $\{x \in \mathbb{R}^n : |x| < 1, x^n > 0\}$ and $\overline{B}^+$ is its closure.

Theorem 4.14 *Let the hypothesis of Theorem 4.11 be in force. Assume in addition that $\partial\Omega$ is of class C^{k+2} and $u - g \in W^{1,2}_0(\Omega)$ for a given $g \in W^{k+2,2}(\Omega)$. Then $u \in W^{k+2,2}(\Omega)$ and there is a constant c depending on k, Ω and $\|A\|_{C^{k,1}}$ such that*

$$\|D^{k+2}u\|_{L^2(\Omega)} \leq c(\|f\|_{W^{k,2}(\Omega)} + \|DF\|_{W^{k,2}(\Omega)} + \|g\|_{W^{k+2,2}(\Omega)}).$$

Proof. Up to redefining u by $u - g$ and changing the data of the system accordingly, we may assume that $u \in W^{1,2}_0(\Omega)$.

Step 1: Reduction to a flat boundary. For a neighborhood U of a point $x_0 \in \partial\Omega$ and a C^{k+2}-diffeomorphism $\gamma : U \to B^+$ as in Definition 4.13, define in B^+. We can assume that $J_\gamma := \det D\gamma > 0$ on $\overline{B}^+$. Setting

$$\begin{aligned}
\widetilde{u}(y) &:= u(\gamma(y)),\\
\widetilde{A}_{ij}^{\alpha\beta}(y) &:= A_{ij}^{\nu\mu}(\gamma(y))J_\gamma(y)(D_\mu\gamma^\beta(y))^{-1}D_\nu(\gamma^{-1})^\alpha\gamma(y),\\
\widetilde{F}_i^\alpha(y) &:= J_\gamma(y)D_\mu(\gamma^{-1})^\alpha(\gamma(y))F_i^\mu(\gamma(y))\\
\widetilde{f}_i(y) &:= J_\gamma(y)f_i(\gamma(y)),
\end{aligned}$$

we have with the change of variable formula we have for $\varphi \in C_x^\infty(B^+)$

$$\begin{aligned}
\int_{B^+}\widetilde{A}_{ij}^{\alpha\beta}(y)D_\beta\widetilde{u}^j(y)D_\alpha\varphi^i(y)dy &= \int_U A_{ij}^{\alpha\beta}(x)D_\beta u^j(x)D_\alpha\varphi^i(\gamma^{-1}(x))dx\\
&= \int_U f_i(x)\varphi^i(\gamma^{-1}(x))dx + \int_U F_i^\alpha(x)D_\alpha\varphi^i(\gamma^{-1}(x))dx\\
&= \int_{B^+}\widetilde{f}_i(y)\varphi^i(y)dy + \int_{B^+}\widetilde{F}_i^\alpha(y)D_\alpha\varphi^i(y)dy,
\end{aligned}$$

i.e.

$$-D_\alpha(\widetilde{A}_{ij}^{\alpha\beta}D_\beta\widetilde{u}^j) = \widetilde{f}_i - \widetilde{F}_i^\alpha \quad \text{in } B^+. \tag{4.17}$$

Thanks to the assumption on γ, $\widetilde{u} \in W^{k+2,2}(B^+)$ if and only if $u \in W^{k+2,2}(U)$. It is then clear that we can first assume $\Omega = B^+$ and

$$u = 0 \text{ on } \{x \in \overline{B}^+ : x^n = 0\}, \tag{4.18}$$

and prove that $u \in W^{k+2,2}(B^+_{1/2})$, where

$$B^+_{1/2} = \{x \in \mathbb{R}^n : |x| < 1/2,\, x_n > 0\}.$$

Then using a covering argument on $\partial\Omega$ we conclude for a general set Ω with C^{k+2} boundary.

Step 2: Existence of second derivatives D_sDu, $s \neq n$. By (4.18), for any $\eta \in C_c^\infty(B_1)$ we have $\eta u \in W_0^{1,2}(B^+, \mathbb{R}^m)$ and similarly, if $s \neq n$, $\varphi := \tau_{-h,s}(\eta^2\tau_{h,s}u) \in W_0^{1,2}(B^+)$, so that φ is an admissible test function. Inserting φ in (4.17) and carrying out the same computations as in Exercise 4.10 yields

$$\|\tau_{h,s}Du\|_{L^2(B^+_{1/2})} \le c\big\{\|Du\|_{L^2(B^+)} + \|\widetilde{f}\|_{L^2(B^+)} + \|D\widetilde{F}\|_{L^2(B^+)}\big\},$$

thus proving, by Proposition 4.8, that all the second derivatives of u except for $D_nD_\beta u$ are in $L^2(B^+_{1/2})$ and bounded, for $\beta = 1, \dots, n$. Since weak derivatives commute, the same reasoning applies with $s = \beta$ if $\beta \neq n$, so that it only remains to prove that $D_{nn}u$ is bounded in $L^2(B^+_{1/2})$.

Step 3: Existence of $D_{nn}u$ bounded in $L^2(B^+_{1/2})$. System (4.17) may be rewritten (ignoring the $\sim$ symbols) as

$$\int_{B^+_{1/2}} A^{nn}_{ij} D_n u^j D_n \varphi^i dx = - \sum_{\substack{\alpha,\beta=1 \\ (\alpha,\beta)\neq(n,n)}}^{n} \sum_{i,j=1}^{m} \int_{B^+_{1/2}} A^{\alpha\beta}_{ij} D_\beta u^j D_\alpha \varphi^i dx + \int_{B^+_{1/2}} f_i \varphi^i dx + \int_{B^+_{1/2}} F^\alpha_i D_\alpha \varphi^i dx. \tag{4.19}$$

After integration by parts, we get

$$\begin{aligned} &- A^{nn}_{ij} (D_{nn} u^j, \varphi^i)_{L^2(B^+_{1/2})} \\ &= \int_{B^+_{1/2}} \underbrace{\Big[\sum_{\substack{\alpha,\beta=1 \\ (\alpha,\beta)\neq(n,n)}}^{n} D_\alpha \Big(A^{\alpha\beta}_{ij} D_\beta u^j \Big) + D_n A^{nn}_{ij} D_n u^j + f_i - D_\alpha F^\alpha_i \Big]}_{\text{bounded in } L^2(B^+_{1/2})} \varphi^i dx. \end{aligned} \tag{4.20}$$

Observing that (A^{nn}_{ij}) is positive definite by ellipticity, hence invertible, (4.20) implies

$$\sup_{\|\varphi\|_{L^2(B^+_{1/2})} \leq 1} (D_{nn} u^j, \varphi^i)_{L^2(B^+_{1/2})} \leq c,$$

hence by duality $D_{nn}u^j$ belongs to $L^2(B^+_{1/2})$ and is bounded as usual.

Step 4. With a covering argument we obtain $D^2 u \in L^2(\Omega)$ and

$$\|D^2(u-g)\|_{L^2(\Omega)} \leq c\big\{ \|D(u-g)\|_{L^2(\Omega)} + \|\widetilde{f}\|_{L^2(\Omega)} + \|D\widetilde{F}\|_{L^2(\Omega)} \big\},$$

i.e.

$$\|D^2 u\|_{L^2(\Omega)} \leq c\big\{ \|Du\|_{L^2(\Omega)} + \|Dg\|_{W^{1,2}(\Omega)} + \|\widetilde{f}\|_{L^2(\Omega)} + \|D\widetilde{F}\|_{L^2(\Omega)} \big\},$$

but we can get rid of the term $\|Du\|_{L^2(\Omega)}$ on the right-hand side by testing the system with $\varphi = (u-g)$, using ellipticity, Hölder's and Poincaré's inequalities:

$$\begin{aligned} \lambda \|Du\|^2_{L^2(\Omega)} &\leq \int_\Omega A^{\alpha\beta}_{ij} D_\beta u^j D_\alpha (u^j - g^j) dx + \int_\Omega A^{\alpha\beta}_{ij} D_\beta u^j D_\alpha g^i dx \\ &\leq c\Big(\|f\|_{L^2(\Omega)} \|u-g\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)} \|D(u-g)\|_{L^2(\Omega)} \\ &\qquad + \|Du\|_{L^2(\Omega)} \|Dg\|_{L^2(\Omega)} \Big) \\ &\leq \frac{\lambda}{2} \|Du\|^2_{L^2(\Omega)} + \frac{c}{\lambda} \Big(\|F\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)} + \|Dg\|^2_{L^2(\Omega)} \Big). \end{aligned}$$

With Poincaré's inequality we easily bound also $\|u\|^2_{L^2(\Omega)}$. For the higher derivatives we may proceed by induction as in Theorem 4.11. $\square$

4.4 The hole-filling technique

Caccioppoli's inequality may be used to obtain a decay estimate for the Dirichlet integral of weak solutions of linear elliptic systems. Here we show how to do this by the *hole-filling technique* of Widman [115]. As a consequence we obtain Hölder continuity for the solutions of elliptic systems with bounded coefficients in dimension 2.

Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$, $\Omega \subset \mathbb{R}^n$ be a solution to the following elliptic system:

$$-D_\alpha(A^{\alpha\beta}_{ij}(x)D_\beta u^j) = 0 \quad \text{in } \Omega, \tag{4.21}$$

$$\lambda|\xi|^2 \le A^{\alpha\beta}_{ij}\xi^i_\alpha\xi^j_\beta \le \Lambda|\xi|^2.$$

Take $x_0 \in \Omega$, $0 < R < \mathrm{dist}(x_0, \partial\Omega)$. Insert in (4.21) the test function $(u-\xi)\eta^2$, where $\xi \in \mathbb{R}^m$ and η is a non-negative cut-off function with $\eta \equiv 1$ on $B_{\frac{R}{2}}(x_0)$, $\eta \equiv 0$ on $\Omega \backslash B_R(x_0)$, $|D\eta| \le \frac{2}{R}$. We obtain

$$\int_\Omega |Du|^2\eta^2 dx \le c\int_\Omega |Du||D\eta||u-\xi|\eta dx, \quad c = c(\lambda, \Lambda)$$

and taking into account the properties of η, Poincaré's inequality and $2ab \le \varepsilon a^2 + \frac{b^2}{\varepsilon}$ the right-hand side above can be bounded by

$$\frac{1}{2}\int_\Omega |Du|^2\eta^2 dx + c_1\int_{B_R(x_0)\backslash B_{\frac{R}{2}}(x_0)} |u-\xi|^2 dx. \tag{4.22}$$

Choosing

$$\xi = \fint_{B_R(x_0)\backslash B_{\frac{R}{2}}(x_0)} u dx$$

we can use the Poincaré-type inequality

$$\int_{B_R(x_0)\backslash B_{\frac{R}{2}}(x_0)} |u-\xi|^2 dx \le c_2 R^2 \int_{B_R(x_0)\backslash B_{\frac{R}{2}}(x_0)} |Du|^2 dx$$

(prove it as exercise: first assume $R = 1$ and apply Proposition 3.21, then rescale) and summing up we find

$$\int_{B_{\frac{R}{2}}(x_0)} |Du|^2 dx \le c\int_{B_R(x_0)\backslash B_{\frac{R}{2}}(x_0)} |Du|^2 dx \tag{4.23}$$

for a dimensional constant c. Adding c times the left-hand side to both sides we get

$$\int_{B_{\frac{R}{2}}(x_0)} |Du|^2 dx \leq \frac{c}{c+1} \int_{B_R(x_0)} |Du|^2 dx, \tag{4.24}$$

and

$$\int_{B_{2^{-k}R}(x_0)} |Du|^2 dx \leq \left(\frac{c}{c+1}\right)^k \int_{B_R(x_0)} |Du|^2 dx$$

for all $k \geq 1$. This yields the existence of some $\alpha = \alpha(\lambda, \Lambda) > 0$ such that

$$\int_{B_\rho(x_0)} |Du|^2 dx \leq c_1 \rho^{2\alpha} dx.$$

As we shall see in the next chapter, when $n = 2$, i.e. $\Omega \subset \mathbb{R}^2$, Morrey's Theorem 5.7 implies that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$.

Remark 4.15 By Gårding's inequality, we get (4.23) also in the case that $A^{\alpha\beta}_{ij}$ only satisfy the Legendre-Hadamard condition and are constant, or satisfy the Legendre-Hadamard condition, are continuous and R is small enough. Hence also in these cases we have Hölder continuity in dimension 2.

Another easy consequence of (4.24) is that *entire solutions* of (4.21), i.e. solutions of (4.21) *in all of* $\mathbb{R}^n$, with *finite energy*,

$$\int_{\mathbb{R}^n} |Du|^2 dx < \infty,$$

are constant. Consider now an entire solution u of (4.21) in dimension $n = 2$. Suppose it is globally bounded; then from (4.22) (with $\xi = 0$) we get

$$\int_{B_R(0)} |Du|^2 dx \leq \frac{c}{R^2} \int_{B_{2R}(0)} |u|^2 dx \leq c_1 \sup_{\mathbb{R}^2} |u|^2$$

hence u has finite energy. Therefore we conclude

Theorem 4.16 (Liouville) *Let $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ be a bounded solution of the elliptic system* (4.21) *with $\Omega = \mathbb{R}^2$. Then u is constant.*

Chapter 5
Schauder estimates

In this chapter we prove *Schauder estimates* according to the work of S. Campanato, without using potential theory.

5.1 The spaces of Morrey and Campanato

The domains $\Omega \subset \mathbb{R}^n$ in this chapter are supposed to satisfy the following property: *there exists a constant* $A > 0$ *such that for all* $x_0 \in \Omega$, $\rho < \operatorname{diam}\Omega$ *we have*

$$|B_\rho(x_0) \cap \Omega| \geq A\rho^n. \tag{5.1}$$

Note that every domain of class C^1 or Lipschitz has the above property.

Definition 5.1 *Set* $\Omega(x_0,\rho) := \Omega \cap B_\rho(x_0)$ *and for every* $1 \leq p \leq +\infty$, $\lambda \geq 0$ *define the* Morrey space $L^{p,\lambda}(\Omega)$

$$L^{p,\lambda}(\Omega) := \left\{ u \in L^p(\Omega) : \sup_{\substack{x_0\in\Omega\\ \rho>0}} \rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u|^p dx < +\infty \right\},$$

endowed with the norm defined by

$$\|u\|^p_{L^{p,\lambda}(\Omega)} := \sup_{\substack{x_0\in\Omega\\ \rho>0}} \rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u|^p dx$$

and the Campanato space $\mathcal{L}^{p,\lambda}(\Omega)$

$$\mathcal{L}^{p,\lambda}(\Omega) := \left\{ u \in L^p(\Omega) : \sup_{\substack{x_0\in\Omega\\ \rho>0}} \rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u - u_{x_0,\rho}|^p dx < +\infty \right\},$$

where $u_{x_0,\rho} := \fint_{\Omega(x_0,\rho)} u dx$. *We give the Campanato space* $\mathcal{L}^{p,\lambda}(\Omega)$ *the seminorm*

$$[u]^p_{p,\lambda} := \sup_{\substack{x_0\in\Omega\\ \rho>0}} \rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u - u_{x_0,\rho}|^p dx$$

and the norm

$$\|u\|_{\mathcal{L}^{p,\lambda}(\Omega)} := [u]_{p,\lambda} + \|u\|_{L^p(\Omega)}.$$

Remark 5.2 In the above definition only small radii are relevant, i.e. we can fix $\rho_0 > 0$ and replace the definition of $\|u\|^p_{L^{p,\lambda}(\Omega)}$ with

$$\sup_{\substack{x_0\in\Omega\\ 0<\rho<\rho_0}} \rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u|^p dx,$$

and similarly we modify $[u]_{p,\lambda}$. These norms, which are more convenient when proving local estimates, are equivalent to the previous ones, as can be seen with a simple covering argument.

Remark 5.3 The spaces of Morrey and Campanato are Banach spaces; however one can show that smooth functions are not dense in these spaces. In any case we shall not use the Banach structure of these spaces.

Proposition 5.4 *For* $0 \le \lambda < n$ *we have* $L^{p,\lambda}(\Omega) \cong \mathcal{L}^{p,\lambda}(\Omega)$.

Proof. We have[1]

$$\int_{\Omega(x_0,\rho)} |u - u_{x_0,\rho}|^p dx \le 2^{p-1} \left\{ \int_{\Omega(x_0,\rho)} |u|^p dx + |\Omega(x_0,\rho)||u_{x_0,\rho}|^p \right\} \tag{5.2}$$

and by Jensen's inequality

$$|u_{x_0,\rho}|^p \le \frac{1}{|\Omega(x_0,\rho)|} \int_{\Omega(x_0,\rho)} |u|^p dx. \tag{5.3}$$

Insert (5.3) in (5.2), divide by ρ^λ to obtain

$$[u]^p_{p,\lambda} \le 2^p \|u\|^p_{L^{p,\lambda}(\Omega)},$$

thus concluding $L^{p,\lambda}(\Omega) \subset \mathcal{L}^{p,\lambda}(\Omega)$.

For the converse write

$$\frac{1}{\rho^\lambda} \int_{\Omega(x_0,\rho)} |u|^p dx \le 2^{p-1} \left\{ \frac{1}{\rho^\lambda} \int_{\Omega(x_0,\rho)} |u - u_{x_0,\rho}|^p dx + \omega_n \rho^{n-\lambda} |u_{x_0,\rho}|^p \right\}. \tag{5.4}$$

[1] We shall freely use the inequality $(a+b)^p \le 2^{p-1}(a^p + b^p)$ valid for every $p \ge 1$.

We need to estimate the term $\rho^{n-\lambda}|u_{x_0,\rho}|^p$ uniformly with respect to x_0 and ρ. For $0 < r < R$ and $x, x_0 \in \Omega$ we have

$$|u_{x_0,R} - u_{x_0,r}|^p \leq 2^{p-1}\big\{|u(x) - u_{x_0,R}|^p + |u(x) - u_{x_0,r}|^p\big\};$$

integrating with respect to x on $\Omega(x_0, r)$ and using (5.1) we obtain

$$|u_{x_0,R}-u_{x_0,r}|^p \leq \frac{2^{p-1}}{Ar^n}\left\{\int_{\Omega(x_0,R)} |u-u_{x_0,R}|^p dx + \int_{\Omega(x_0,r)} |u-u_{x_0,r}|^p dx\right\},$$

thus

$$|u_{x_0,R} - u_{x_0,r}|^p \leq \frac{c_1(p,A)}{r^n}(R^\lambda + r^\lambda)[u]^p_{p,\lambda} \leq \frac{2c_1(p,A)}{r^n} R^\lambda [u]^p_{p,\lambda}.$$

and taking the p-th root

$$|u_{x_0,R} - u_{x_0,r}| \leq c_2[u]_{p,\lambda} R^{\frac{\lambda}{p}} r^{-\frac{n}{p}}. \tag{5.5}$$

Set $R_k = \frac{R}{2^k}$; (5.5) implies

$$|u_{x_0,R_k} - u_{x_0,R_{k+1}}| \leq c_2 R^{\frac{\lambda-n}{p}} [u]_{p,\lambda} 2^{k\frac{n-\lambda}{p}+\frac{n}{p}}, \tag{5.6}$$

and taking the sum from 0 to h we infer

$$|u_{x_0,R} - u_{x_0,R_{h+1}}| \leq c_3(n,p,\lambda,A)[u]_{p,\lambda} R_{h+1}^{\frac{\lambda-n}{p}}. \tag{5.7}$$

Choose h and R with $\operatorname{diam}\Omega \leq R \leq 2\operatorname{diam}\Omega$ and $R_{h+1} = \rho$. Then we have

$$\begin{aligned}|u_{x_0,\rho}|^p &\leq 2^{p-1}\big\{|u_{x_0,R}|^p + |u_{x_0,R} - u_{x_0,\rho}|^p\big\}\\ &\leq 2^{p-1}\big\{|u_{x_0,R}|^p + c_3^p \rho^{\lambda-n}[u]^p_{p,\lambda}\big\},\end{aligned}$$

which inserted in (5.4), and taking into account the condition on R, yields

$$\frac{1}{\rho^\lambda}\int_{\Omega(x_0,\rho)} |u|^p dx \leq c_4\big\{[u]^p_{p,\lambda} + |u_{x_0,R}|^p\big\} \leq c_5\|u\|^p_{\mathcal{L}^{p,\lambda}(\Omega)},$$

where we also used

$$\rho^{n-\lambda}|u_{x_0,R}|^p \leq c_5|u_{x_0,R}|^p \leq c_6\|u\|^p_{L^p(\Omega)}.$$

□

5.1.1 A characterization of Hölder continuous functions

Theorem 5.5 (Campanato) *For $n < \lambda \le n + p$ and $\alpha = \frac{\lambda - n}{p}$ we have $\mathcal{L}^{p,\lambda}(\Omega) \cong C^{0,\alpha}(\overline{\Omega})$. Moreover the seminorm*

$$[u]_{C^{0,\alpha}} := \sup_{x,y\in\Omega,\, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$$

is equivalent to $[u]_{p,\lambda}$. If $\lambda > n + p$ and $u \in \mathcal{L}^{p,\lambda}(\Omega)$, then u is constant.

Proof. Assume $u \in C^{0,\alpha}(\overline{\Omega})$. For $x \in \Omega(x_0, \rho)$ we have

$$|u(x) - u(x_0)| \le [u]_{C^{0,\alpha}} \rho^\alpha,$$

hence

$$|u(x) - u_{x_0,\rho}| \le [u]_{C^{0,\alpha}} \rho^\alpha.$$

Consequently

$$\frac{1}{\rho^\lambda} \int_{\Omega(x_0,\rho)} |u(x) - u_{x_0,\rho}|^p dx \le \omega_n [u]^p_{C^{0,\alpha}} \rho^{n-\lambda+\alpha p} = \omega_n [u]^p_{C^{0,\alpha}}.$$

Conversely, take $u \in \mathcal{L}^{p,\lambda}(\Omega)$. For $x_0 \in \Omega$, $R > 0$, $R_k := \frac{R}{2^k}$, we get from (5.7)

$$|u_{x_0,R_k} - u_{x_0,R_h}| \le c[u]_{p,\lambda} R_k^{\frac{\lambda-n}{p}}, \quad k < h, \tag{5.8}$$

consequently $\{u_{x_0,R_k}\}$ is a Cauchy sequence. Set

$$\widetilde{u}(x_0) := \lim_{h\to+\infty} u_{x_0,R_h}.$$

This limit doesn't depend on the choice of R, as can be easily verified using (5.5).

From the differentiation theorem of Lebesgue we know that $u_{x,\rho} \to u(x)$ in $L^1(\Omega)$ as $\rho \to 0$, so that $u = \widetilde{u}$ almost everywhere. Taking the limit as $h \to +\infty$ in (5.8) we conclude

$$|u_{x,R} - u(x)| \le c[u]_{p,\lambda} R^{\frac{\lambda-n}{p}}, \tag{5.9}$$

from which we see that the convergence of $u_{x,R}$ to $\widetilde{u}(x)$ is *uniform*. By the absolute continuity of the Lebesgue integral we have that $u_{x,R}$ is continuous with respect to x, so that the uniform limit u is also continuous.

To show Hölder continuity we take $x, y \in \Omega$, $R := |x - y|$ and estimate

$$|u(x) - u(y)| \le |u_{x,2R} - u(x)| + |u_{x,2R} - u_{y,2R}| + |u_{y,2R} - u(y)|. \tag{5.10}$$

The first and the third terms are estimated by (5.9). For the second term we have

$$|u_{x,2R} - u_{y,2R}| \le |u_{x,2R} - u(z)| + |u(z) - u_{y,2R}|$$

which, integrated with respect to z over $\Omega(x,2R) \cap \Omega(y,2R)$, gives

$$|u_{x,2R} - u_{y,2R}| \le \frac{\int_{\Omega(x,2R)} |u(z) - u_{x,2R}|dz + \int_{\Omega(y,2R)} |u(z) - u_{y,2R}|dz}{|\Omega(x,2R) \cap \Omega(y,2R)|}.$$

By Hölder's inequality (applied to the functions $1 \cdot |u(z) - u_{y,2R}|$ and $1 \cdot |u(z) - u_{x,2R}|$) we obtain

$$|u_{x,2R} - u_{y,2R}| \le c\frac{1}{|\Omega(x,2R) \cap \Omega(y,2R)|}[u]_{p,\lambda} R^{\frac{\lambda-n}{p}+n}. \qquad (5.11)$$

On the other hand $\Omega(x,R) \subset \Omega(x,2R) \cap \Omega(y,2R)$, thus by (5.1) we get

$$|\Omega(x,2R) \cap \Omega(y,2R)| \ge AR^n,$$

therefore (5.11) becomes

$$|u_{x,2R} - u_{y,2R}| \le c_1 [u]_{p,\lambda} R^{\frac{\lambda-n}{p}},$$

which together with the preceding estimate in (5.10) yields

$$|u(x) - u(y)| \le c_2 [u]_{p,\lambda} R^{\frac{\lambda-n}{p}} = c_2 [u]_{p,\lambda} |x-y|^{\frac{\lambda-n}{p}}.$$

Hence $u \in C^{0,\alpha}(\overline{\Omega})$ and $[u]_{C^{0,\alpha}} \le c_2 [u]_{p,\lambda}$. Since any $u \in C^{0,\alpha}(\overline{\Omega})$ is constant if $\alpha > 1$, also the last claim of the theorem follows. □

Corollary 5.6 *Assume that Ω has the extention property, for instance it is Lipschitz, and let $u \in W^{1,p}(\Omega)$, $p > n$. Then $u \in C^{0,1-\frac{n}{p}}(\overline{\Omega})$, and $\|u\|_{C^{0,1-n/p}} \le c\|u\|_{W^{1,p}}$, where $c = c(\Omega, p)$.*

Proof. Extend u to a function $\tilde{u} \in W^{1,p}(\mathbb{R}^n)$, with

$$\|\tilde{u}\|_{W^{1,p}(\mathbb{R}^n)} \le c_1 \|u\|_{W^{1,p}(\Omega)}$$

for a constant $c_1(\Omega, p)$. By Poincaré's inequality, Proposition 3.21, and Hölder's inequality we have

$$\begin{aligned} \int_{\Omega(x_0,\rho)} |u - u_{x_0,\rho}|dx &\le c_2 \rho \int_{\mathbb{R}^n} |D\tilde{u}|dx \\ &\le c_3 \Big(\int_{B_\rho(x_0)} |Du|^p dx \Big)^{\frac{1}{p}} \rho^{n-\frac{n}{p}+1}, \end{aligned}$$

that is, $u \in \mathcal{L}^{1,n-\frac{n}{p}+1}(\Omega) \cong C^{0,1-\frac{n}{p}}(\overline{\Omega})$. □

As a corollary, we obtain the celebrated theorem of Morrey on the *growth of the Dirichlet integral*

Theorem 5.7 (Morrey) *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $Du \in L^{p,n-p+\varepsilon}_{\mathrm{loc}}(\Omega)$, for some $\varepsilon > 0$. Then $u \in C^{0,\frac{\varepsilon}{p}}_{\mathrm{loc}}(\Omega)$.*

Proof. By Poincaré's inequality, Proposition 3.21, we have for any ball $B_\rho(x_0) \Subset \Omega$

$$\int_{B_\rho(x_0)} |u - u_{x_0,\rho}|^p dx \leq c\rho^p \int_{B_\rho(x_0)} |Du|^p dx \leq c\rho^{n+\varepsilon} \|Du\|^p_{L^p(B_\rho(x_0))},$$

so that by standard covering arguments $u \in \mathcal{L}^{p,n+\varepsilon}_{\mathrm{loc}}(\Omega)$ (we are also using that in the definition of Morrey and Campanato spaces only small radii are relevant) and the result follows from Campanato's theorem. $\square$

5.2 Elliptic systems with constant coefficients: two basic estimates

The following proposition is a simple consequence of the L^2-regularity and, in particular, of the Caccioppoli inequality, and will be the basic tool in proving Schauder estimates.

Proposition 5.8 *Let $A^{\alpha\beta}_{ij}$ be constant and satisfy the Legendre-Hadamard condition* (3.17). *Then there exists a constant $c = c(n, m, \lambda, \Lambda)$ such that any solution $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ of*

$$D_\alpha(A^{\alpha\beta}_{ij} D_\beta u) = 0 \tag{5.12}$$

satisfies

$$\int_{B_\rho(x_0)} |u|^2 dx \leq c\left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |u|^2 dx, \tag{5.13}$$

and

$$\int_{B_\rho(x_0)} |u - u_{x_0,\rho}|^2 dx \leq c\left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x_0)} |u - u_{x_0,R}|^2 dx, \tag{5.14}$$

for arbitrary balls $B_\rho(x_0) \Subset B_R(x_0) \Subset \Omega$.

Proof. Both inequalities are trivial for $r \geq \frac{R}{2}$, so that we may assume $r < \frac{R}{2}$.

Let us first prove (5.13). By Theorem 4.11 we have for $k \geq 1$ $u \in W^{k,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ and

$$\|u\|_{W^{k,2}(B_{R/2})} \leq c(k, R, n, m, \lambda, \Lambda)\|u\|_{L^2(B_R)}.$$

Thus, for k large enough (depending on n), on account of Sobolev Theorem 3.26

$$
\begin{aligned}
\int_{B_\rho(x_0)} |u|^2 dx &\le \omega_n \rho^n \sup_{B_\rho(x_0)} |u|^2 dx \\
&\le \omega_n \rho^n \sup_{B_{R/2}(x_0)} |u|^2 dx \\
&\le c_1(n,R)\rho^n \|u\|^2_{W^{k,2}(B_{R/2}(x_0))} \\
&\le c_2(R,n,m,\lambda,\Lambda)\rho^n \|u\|^2_{L^2(B_R(x_0))}.
\end{aligned}
$$

A simple scaling argument (see the exercise below) finally yields

$$
c_2(R,n,m,\lambda,\Lambda) = \frac{1}{R^n} c(n,m,\lambda,\Lambda).
$$

Inequality (5.14) follows from (5.13) applied to the partial derivatives $D_s u$ (which are also solutions of (5.12)) together with the inequalities of Caccioppoli (4.5) and Poincaré:

$$
\begin{aligned}
\int_{B_\rho(x_0)} |u - u_{x_0,\rho}|^2 dx &\le c_1 \rho^2 \int_{B_\rho(x_0)} |Du|^2 dx \\
&\le c_2 \rho^2 \Big(\frac{\rho}{R}\Big)^n \int_{B_{R/2}(x_0)} |Du|^2 dx \\
&\le c_3 \rho^2 \Big(\frac{\rho}{R}\Big)^n \frac{1}{R^2} \int_{B_R(x_0)} |u - u_{x_0,\rho}|^2 dx.
\end{aligned}
$$

□

Exercise 5.9 Make precise the scaling argument in the proof of the above proposition.
[Hint. We can assume $x_0 = 0$. For a given solution u of (5.12) in $B_R(0)$, the rescaled function $\widetilde{u}(x) = u(Rx)$ is a solution of (5.12) in $B_1(0)$, so that (5.13) applies to $\widetilde{u}$ with $\frac{\rho}{R}$ instead of ρ.]

Exercise 5.10 Prove that for every $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$, $B_\rho(x_0) \Subset \Omega$

$$
\int_{B_\rho(x_0)} |f(x) - f_{x_0,\rho}|^2 dx = \inf_{\lambda \in \mathbb{R}^m} \int_{B_\rho(x_0)} |f(x) - \lambda|^2 dx. \tag{5.15}
$$

[Hint: Differentiate the right-hand side with respect to λ.]

Remark 5.11 Since $x \mapsto |x|^2$ is a convex function, whenever u is harmonic, $v = |u|^2$ is subharmonic, see Exercise 1.2. Therefore the energy inequality (5.13) in this case follows from the monotonicity formula (1.7) for subharmonic functions:

$$
⨍_{B_\rho(x_0)} |u|^2 dx \le ⨍_{B_R(x_0)} |u|^2 dx, \quad B_\rho(x_0) \Subset B_R(x_0) \Subset \Omega.
$$

In particular we can choose $c = 1$ in this special case.

5.2.1 A generalization of Liouville's theorem

Propositon 5.8 extends to all partial derivatives of u since they also solve (5.12). A consequence of this is that the only entire solutions of an elliptic system with constant coefficients that grow at most polynomially at infinity are polynomials.

Theorem 5.12 *Let $u : \mathbb{R}^n \to \mathbb{R}^m$ be an entire solution to the elliptic system* (5.12)*, and assume that there exists a constant $M > 0$ and an integer $k \geq 0$ such that*

$$|u(x)| \leq M(1 + |x|^k), \quad \forall x \in \mathbb{R}^n.$$

Then u is a polynomial of degree at most k.

Proof. Fix $\rho > 0$ and let P be a polynomial of degree at most k such that for any multi-index γ with $|\gamma| \leq k$ we have on $B_\rho = B_\rho(0)$

$$\int_{B_\rho} D_\gamma(u - P)dx = 0. \tag{5.16}$$

Such a polynomial can be easily determined, starting with the condition (5.16) for $|\gamma| = k$, which determines the coefficients of the monomials in P of highest degree, and then inductively lowering $|\gamma|$. Repeatedly applying Poincaré's inequality (3.2) to $v = D_\gamma u - D_\gamma P$, $|\gamma| = 0, \ldots, k$, using (5.13) for some $R \geq 2\rho$, $(k+1)$-times Caccioppoli's inequality (4.5), and the bound on u yields

$$\begin{aligned}\int_{B_\rho} |u - P|^2 dx &\leq c_1 \rho^{2k+2} \int_{B_\rho} |D^{k+1}u|^2 dx \\ &\leq c_2 \Big(\frac{\rho}{R}\Big)^n \rho^{2k+2} \int_{B_R} |D^{k+1}u|^2 dx \\ &\leq c_3 \Big(\frac{\rho}{R}\Big)^{n+2k+2} \int_{B_{2^{k+1}R}} |u|^2 dx \leq c_4 M^2 \Big(\frac{\rho}{R}\Big)^{n+2k+2} R^{n+2k}.\end{aligned}$$

Letting $R \to +\infty$ we conclude that $u = P$ in B_ρ, and in particular $D^{k+1}u = 0$ in B_ρ. Since ρ was arbitrary $D^{k+1}u = 0$ in $\mathbb{R}^n$, hence it is a polynomial of degree at most k. $\square$

5.3 A lemma

The following lemma turns out to be very useful.

Lemma 5.13 *Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-negative and non-decreasing function satisfying*

$$\phi(\rho) \leq A\Big[\Big(\frac{\rho}{R}\Big)^\alpha + \varepsilon\Big]\phi(R) + BR^\beta,$$

for some $A, \alpha, \beta > 0$, with $\alpha > \beta$ and for all $0 < \rho \le R \le R_0$, where $R_0 > 0$ is given. Then there exist constants $\varepsilon_0 = \varepsilon_0(A, \alpha, \beta)$ and $c = c(A, \alpha, \beta)$ such that if $\varepsilon \le \varepsilon_0$, we have

$$\phi(\rho) \le c\Big[\frac{\phi(R)}{R^\beta} + B\Big]\rho^\beta. \tag{5.17}$$

for all $0 \le \rho \le R \le R_0$.

Proof. Set $\rho := \tau R$, $0 < \tau < 1$; then

$$\phi(\tau R) \le \tau^\alpha A\Big[1 + \frac{\varepsilon}{\tau^\alpha}\Big]\phi(R) + BR^\beta. \tag{5.18}$$

Set $\gamma := \frac{\alpha+\beta}{2}$. We may assume, without loss of generality, that $2A > 1$, so that we may choose $\tau \in (0,1)$ satisfying $2A\tau^\alpha = \tau^\gamma$. Choose $\varepsilon_0 = \varepsilon_0(A, \alpha, \beta) > 0$ such that $\frac{\varepsilon_0}{\tau^\alpha} < 1$. Then (5.18) gives for $\varepsilon \le \varepsilon_0$

$$\phi(\tau R) \le \tau^\gamma \phi(R) + BR^\beta.$$

Iterating we find for $k \ge 0$

$$\begin{aligned}
\phi(\tau^k R) \le& \tau^\gamma \phi(\tau^{k-1}R) + B\tau^{(k-1)\beta}R^\beta \\
\le& \tau^{k\gamma}\phi(R) + B\tau^{(k-1)\beta}R^\beta \sum_{j=0}^{k-1} \tau^{j(\gamma-\beta)} \\
\le& \Big[\tau^{-\beta} + \tau^{-2\beta}\sum_{j=0}^{\infty}\tau^{j(\gamma-\beta)}\Big]\tau^{(k+1)\beta}(\phi(R) + BR^\beta) \\
=& c\tau^{(k+1)\beta}(\phi(R) + BR^\beta),
\end{aligned}$$

with a constant $c(A, \alpha, \beta)$. Choose $k \in \mathbb{N}$ such that $\tau^{k+1}R \le \rho \le \tau^k R$. Then

$$\phi(\rho) \le \phi(\tau^k R) \le c\tau^{(k+1)\beta}(\phi(R) + BR^\beta),$$

which gives (5.17), since $\tau^{k+1} \le \rho/R$. $\square$

5.4 Schauder estimates for elliptic systems in divergence form

5.4.1 Constant coefficients

Theorem 5.14 *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ be a solution to*

$$D_\alpha(A_{ij}^{\alpha\beta} D_\beta u^j) = -D_\alpha F_i^\alpha, \tag{5.19}$$

with $A_{ij}^{\alpha\beta}$ constant and satisfying the Legendre-Hadamard condition (3.17). *If $F_i^\alpha \in \mathcal{L}^{2,\mu}_{\mathrm{loc}}(\Omega)$, $0 \le \mu < n+2$, then $Du \in \mathcal{L}^{2,\mu}_{\mathrm{loc}}(\Omega)$, and*

$$\|Du\|_{\mathcal{L}^{2,\mu}(K)} \le c\Big(\|Du\|_{L^2(\Omega)} + [F]_{\mathcal{L}^{2,\mu}(\widetilde{\Omega})}\Big), \tag{5.20}$$

for every compact $K \Subset \widetilde{\Omega} \Subset \Omega$, with $c = c(n, m, K, \widetilde{\Omega}, \lambda, \Lambda, \mu)$.

Corollary 5.15 *In the hypothesis of the theorem, if $F_i^\alpha \in C^{k,\sigma}(\overline{\Omega})$, $k \ge 1$, then $u \in C^{k+1,\sigma}_{\mathrm{loc}}(\Omega)$ and*

$$\|u\|_{C^{k+1,\sigma}(K)} \le c\Big(\|Du\|_{L^2(\Omega)} + \|F\|_{C^{k,\sigma}(\overline{\Omega})}\Big),$$

with $c = c(n, m, K, \Omega, \lambda, \Lambda, \sigma)$

Proof. Thanks to Theorem 4.11, $u \in W^{k+1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ so that we may differentiate the system k times. If γ is a multi-index with $|\gamma| \le k$, then we obtain

$$D_\alpha(A_{ij}^{\alpha\beta} D_\beta(D_\gamma u^j)) = -D_\alpha(D_\gamma F_i^\alpha).$$

Theorems 5.5 and 5.14 then yield the result. The details of the estimate are left for the reader. □

Proof of Theorem 5.14. For a given ball $B_R(x_0) \subset \widetilde{\Omega}$ we write $u = v + w$ v is the solution (which exists and is unique by Corollary 3.46) to

$$\begin{cases} D_\alpha(A_{ij}^{\alpha\beta} D_\beta v^j) = 0 & \text{in } B_R(x_0) \\ v = u & \text{on } \partial B_R(x_0). \end{cases} \tag{5.21}$$

By Proposition 5.12 we get

$$\int_{B_\rho(x_0)} |Dv - (Dv)_{x_0,\rho}|^2 dx \le c\Big(\frac{\rho}{R}\Big)^{n+2} \int_{B_R(x_0)} |Dv - (Dv)_{x_0,R}|^2 dx,$$

consequently, using (5.15)

$$\begin{aligned} &\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 dx \\ &= \int_{B_\rho(x_0)} |Dv - (Dv)_{x_0,\rho} + Dw - (Dw)_{x_0,\rho}|^2 dx \\ &\le c_1\Big(\frac{\rho}{R}\Big)^{n+2} \int_{B_R(x_0)} |Dv - (Dv)_{x_0,R}|^2 dx + 2\int_{B_\rho(x_0)} |Dw - (Dw)_{x_0,\rho}|^2 dx \\ &\le c_2\Big(\frac{\rho}{R}\Big)^{n+2} \int_{B_R(x_0)} |Du - (Du)_{x_0,R}|^2 dx + c_3\int_{B_R(x_0)} |Dw - (Dw)_{x_0,R}|^2 dx \\ &\le c_2\Big(\frac{\rho}{R}\Big)^{n+2} \int_{B_R(x_0)} |Du - (Du)_{x_0,R}|^2 dx + c_3\int_{B_R(x_0)} |Dw|^2 dx. \end{aligned} \tag{5.22}$$

In order to estimate $\int_{B_R(x_0)} |Dw|^2 dx$ we observe that by (5.19) and (5.21) we have

$$\begin{aligned}\int_{B_R(x_0)} A_{ij}^{\alpha\beta} D_\beta w^j D_\alpha \varphi^i dx &= \int_{B_R(x_0)} F_i^\alpha D_\alpha \varphi^i dx \\ &= \int_{B_R(x_0)} \big(F_i^\alpha - (F_i^\alpha)_{x_0,R}\big) D_\alpha \varphi^i dx,\end{aligned}$$

for every $\varphi \in W_0^{1,2}(B_R(x_0), \mathbb{R}^m)$. Choose $\varphi = w$ as test function; on account of Proposition 3.45

$$\begin{aligned}\lambda \int_{B_R(x_0)} |Dw|^2 dx &\le \int_{B_R(x_0)} A_{ij}^{\alpha\beta} D_\alpha w^i D_\beta w^j dx \\ &= \int_{B_R(x_0)} \big(F_i^\alpha - (F_i^\alpha)_{x_0,R}\big) D_\alpha w^i dx \\ &\le \Big(\int_{B_R(x_0)} \sum_{i,\alpha} \big|F_i^\alpha - (F_i^\alpha)_{x_0,R}\big|^2 dx\Big)^{\frac{1}{2}} \Big(\int_{B_R(x_0)} \big|Dw\big|^2 dx\Big)^{\frac{1}{2}},\end{aligned} \tag{5.23}$$

thus, simplifying,

$$\int_{B_R(x_0)} |Dw|^2 dx \le c_3 \int_{B_R(x_0)} \sum_{\alpha,i} |F_i^\alpha - (F_i^\alpha)_{x_0,R}|^2 dx \le [F]_{2,\lambda}^2 R^\mu. \tag{5.24}$$

Inserting (5.24) into (5.22) we obtain

$$\phi(\rho) := \int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 dx \le A\Big(\frac{\rho}{R}\Big)^{n+2} \phi(R) + BR^\mu.$$

Lemma 5.13 with $\alpha = n+2$ and $\beta = \mu$ yields

$$\phi(\rho) \le c\bigg[\Big(\frac{\rho}{R}\Big)^\mu \phi(R) + B\rho^\mu\bigg] \le c_1\bigg[\Big(\frac{\rho}{R}\Big)^\mu \|Du\|_{L^2(\Omega)}^2 + [F]_{2,\lambda}^2 \rho^\mu\bigg],$$

i.e. the first part of the claim. Estimate (5.20) follows at once by covering K with balls. □

Exercise 5.16 State and prove a similar result when (5.19) is replaced by

$$D_\alpha(A_{ij}^{\alpha\beta} D_\beta u^j) = f_i - D_\alpha F_i^\alpha,$$

with $f_i \in L^{2,\mu}(\Omega)$, $F_i^\alpha \in \mathcal{L}^{2,\mu+2}(\Omega)$ for some $0 \le \mu < n$.
[Hint: Use Hölder's and Caccioppoli's inequalities to bound the term

$$\int_{B_R(x_0)} f_i w^i dx$$

arising in (5.23).]

5.4.2 Continuous coefficients

Theorem 5.17 *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ be a solution to*

$$D_\alpha(A^{\alpha\beta}_{ij} D_\beta u^j) = -D_\alpha F^\alpha_i, \tag{5.25}$$

with $A^{\alpha\beta}_{ij} \in C^0_{\mathrm{loc}}(\Omega)$ satisfying the Legendre-Hadamard condition (3.17)*. Then, if $F^\alpha_i \in L^{2,\lambda}_{\mathrm{loc}}(\Omega)$ for some $0 \le \lambda < n$, we have $Du \in L^{2,\lambda}_{\mathrm{loc}}(\Omega)$ and the following estimate*

$$\|Du\|_{L^{2,\lambda}(K)} \le c\Big(\|Du\|_{L^2(\widetilde{\Omega})} + \|F\|^2_{L^{2,\lambda}(\widetilde{\Omega})}\Big) \tag{5.26}$$

holds for every compact $K \Subset \widetilde{\Omega} \Subset \Omega$, where $c = c(n, m, \lambda, \Lambda, K, \widetilde{\Omega}, \omega)$ and ω is the modulus of continuity of $(A^{\alpha\beta}_{ij})$ in $\widetilde{\Omega}$:

$$\omega(R) := \sup_{\substack{x,y \in \widetilde{\Omega} \\ |x-y| \le R}} |A(x) - A(y)|,$$

Proof. Fix $x_0 \in K$ and $B_R(x_0) \subset \widetilde{\Omega}$ and write,

$$\begin{aligned} D_\alpha\big(A^{\alpha\beta}_{ij}(x_0) D_\beta u^j\big) &= -D_\alpha\big\{\big(A^{\alpha\beta}_{ij}(x) - A^{\alpha\beta}_{ij}(x_0)\big) D_\beta u^j + F^\alpha_i\big\} \\ &=: -D_\alpha G^\alpha_i. \end{aligned} \tag{5.27}$$

This is often referred to as *Korn's trick.* With the same computation of the proof of Theorem 5.14 (using (5.13) instead of (5.14)) we obtain

$$\int_{B_\rho(x_0)} |Du|^2 dx \le c\Big(\frac{\rho}{R}\Big)^n \int_{B_R(x_0)} |Du|^2 dx + c\int_{B_R(x_0)} |Du - Dv|^2 dx, \tag{5.28}$$

v being the solution of (5.21) with $A^{\alpha\beta}_{ij} = A^{\alpha\beta}_{ij}(x_0)$. As for (5.24) we have

$$\begin{aligned} \int_{B_R(x_0)} |D(u-v)|^2 dx &\le c\int_{B_R(x_0)} |G|^2 dx \\ &\le c\int_{B_R(x_0)} |F|^2 dx + c\omega(R)^2 \int_{B_R(x_0)} |Du|^2 dx. \end{aligned}$$

Together with inequality (5.28), this gives

$$\int_{B_\rho(x_0)} |Du|^2 dx \le A\Big\{\Big(\frac{\rho}{R}\Big)^n + \omega(R)^2\Big\} \int_{B_R(x_0)} |Du|^2 dx + \|F\|_{L^{2,\lambda}} R^\lambda.$$

Lemma 5.13 applied with $\phi(\rho) = \int_{B_\rho(x_0)} |Du|^2 dx^2$, $\alpha = n$, $\beta = \lambda$ and choose $R \le R_0$ so that $\omega(R_0)$ is small enough yields the result. $\square$

Corollary 5.18 *In the same hypothesis of the theorem, if $\lambda > n-2$, then $u \in C^{0,\sigma}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$, $\sigma = \frac{\lambda-n+2}{2}$.*

Proof. $Du \in L^{2,\lambda}_{\mathrm{loc}}(\Omega)$ by the theorem and the conclusion follows from Morrey's Theorem 5.7. $\square$

In particular if F and A are continuous, then u is Hölder continuous.

5.4.3 Hölder continuous coefficients

Theorem 5.19 *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ be a solution to*

$$D_\alpha(A^{\alpha\beta}_{ij} D_\beta u^j) = -D_\alpha F^\alpha_i, \tag{5.29}$$

with $A^{\alpha\beta}_{ij} \in C^{0,\sigma}_{\mathrm{loc}}(\Omega)$ satisfying the Legendre-Hadamard condition (3.17) *for some $\sigma \in (0,1)$. If $F^\alpha_i \in C^{0,\sigma}_{\mathrm{loc}}(\Omega)$, then we have $Du \in C^{0,\sigma}_{\mathrm{loc}}(\Omega)$. Moreover for every compact $K \Subset \widetilde{\Omega} \Subset \Omega$*

$$\|Du\|_{C^{0,\sigma}(K)} \leq c\Big(\|Du\|_{L^2(\widetilde{\Omega})} + \|F\|_{C^{0,\sigma}(\widetilde{\Omega})}\Big), \tag{5.30}$$

c depending on K, $\widetilde{\Omega}$, the ellipticity and the Hölder norm of the coefficients $A^{\alpha\beta}_{ij}$.

Proof. First observe that the hypothesis implies that $\omega(R) \leq cR^\sigma$, if ω is the modulus of continuity of the coefficients $A^{\alpha\beta}_{ij}(x)$. Moreover $F^\alpha_i \in \mathcal{L}^{2,n+2\sigma}(\Omega)$ by Campanato's theorem. We define G^α_i as in (5.27) and with the same argument used to obtain (5.22) and (5.24) we get for any ball $B_R(x_0) \Subset \Omega$

$$\begin{aligned}\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 dx \leq & c\Big(\frac{\rho}{R}\Big)^{n+2} \int_{B_R(x_0)} |Du - (Du)_{x_0,R}|^2 dx \\ & + c\int_{B_R(x_0)} |Dw|^2 dx,\end{aligned} \tag{5.31}$$

where $w := u - v$ and v is the solution to (5.21) with $A^{\alpha\beta}_{ij} = A^{\alpha\beta}_{ij}(x_0)$. By Proposition 3.45 and w being an admissible test function we get, as in (5.23),

$$\int_{B_R(x_0)} |Dw|^2 dx \leq c_1 \int_{B_R(x_0)} |F - F_{x_0,R}|^2 dx + c_1\omega(R)^2 \int_{B_R(x_0)} |Du|^2 dx.$$

Together with (5.31) and remembering that $Du \in L^{2,n-\varepsilon}_{\mathrm{loc}}(\Omega)$ for every $\varepsilon > 0$ by Theorem 5.17, we obtain

$$\begin{aligned}\phi(\rho) :=& \int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 dx \\ &\leq c\Big(\frac{\rho}{R}\Big)^{n+2} \int_{B_R(x_0)} |Du - (Du)_{x_0,R}|^2 dx \\ &\quad + \underbrace{c_1 \int_{B_R(x_0)} |F - F_{x_0,R}|^2 dx}_{\leq [F]^2_{2,n+2\sigma} R^{n+2\sigma}} + \underbrace{\omega(R)^2}_{\leq c_2 R^{2\sigma}} \underbrace{\int_{B_R(x_0)} |Du|^2 dx}_{\leq c(\varepsilon) R^{n-\varepsilon}} \\ &\leq c\Big(\frac{\rho}{R}\Big)^{n+2} \phi(R) + BR^{n+2\sigma-\varepsilon},\end{aligned} \tag{5.32}$$

which, by Lemma 5.13 implies $Du \in \mathcal{L}^{2,n+2\sigma-\varepsilon}_{\mathrm{loc}}(\Omega) \cong C^{0,\sigma-\frac{\varepsilon}{2}}_{\mathrm{loc}}(\Omega)$. In particular Du is locally bounded and we have

$$\int_{B_R(x_0)} |Du|^2 dx \leq \omega_n \sup_{B_R(x_0)} |Du|^2 R^n.$$

Consequently (5.32) improves to

$$\phi(\rho) \leq c\Big(\frac{\rho}{R}\Big)^{n+2} \phi(R) + BR^{n+2\sigma}$$

and, again by Lemma 5.13,

$$\phi(\rho) \leq c\Big(\frac{\phi(R)}{R^{n+2\sigma}} + B\Big)\rho^{n+2\sigma}. \tag{5.33}$$

We therefore conclude that $Du \in \mathcal{L}^{2,n+2\sigma}_{\mathrm{loc}}(\Omega) \cong C^{0,\sigma}_{\mathrm{loc}}(\Omega)$ and the estimate easily follows. □

5.4.4 Summary and generalizations

Theorem 5.20 *Assume that $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ is a solution to*

$$D_\alpha(A^{\alpha\beta}_{ij} D_\beta u^j) = f_i - D_\alpha F^\alpha_i,$$

where $k \geq 1$ and

(i) $A^{\alpha\beta}_{ij} \in C^k_{\mathrm{loc}}(\Omega)$ *(resp. $C^{k,\sigma}_{\mathrm{loc}}(\Omega)$ for some $0 < \sigma < 1$),*

(ii) $D^k F^\alpha_i \in L^{2,\lambda}_{\mathrm{loc}}(\Omega)$, *for some $\lambda < n$ (resp. $\mathcal{L}^{2,\lambda}_{\mathrm{loc}}(\Omega)$, $n \leq \lambda \leq n + 2\sigma$),*

(iii) $D^{k-1} f_i \in L^{2,\lambda}_{\mathrm{loc}}(\Omega)$, *for some $\lambda < n$ (resp. $\mathcal{L}^{2,\lambda}_{\mathrm{loc}}(\Omega)$, $n \leq \lambda \leq n+2\sigma$).*

Then $D^{k+1}u \in L^{2,\lambda}_{\mathrm{loc}}(\Omega)$ *(resp.* $\mathcal{L}^{2,\lambda}_{\mathrm{loc}}(\Omega)$*).*

In particular if $A^{\alpha\beta}_{ij} \in C^{k,\sigma}_{\mathrm{loc}}(\Omega)$, $F^\alpha_i \in C^{k,\sigma}_{\mathrm{loc}}(\Omega)$ *and* $f_i \in C^{k-1,\sigma}_{\mathrm{loc}}(\Omega)$, *then* $u \in C^{k+1,\sigma}_{\mathrm{loc}}(\Omega)$.

Proof. The proof is just sketched. With the same techniques used so far (freezing the coefficients $A^{\alpha\beta}_{ij}$ and solving the homogeneous system $D_\alpha(A^{\alpha\beta}_{ij} D_\beta v^j) = 0$ in $B_R(x_0) \Subset \Omega$) we obtain, for the case $\lambda < n$,

$$\begin{aligned}\int_{B_\rho(x_0)} |Du|^2 dx \leq c\left[\left(\frac{\rho}{R}\right)^n + \omega(R)^2\right] \int_{B_R(x_0)} |Du|^2 dx \\ + c\int_{B_R(x_0)} \left[|F|^2 + R^2|f|^2\right] dx,\end{aligned}$$

and for $n \leq \lambda \leq n + 2\sigma$,

$$\begin{aligned}\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 dx \leq & c\left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x_0)} |Du - (Du)_{x_0,R}|^2 dx \\ & + c\omega(R)^2 \int_{B_R(x_0)} |Du|^2 dx \\ & + c\int_{B_R(x_0)} \left[|F - F_{x_0,R}|^2 + R^2|f|^2\right] dx.\end{aligned}$$

If we differentiate the system and use induction as in Corollary 5.15, the result follows from from Lemma 5.13 as usual. □

5.4.5 Boundary regularity

Here we sketch how to get Schauder estimates at the boundary.

Theorem 5.21 *Let* $u \in W^{1,2}(\Omega)$ *be a solution to*

$$\begin{cases} -D_\alpha(A^{\alpha\beta}_{ij} D_\beta u^j) = -D_\alpha F^\alpha_i & \text{in } \Omega \\ u - g \in W^{1,2}_0(\Omega) \end{cases} \tag{5.34}$$

with $A^{\alpha\beta}_{ij} \in C^{k,\sigma}(\overline{\Omega})$ *satisfying the Legendre-Hadamard condition* (3.17), $F \in C^{k,\sigma}(\overline{\Omega})$, $g \in C^{k+1,\sigma}(\overline{\Omega})$, $\sigma \in (0,1)$. *Then we have* $u \in C^{k+1,\sigma}(\overline{\Omega})$ *and*

$$\|u\|_{C^{k+1,\sigma}(\overline{\Omega})} \leq c(\Omega, \sigma, \lambda, \|A\|_{C^{k,\alpha}(\overline{\Omega})})\big\{\|F\|_{C^{k,\sigma}(\overline{\Omega})} + \|g\|_{C^{k+1,\sigma}(\overline{\Omega})}\big\}, \tag{5.35}$$

where λ *is the ellipticity constant in* (3.17).

Proof. We give the proof in the case $A_{ij}^{\alpha\beta}$ are constant.

Step 1. Reduction to zero boundary value. It is enough to study the regularity of $v := u - g$ which solves

$$\begin{cases} -D_\alpha(A_{ij}^{\alpha\beta} D_\beta v^j) = -D_\alpha\big(F_i^\alpha + A_{ij}^{\alpha\beta} D_\beta g^j\big) := -DG_i^\alpha & \text{in } \Omega \\ v \in W_0^{1,2}(\Omega) \end{cases} \tag{5.36}$$

Observe that $G_i^\alpha \in C^{0,\sigma}(\Omega)$.

Step 2. Reduction to a flat boundary. As in step 1, Theorem 4.14, we may consider the boundary to be flat working locally.[2] We may thus assume that $\Omega = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n \geq 0\}$.

Step 3. Generalization of (5.32) to the boundary. Inequality (5.32) continues to hold true if $x_0 \in \Gamma = \{x \in \mathbb{R}^n_+ : x_n = 0\}$ and instead of $B_\rho(x_0)$ and $B_R(x_0)$ we write $B_\rho^+(x_0)$ and $B_R^+(x_0)$. This is because inequalities (5.13) and (5.14) generalize to the boundary as

$$\int_{B_\rho^+(x_0)} |v|^2 dx \leq c\Big(\frac{\rho}{R}\Big)^n \int_{B_R^+(x_0)} |v|^2 dx, \tag{5.37}$$

$$\int_{B_\rho^+(x_0)} |v - v_{x_0,\rho}|^2 dx \leq c\Big(\frac{\rho}{R}\Big)^{n+2} \int_{B_R^+(x_0)} |v - v_{x_0,\rho}|^2 dx, \tag{5.38}$$

because $v = 0$ on Γ, being this time $v_{x_0,\rho}$ the average of v on $B_\rho^+(x_0)$.

Step 4. Global estimates. To see that $Du \in C^{1,\sigma}(\overline{\Omega})$ we need to show that

$$\phi(\rho, x_0) := \int_{\Omega(x_0,\rho)} |Du - (Du)_{x_0,\rho}|^2 dx \leq c\rho^{n+2\sigma},$$

with c independent of x_0 and ρ. Assume first that $G = 0$ (this is the case if $g = 0$ and $F = 0$). Then we fix $R > 0$ satisfying the following property: for every $y_0 \in \partial\Omega$ the neighborhood $B_{2R}(y_0) \cap \Omega$ is diffeomorphic to $B_{2R}^+(0)$. Choose $x_0 \in \Omega$, $0 < \rho \leq R$.

Case 1: $x_0 \in \Omega_R := \Big\{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > R\Big\}$. Then

$$\begin{aligned} \int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 &\leq c\Big(\frac{\rho}{R}\Big)^{n+2} \int_{B_R(x_0)} |Du - (Du)_{x_0,R}|^2 \\ &\leq c\rho^{n+2} \frac{\|Du\|^2_{L^2(\Omega)}}{R^{n+2}}. \end{aligned}$$

In particular

$$[Du]^2_{\mathcal{L}^{2,n+2}(\Omega_R)} \leq c \frac{\|Du\|^2_{L^2(\Omega)}}{R^{n+2}}.$$

[2]This is why the constant in (5.35) depends on Ω.

Case 2: $r := \mathrm{dist}(x_0, \partial\Omega) \leq R$, and $\rho \leq r$. Call y_0 the projection of x_0 onto Γ. Then

$$\begin{aligned}\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 dx &\leq c\Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 dx \\ &\leq c\Big(\frac{\rho}{r}\Big)^{n+2} \int_{B^+_{2r}(y_0)} |Du - (Du)_{y_0,2r}|^2 dx \\ &\leq c_1\Big(\frac{\rho}{r}\Big)^{n+2}\Big(\frac{2r}{2R}\Big)^{n+2} \int_{B^+_{2R}(y_0)} |Du - (Du)_{y_0,2R}|^2 dx \\ &\leq c_2\Big(\frac{\rho}{R}\Big)^{n+2} \|Du\|^2_{L^2(\Omega)}.\end{aligned}$$

Case 3: $r := \mathrm{dist}(x_0, \partial\Omega) \leq R$, and $\rho > r$. Set y_0 to be the projection of x_0 on Γ. We have

$$\begin{aligned}\int_{\Omega(x_0,\rho)} |Du - (Du)_{x_0,\rho}|^2 dx &\leq \int_{B_{2\rho}(y_0)} |Du - (Du)_{y_0,2\rho}|^2 dx \\ &\leq c\Big(\frac{2\rho}{2R}\Big)^{n+2} \int_{B^+_{2R}(y_0)} |Du - (Du)_{y_0,2R}|^2 dx \\ &\leq c_3\Big(\frac{\rho}{R}\Big)^{n+2} \|Du\|^2_{L^2(\Omega)}.\end{aligned}$$

Therefore $u \in \mathcal{L}^{2,n+2}(\Omega)$, hence it is Hölder continuous by Campanato's theorem.

Step 5. Drop the assumption $G = 0$. Divide the estimates in the three cases as above. For the first case we have the same estimates as in Theorem 5.19. For the second case set v equal to the solution to the homogenous system in $B_r(x_0)$ with boundary data u and estimate

$$\begin{aligned}&\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 dx \\ &\qquad \leq c\Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 dx + \int_{B_r(x_0)} |D(u-v)|^2 dx.\end{aligned}$$

Next define w to be the solution to the homogeneous system in $B^+_{2R}(y_0)$ with boundary data u and find

$$\begin{aligned}\int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 dx &\leq \int_{B^+_{2r}(y_0)} |Du - (Du)_{y_0,2r}|^2 dx \\ &\leq c\Big(\frac{r}{R}\Big)^{n+2} \int_{B_{2R}(y_0)} |Du - (Du)_{y_0,2R}|^2 dx \\ &\quad + \int_{B_{2R}(y_0)} |D(u-w)|^2 dx.\end{aligned}$$

Next estimate $\int_{B_{2R}(y_0)} |D(u-w)|^2$ and $\int_{B_r(x_0)} |D(u-v)|^2$ as in (5.32) and apply Lemma 5.13. The third case is similar and is left for the reader. □

5.5 Schauder estimates for elliptic systems in non-divergence form

Schauder theory for elliptic systems in non-divergence form

$$A_{ij}^{\alpha\beta} D_{\alpha\beta} u^j = f_i$$

develops similarly to the case of systems in divergence form, see [20] [39]. First of all we observe that if $A_{ij}^{\alpha\beta}$ are constant, then

$$A_{ij}^{\alpha\beta} D_{\alpha\beta} u^j = D_\alpha (A_{ij}^{\alpha\beta} D_\beta u^j).$$

Consequently inequalities (5.13) and (5.14) hold true for u and its derivatives.

Theorem 5.22 *Assume that $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ is a solution to*

$$A_{ij}^{\alpha\beta} D_{\alpha\beta} u^j = f_i,$$

where $A_{ij}^{\alpha\beta}$ satisfies the Legendre-Hadamard condition and for some given $k \geq 0$

(i) $A_{ij}^{\alpha\beta} \in C^k_{\mathrm{loc}}(\Omega)$ *(resp. $C^{k,\sigma}_{\mathrm{loc}}(\Omega)$ for some $\sigma \in (0,1)$),*

(ii) $D^k f_i \in L^{2,\lambda}_{\mathrm{loc}}(\Omega)$, *for some $\lambda < n$ (resp. $\mathcal{L}^{2,\lambda}_{\mathrm{loc}}(\Omega)$, for some given $\lambda \in [n, n+2\sigma]$).*

Then $D^{k+2} u \in L^{2,\lambda}_{\mathrm{loc}}(\Omega)$ (resp. $\mathcal{L}^{2,\lambda}_{\mathrm{loc}}(\Omega)$).

In particular if $A_{ij}^{\alpha\beta}, f \in C^{k,\sigma}_{\mathrm{loc}}(\Omega)$, then $u \in C^{k+2,\sigma}_{\mathrm{loc}}(\Omega)$.

Proof. Repeating the arguments above, we freeze the coefficients:

$$A_{ij}^{\alpha\beta}(x_0) D_{\alpha\beta} u^j = -\big[A_{ij}^{\alpha\beta}(x) - A_{ij}^{\alpha\beta}(x_0)\big] D_{\alpha\beta} u^j + f_i.$$

By solving the homogeneous equation we get

$$\int_{B_\rho(x_0)} |D^2 u|^2 dx \leq c\left[\left(\frac{\rho}{R}\right)^n + c\omega(R)^2\right] \int_{B_R(x_0)} |D^2 u|^2 dx + c \int_{B_R(x_0)} |f|^2 dx,$$

where ω is the modulus of continuity of $(A_{ij}^{\alpha\beta})$. Then

$$\begin{aligned}\int_{B_\rho(x_0)} |D^2 u - (D^2 u)_{x_0,\rho}|^2 dx &\leq c\left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |D^2 u - (D^2 u)_{x_0,R}|^2 dx \\ &\quad + \omega(R)^2 \int_{B_R(x_0)} |D^2 u|^2 dx + \int_{B_R(x_0)} |f - f_{x_0,R}|^2 dx.\end{aligned}$$

The conclusion follows then from Lemma 5.13 for $k = 0$ and by differentiating the system for $k > 0$. □

As for systems in divergence form, one also proves boundary estimates, concluding:

Theorem 5.23 *Let u be a solution of*

$$\begin{cases} A_{ij}^{\alpha\beta} D_{\alpha\beta} u^j = f_i & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{5.39}$$

of class $W^{2,2}(\Omega, \mathbb{R}^m)$ or $C^2(\overline{\Omega}, \mathbb{R}^m)$, where the coefficients $A_{ij}^{\alpha\beta} \in C^{0,\sigma}(\overline{\Omega})$ satisfy the Legendre-Hadamard condition , $f_i \in C^{0,\sigma}(\overline{\Omega})$, $g \in C^{2,\sigma}(\overline{\Omega}, \mathbb{R}^m)$, and finally $\partial\Omega$ is of class $C^{2,1}$. Then $u \in C^{2,\sigma}(\overline{\Omega}, \mathbb{R}^m)$ and

$$\|u\|_{C^{2,\sigma}(\overline{\Omega})} \leq c\Big\{ \|f\|_{C^{0,\sigma}(\overline{\Omega})} + \|g\|_{C^{2,\sigma}(\overline{\Omega})} + \|D^2 u\|_{L^2(\Omega)} \Big\}, \tag{5.40}$$

where c is a constant depending on n, σ, Ω, the ellipticity constants and the $C^{0,\sigma}$ norm of the coefficients $A_{ij}^{\alpha\beta}(x)$.

It is worth noticing the (5.40) is just an *a priori estimate*; in fact presently we do not know how to show existence of a $W^{2,2}$ or C^2 solution of (5.39).

5.5.1 Solving the Dirichlet problem

Consider the linear elliptic operator

$$Lu := A_{ij}^{\alpha\beta} D_{\alpha\beta} u^j,$$

where the $A_{ij}^{\alpha\beta}$ are Hölder continuous and satisfy the Legendre-Hadamard condition. The a priori estimate (5.40) is one of the key points in proving existence of a classical solution $u \in C^{2,\sigma}(\overline{\Omega}, \mathbb{R}^m)$ of the boundary value problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{5.41}$$

where $f \in C^{0,\sigma}(\overline{\Omega}, \mathbb{R}^m)$, $g \in C^{2,\sigma}(\overline{\Omega}, \mathbb{R}^m)$, and $\partial\Omega$ is of class $C^{2,1}$. This is done via the so-called *continuity method*.

We consider the boundary value problem

$$\begin{cases} L_t u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{5.42}$$

where

$$L_t := (1-t)\Delta u + tLu, \qquad t \in [0,1].$$

As we saw in Chapter 1, Problem (5.42) is uniquely solvable if $t = 0$. Therefore (5.41) is uniquely solvable if the set

$$\Sigma := \big\{t \in [0,1] : (5.42) \text{ is uniquely solvable for any } f \in C^{0,\sigma}(\overline{\Omega}, \mathbb{R}^m), g \in C^{2,\sigma}(\overline{\Omega}, \mathbb{R}^m)\big\}$$

is both open and closed, as in this case $\Sigma = [0,1]$.

To prove this we shall use the a priori estimate (5.40), in fact an improvement of (5.40), i.e. the a priori estimate

$$\|u\|_{C^{2,\sigma}(\overline{\Omega})} \le c\|f\|_{C^{0,\sigma}(\overline{\Omega})}, \tag{5.43}$$

where without loss of generality we can also assume $g = 0$ (it is enough to consider the equation solved by $u - g$).

The estimate (5.43) is for instance a consequence of uniqueness, as stated in the following

Theorem 5.24 *Suppose that the boundary value problem*

$$L_t u = 0 \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega \tag{5.44}$$

has only the zero solution. Then for a solution $u \in W^{2,2}(\Omega, \mathbb{R}^m)$ of

$$L_t u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega \tag{5.45}$$

we have

$$\|D^2 u\|_{L^2(\Omega)} \le c\|f\|_{C^{0,\sigma}(\overline{\Omega})}. \tag{5.46}$$

Proof. We argue by contradiction. Assume that (5.46) does not hold. Then we can find a sequence $(u_k) \subset W^{2,2}(\Omega, \mathbb{R}^m)$ of functions solving (5.45) with $f = f_k \to 0$ in $C^{0,\sigma}(\overline{\Omega})$ and satisfying $\|D^2 u_k\|_{L^2} = 1$. Then by (5.40), (u_k) is uniformly bounded in $C^{2,\sigma}(\overline{\Omega}, \mathbb{R}^m)$, hence up to a subsequence we have $u_k \to u$ in $C^2(\overline{\Omega}, \mathbb{R}^m)$, where u solves

$$L_t u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad \|D^2 u\|_{L^2} = 1,$$

contradicting uniqueness. □

Notice that (5.46) implies that (5.40) can be improved to

$$\|u\|_{C^{2,\sigma}(\overline{\Omega})} \le c(\|f\|_{C^{0,\sigma}(\overline{\Omega})} + \|g\|_{C^{2,\sigma}(\overline{\Omega})}). \tag{5.47}$$

Of course, by Hopf maximum principle, see Exercise 1.4, (5.44) has zero as unique solution if $Lu = 0$ is a second order elliptic *scalar equation.*

Assuming (5.47), we now prove that Σ is both open and closed, thus

Theorem 5.25 *Assume that for every $t \in [0,1]$ Problem* (5.44) *has at most one solution. For instance suppose that*

$$Lu := A^{\alpha\beta} D_{\alpha\beta} u, \qquad u \text{ scalar},$$

where the coefficients of L are Hölder continuous, $f \in C^{0,\sigma}(\overline{\Omega}, \mathbb{R}^m)$, $g \in C^{2,\sigma}(\overline{\Omega}, \mathbb{R}^m)$ and $\partial\Omega$ is of class $C^{2,1}$. Then the boundary value problem (5.41) *has a unique solution $u \in C^{2,\sigma}(\overline{\Omega}, \mathbb{R}^m)$. Moreover*

$$\|u\|_{C^{2,\sigma}(\overline{\Omega})} \le c\big(\|f\|_{C^{0,\sigma}(\overline{\Omega})} + \|g\|_{C^{2,\sigma}(\overline{\Omega})}\big).$$

Proof. We have to prove that Σ is open and closed, since as already observed $0 \in \Sigma$.
Σ *is closed*: Let $t_k \in \Sigma$ and $t_k \to t$. For

$$f \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^m), \quad g \in C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^m)$$

we can find $u_k \in C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^m)$ solving

$$L_{t_k} u^{(k)} = f \text{ in } \Omega \qquad u^{(k)} = g \text{ on } \partial\Omega.$$

From (5.47) we infer, up to a subsequence, $u^{(k)} \to u$ in $C^2(\overline{\Omega})$ and

$$L_t u = f \text{ in } \Omega \qquad u = g \text{ on } \partial\Omega,$$

hence $t \in \Sigma$.

Σ *is open*: Let $t_0 \in \Sigma$. For $w \in C^{2,\sigma}(\overline{\Omega})$ let $T_t w = u_w$ be the unique solution of

$$L_{t_0} u_w = (L_{t_0} - L_t) w + f \quad \text{in } \Omega, \qquad u_w = g \quad \text{on } \partial\Omega.$$

From (5.47), and noticing that

$$L_{t_0} - Lt = (t - t_0)\Delta + (t_0 - t) L$$

we then infer

$$\|T_t w_1 - T_t w_2\|_{C^{2,\sigma}(\overline{\Omega})} \le c\,|t - t_0|\, \|w_1 - w_2\|_{C^{2,\sigma}(\overline{\Omega})},$$

i.e.

$$T_t : C^{2,\sigma}(\overline{\Omega}, \mathbb{R}^m) \to C^{2,\sigma}(\overline{\Omega}, \mathbb{R}^m)$$

is a contraction for $|t - t_0| < \delta$, δ small, hence is has a fixed point, which is a solution of (5.42). Consequently $(t_0 - \delta, t_0 + \delta) \subset \Sigma$. $\square$

Chapter 6
Some real analysis

We collect in this chapter some facts of *real analysis* that will be relevant for us in the sequel.

6.1 The distribution function and an interpolation theorem

Two functions are particularly useful when studying the size of a measurable function f: the *distribution function* of f and the *maximal function* of f.

6.1.1 The distribution function

Let Ω be an open set and $f : \Omega \to \mathbb{R}$ a measurable function. Given $t \geq 0$ set $A(f,t)$, or $A_f(t)$ or A_t

$$A(f,t) := \{x \in \Omega : |f(x)| > t\}.$$

The *distribution function* of f, $\lambda(f,t)$ or $\lambda_f(t)$ or simply $\lambda(t)$, is defined as the function $\lambda : [0,+\infty) \to \overline{\mathbb{R}}$ given by

$$\lambda(t) := |A(f,t)|.$$

Trivially:

1. $\lambda(t)$ is non increasing, continuous on the right and jumps at every value t that is assumed by $|f|$ on a set of positive measure:

$$\lambda_f(t) - \lambda_f(t^-) = \operatorname{meas}\{x \in \Omega \mid |f(x)| = t\}.$$

2. $\lambda_f(t) \to 0$ as $t \to \infty$ if $f \in L^1(\Omega)$, and

$$\|f\|_{L^\infty(\Omega)} = \inf\{t \geq 0 \mid \lambda_f(t) = 0\}.$$

Proposition 6.1 *For all $p > 0$ we have*

$$\int_{A_t} |f|^p dx = p\int_t^\infty s^{p-1}|A_s|ds + t^p|A_t|. \tag{6.1}$$

In particular

$$\int_\Omega |f|^p dx = p\int_0^\infty s^{p-1}|A_s|ds. \tag{6.2}$$

Proof. If χ_{A_t} is the characteristic function of A_t, by Fubini-Tonelli's theorem we have

$$\begin{aligned}\int_\Omega |f|^p dx &= \int_\Omega dx \int_0^{|f(x)|} p\,t^{p-1}dt \\ &= \int_\Omega \int_0^\infty p\,t^{p-1}\chi_{A_t} dt dx \\ &= p\int_0^\infty t^{p-1}\int_\Omega |\chi_{A_t}| dx dt,\end{aligned}$$

i.e. (6.2). Applying (6.2) to $\max\left\{|f|^p - t^p, 0\right\}$ we find (6.1). $\square$

Let $f \in L^p(\Omega)$, $p \geq 1$. Then

$$t^p\lambda_f(t) \leq \int_{A_f(t)} |f|^p dx \leq \|f\|^p_{L^p(\Omega)},$$

i.e. f satisfies the so called *p-weak estimate*

$$\lambda_f(t) \leq \left(\frac{\|f\|_{L^p(\Omega)}}{t}\right)^p.$$

Definition 6.2 *We say that a measurable function $u : \Omega \to \mathbb{R}$ is* weakly p-summable *or belongs to the* weak L^p-space, *denoted $L^p_w(\Omega)$, if*

$$\|f\|_{L^p_w(\Omega)} := \sup_{t>0} t\,\lambda_f(t)^{\frac{1}{p}} < \infty.$$

If $p = \infty$, we set $L^\infty_w(\Omega) = L^\infty(\Omega)$.

Notice that $\|f\|_{L^p_w(\Omega)}$ is not a norm and that $L^p(\Omega) \subset L^p_w(\Omega)$, while $L^p_w(\Omega) \subset L^q(\Omega)$ for every $q < p$ if Ω is bounded.

Exercise 6.3 Let f be measurable and $g \in L^p(\Omega)$; suppose that

$$\lambda_f(t) \leq \lambda_g(t) \quad \forall t > 0, \quad \text{or} \quad \lambda_f(t) \leq \frac{c}{t}\int_{A_g(t)} g dx \quad \forall t > 0.$$

Show that $f \in L^p(\Omega)$.

Exercise 6.4 Let $1 < p < \infty$. Show that $f \in L^p_w(\Omega)$ if and only if

$$\sup\left\{\frac{1}{|E|^{1-1/p}}\int_E |f|\,dx \;\middle|\; E \subset \Omega \text{ measurable}\right\} < \infty.$$

6.1.2 Riesz-Thorin's theorem

Let $T : \mathcal{A} \to \mathcal{B}$ be a linear operator from some linear space $\mathcal{A}$ into a linear space $\mathcal{B}$. Suppose that T maps continuously the Banach subspaces A_0 and A_1 of $\mathcal{A}$ into the Banach subspaces, respectively, B_0 and B_1 of $\mathcal{B}$. In this setting it often happens that there exists two families of Banach spaces, called *spaces of linear interpolation*, $A_t \subset \mathcal{A}$ and $B_t \subset \mathcal{B}$, $t \in [0,1]$ such that T maps continuously A_t into B_t for every $t \in [0,1]$. Results of this type are called *interpolation theorems*. The simplest one is that expressed by the interpolation inequality

$$\|f\|_{L^q(\Omega)} \leq \|f\|^{\theta}_{L^p(\Omega)} \|f\|^{1-\theta}_{L^r(\Omega)},$$

where

$$\theta \in [0,1], \quad 1 \leq p \leq q \leq r \leq \infty \quad \text{and} \quad \frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}$$

that is a simple consequence of Hölder's inequality.

The first interpolation theorem probably is *Riesz's convexity theorem* together with its complex extension, known as *Riesz-Thorin interpolation theorem*. We shall not need it in the sequel, thus we state it without proof.

Let $T : L^{p_0}(\Omega) + L^{p_1}(\Omega) \to \mathcal{M}$ be a linear map from $L^{p_0}(\Omega) + L^{p_1}(\Omega)$, the space of functions that can be written as $f+g$, $f \in L^{p_0}(\Omega)$, $g \in L^{p_1}(\Omega)$, into the space of measurable functions. We say that T *is of type* (p,q), $p_0 \leq p \leq p_1$ if it maps $L^p(\Omega)$ continuously into $L^q(\Omega)$, i.e.

$$\|Tf\|_{L^q(\Omega)} \leq M \|f\|_{L^p(\Omega)} \quad \forall\, f \in L^p(\Omega). \tag{6.3}$$

The greatest lower bound of the constants M such that (6.3) holds is called the (p,q)-*norm of* T.

Theorem 6.5 (M. Riesz convexity theorem) *Let the operator T be of type (p_0,q_0) and (p_1,q_1), where $p_0 \leq q_0$, $p_1 \leq q_1$ and $p_0 \leq p_1$. For all $\theta \in [0,1]$ define p_θ and q_θ by*

$$\frac{1}{p_\theta} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \frac{1}{q_\theta} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then T is of type (p_θ, q_θ) for all $\theta \in [0,1]$. Moreover if M_θ is the (p_θ,q_θ)-norm of T we have

$$M_\theta \leq M_0^{1-\theta} M_1^{\theta}.$$

The theorem states that T is of type (p_θ, q_θ) if it is of type (p_0,q_0), (p_1,q_1) and if $\left(\frac{1}{p_0}, \frac{1}{q_0}\right)$ and $\left(\frac{1}{p_1}, \frac{1}{q_1}\right)$ lie in the triangle of vertices $(0,0)$, $(0,1)$, $(1,1)$ and if $\left(\frac{1}{p_\theta}, \frac{1}{q_\theta}\right)$ lies on the segment joining the points $\left(\frac{1}{p_0}, \frac{1}{q_0}\right)$ and $\left(\frac{1}{p_1}, \frac{1}{q_1}\right)$. If we work with complex L^p-spaces and complex norms, the theorem extends to the case in which the points $\left(\frac{1}{p_0}, \frac{1}{q_0}\right)$ and $\left(\frac{1}{p_1}, \frac{1}{q_1}\right)$ may belong to the whole square $[0,1]\times[0,1]$ and, in this case, it is called *Riesz-Thorin* theorem. Two classical applications of the Riesz-Thorin theorem are the following.

Convolution operators. Given $f \in L^p(\mathbb{R}^n;\mathbb{C})$, the convolution operator

$$\Lambda_f g := f * g := \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

maps $L^1(\mathbb{R}^n;\mathbb{C})$ into $L^p(\mathbb{R}^n;\mathbb{C})$ and

$$\|f * g\|_{L^p} \leq \|f\|_{L^p}\|g\|_{L^1},$$

and $L^{p'}(\mathbb{R}^n;\mathbb{C})$ into $L^\infty(\mathbb{R}^n;\mathbb{C})$, and

$$\|f * g\|_{L^\infty} \leq \|f\|_{L^p}\|g\|_{L^{p'}}.$$

A consequence of Riesz-Thorin theorem is that Λ_f *maps continuously* any $L^r(\mathbb{R}^n;\mathbb{C})$ into $L^q(\mathbb{R}^n;\mathbb{C})$ and

$$\|f * g\|_{L^q} \leq \|f\|_{L^p}\|g\|_{L^r}$$

provided

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1.$$

This is known as *Young's inequality.*

Fourier transform. As a consequence of Young's inequality the Fourier transform of $f \in L^1(\mathbb{R}^n;\mathbb{C})$

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x)dx$$

maps $L^1(\mathbb{R}^n;\mathbb{C})$ into $L^\infty(\mathbb{R}^n;\mathbb{C})$ with norm not exceeding 1, and actually, by *Riemann-Lebesgue's theorem*[1], $L^1(\mathbb{R}^n;\mathbb{C})$ is mapped into $C^0_0(\mathbb{R}^n;\mathbb{C})$. If $f \in L^1(\mathbb{R}^n;\mathbb{C}) \cap L^2(\mathbb{R}^n;\mathbb{C})$, then $\widehat{f} \in L^2(\mathbb{R}^n;\mathbb{C})$ and $\|\widehat{f}\|_{L^2} = \|f\|_{L^2}$ by *Plancherel theorem.* Consequently, the Fourier transform extends as an isometry from $L^2(\mathbb{R}^n;\mathbb{C})$ into itself.

---

[1] Prove as exercise the following

Theorem 6.6 (Riemann-Lebesgue) *For any $f \in L^1(\mathbb{R}^n;\mathbb{C})$ we have*

$$\lim_{|\xi|\to\infty} \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x)dx = 0.$$

[Hint: Start with $f \in C^0_c(\mathbb{R}^n;\mathbb{C})$.]

Riesz-Thorin theorem then yields that the Fourier transform maps every $L^p(\mathbb{R}^n;\mathbb{C})$, $1 \le p \le 2$ into its dual space $L^{p'}(\mathbb{R}^n;\mathbb{C})$ and

$$\|\widehat{f}\|_{L^{p'}} \le \|f\|_{L^p}.$$

This is known as *Hausdorff-Young inequality*. By duality this holds also for $2 \le p < \infty$.

6.1.3 Marcinkiewicz's interpolation theorem

Let $\Omega \subset \mathbb{R}^n$ be a measurable set. Suppose that T maps measurable functions on Ω into measurable functions on Ω. We say that T is *quasi-linear* if

$$|T(f+g)| \le Q\big(|Tf| + |Tg|\big)$$

for all f and g, Q being a constant independent of f and g. We say that T is of *weak-(p,q) type*, $1 \le p < \infty$, if there is a constant $A \ge 0$ such that

$$\lambda_{Tf}(s) \le \left(\frac{A\|f\|_{L^p(\Omega)}}{s}\right)^q, \quad \forall f \in L^p(\Omega).$$

Theorem 6.7 *Let T be a quasi-linear operator both of weak-(p_0,p_0) and weak-(p_1,p_1) type, $1 \le p_0 < p_1 \le \infty$. Then T is of strong-(p,p) type for all $p_0 < p < p_1$.*

Proof. Let $u \in L^p(\Omega)$ and $s > 0$ be fixed, and let

$$\begin{aligned} E_s &:= \{x \in \Omega : |Tu(x)| > s\} \\ v &:= u\chi_{\{x\in\Omega:|u(x)|\le \frac{s}{2QA_1}\}}, \\ w &:= u\chi_{\{x\in\Omega:|u(x)|> \frac{s}{2QA_1}\}}. \end{aligned}$$

As $u = v + w$, we have $|Tu| \le Q(|Tv| + |Tw|)$, hence

$$E_s \subset \left\{x \in \Omega : |Tv(x)| > \frac{s}{2Q}\right\} \cup \left\{x \in \Omega : |Tw| > \frac{s}{2Q}\right\} =: F_s \cup G_s. \quad (6.4)$$

Let A_0 and A_1 be the (p_0,p_0) and (p_1,p_1) norms of T respectively, i.e.

$$\lambda_{Tf}(s) \le \left(\frac{A_i\|f\|_{L^{p_i}}}{s}\right)^{p_i}, \quad \forall f \in L^{p_i}(\Omega),\ i = 0,1.$$

We have

$$|G_s| \le \left(\frac{2A_0\|w\|_{p_0}Q}{s}\right)^{p_0} = \frac{c_1}{s^{p_0}} \int_{\{|u|>s/(2QA_1)\}} |u(x)|^{p_0}dx,$$

and so

$$\begin{aligned}\int_0^\infty p\lambda^{p-1}|G_s|ds &\le c_1 p\int_0^\infty s^{p-p_0-1}\int_{\{|u|>s/(2QA_1)\}}|u(x)|^{p_0}dxds\\ &= c_1 p\int_\Omega |u(x)|^{p_0}\int_0^{2QA_1|u(x)|} s^{p-p_0-1}dsdx\\ &= \frac{c_2 p}{p-p_0}\int_\Omega |u(x)|^p dx\\ &= \frac{c_2 p}{p-p_0}\|u\|_{L^p}^p.\end{aligned}\tag{6.5}$$

If $p_1=\infty$, then $\|Tv\|_{L^\infty}\le A_1\|v\|_{L^\infty}\le \frac{s}{2Q}$, and so $F_s=\emptyset$; otherwise we have

$$|F_s|\le\left(\frac{2A_1\|v\|_{p_1}Q}{s}\right)^{p_1}=\frac{c_3}{s^{p_1}}\int_{\{|u|\le s/(2QA_1)\}}|u(x)|^{p_1}dx,$$

so that

$$\begin{aligned}\int_0^\infty ps^{p-1}|F_s|ds &\le c_3 p\int_0^\infty s^{p-p_1-1}\int_{\{|u|\le s/(2QA_1)\}}|u(x)|^{p_1}dxds\\ &= c_3 p\int_\Omega |u(x)|^{p_1}\int_{2QA_1|u(x)|}^\infty s^{p-p_1-1}dsdx\\ &= \frac{c_4 p}{p_1-p}\int_\Omega |u(x)|^p dx\\ &= \frac{c_4 p}{p_1-p}\|u\|_{L^p}^p.\end{aligned}\tag{6.6}$$

By (6.2), (6.4), (6.5) and (6.6) we have

$$\begin{aligned}\|Tu\|_{L^p}^p &= \int_0^\infty ps^{p-1}|E_s|ds\\ &\le \int_0^\infty ps^{p-1}(|F_s|+|G_s|)ds\\ &\le \left(\frac{c_2 p}{p-p_0}+\frac{c_4 p}{p_1-p}\right)\|u\|_{L^p}^p,\end{aligned}$$

for $p_1<\infty$, while for $p=\infty$.

$$\|Tu\|_{L^p}^p\le\int_0^\infty ps^{p-1}|G_s|ds\le\frac{c_2 p}{p-p_0}\|u\|_{L^p}^p. \tag{6.7}$$

$\square$

A variant of the previous proof actually yields

Theorem 6.8 *Let p_0, p_1, q_0, q_1 be such that $1 \le p_i \le q_i \le \infty$, $i = 0, 1$, $p_0 \le p_1$ and $q_0 \ne q_1$. Suppose T is quasi-linear and simultaneously of weak-(p_0, q_0) and weak-(p_1, q_1) type. For $\theta \in (0, 1)$ define p_θ and q_θ by*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then T is of strong-(p_θ, q_θ) type.

6.2 The maximal function and the Calderon-Zygmund argument

We discuss here two more tools that are very useful to deal with the measure of the size of a function.

6.2.1 The maximal function

The Hardy-Littlewood maximal function of a locally summable function f in $\mathbb{R}^n$, i.e. a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, is defined for all $x \in \mathbb{R}^n$ by

$$Mf(x) := \sup_{r>0} \fint_{B_r(x)} |f(y)|dy.$$

We clearly have

1. Mf is lower semicontinuous, $A(Mf, t)$ is open for all t, hence measurable, homogeneous of degree one and quasi-linear,
$$M(f+g) \le M(f) + M(g).$$
2. M maps $L^\infty(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ continuously
$$\|Mf\|_{L^\infty} \le \|f\|_{L^\infty},$$
and, on account of Lebesgue differentiation theorem,
$$|f(x)| \le Mf(x), \quad \text{for a.e. } x \in \mathbb{R}^n.$$
3. Mf is not bounded in L^1, indeed Mf is never in L^1 except for $f \equiv 0$ since Mf decays at infinity no faster than $|x|^{-n}$ (up to constant).

Remark 6.9 Sometimes it is convenient to define the maximal function as

$$Mf(x) := \sup_{r>0} \fint_{Q_r(x)} |f(y)|dy,$$

where $Q_r(x)$ is the cube centered at x with sides parallel to the coordinate axis and side $2r$:

$$Q_r(x) := \big\{y \in \mathbb{R}^n \;\big|\; |y_i - x_i| < r, \quad i = 1, \dots, n\big\}.$$

The maximal theorem of Hardy and Littlewood

The key result about the maximal function is the following

Theorem 6.10 (Hardy-Littlewood) *Given a function $f \in L^1(\mathbb{R}^n)$ we have $Mf \in L^1_w(\mathbb{R}^n)$; more precisely*

$$\begin{aligned}\lambda_{Mf}(t) &:= \left|\left\{x \in \mathbb{R}^n \;\middle|\; Mf(x) > t\right\}\right| \\ &\leq \frac{c(n)}{t} \int_{\left\{x \middle| |f(x)| > \frac{t}{2}\right\}} |f(y)| dy.\end{aligned} \tag{6.8}$$

In particular $Mf(x) < \infty$ for a.e. x.

Proof. If $x \in A(Mf, t)$, then for some $r(x) > 0$

$$\fint_{B_{r(x)}(x)} |f(y)| dy > t,$$

or equivalently

$$|B_{r(x)}(x)| < \frac{1}{t} \int_{B_{r(x)}(x)} |f(y)| dy.$$

Using a simple covering argument[2] we can choose points $x_i \in \mathbb{R}^n$, $i = 1, 2, \dots$

$$\begin{aligned}|A(Mf, t)| &\leq \sum_{i=1}^{\infty} |B_{r(x_i)}(x_i)| \\ &\leq \frac{1}{t} \sum_{i=1}^{\infty} \int_{B_{r(x_i)}(x_i)} |f(y)| dy \\ &\leq \frac{\xi(n)}{t} \int_{\mathbb{R}^n} |f(y)| dy.\end{aligned}$$

Next we set

$$\widetilde{f}(x) := f\chi_{A(f,t/2)} = \begin{cases} f(x) & \text{if } |f(x)| > \frac{t}{2}, \\ 0 & \text{if } |f(x)| \leq \frac{t}{2}. \end{cases}$$

[2]We use the following simple version of Besicovitch-Vitali covering theorem:

Lemma 6.11 *Let $E \subset \mathbb{R}^n$ and $r : E \to \mathbb{R}^n$ be a bounded function. There exists a countable family $\{x_i : i \in \mathbb{N}\}$ in E such that*

(i) $E \subset \bigcup_{i=0}^{\infty} B_{r(x_i)}(x_i)$,

(ii) *every point of E belongs at most to $\xi(n)$ balls $B_{r(x_i)}(x_i)$, where $\xi(n)$ is a dimensional constant;*

or even the version in which (ii) is replaced by

(ii)' *the balls $B_{\frac{1}{3}r(x_i)}(x_i)$ are disjoint.*

Of course $\widetilde{f} \in L^1(\mathbb{R}^n)$ and $Mf(x) \leq M\widetilde{f}(x) + \frac{t}{2}$, therefore

$$A(Mf,t) \subset A(M\widetilde{f}, t/2),$$

consequently

$$\begin{aligned} |A(Mf,t)| &\leq |A(M\widetilde{f}, t/2)| \\ &\leq \frac{2\xi(n)}{t} \int_{\mathbb{R}^n} |\widetilde{f}(y)|dy \\ &= \frac{2\xi(n)}{t} \int_{\{x\in\mathbb{R}^n : |f(x)|>t/2\}} |f(y)|dy. \end{aligned}$$

□

Since $\{x \in \mathbb{R}^n : Mf^p(x) > t\} = \{x \in \mathbb{R}^n : Mf(x) > t^{\frac{1}{p}}\}$, (6.8) also yields

$$\Big|\Big\{x \in \mathbb{R}^n \;\Big|\; Mf(x) > t\Big\}\Big| \leq \frac{c(n)}{t^p} \int_{\{x\in\mathbb{R}^n : |f(x)|>t/2\}} |f(y)|^p dy$$

if $f \in L^p(\mathbb{R}^n)$, $p \geq 1$.

From Marcinkiewicz's theorem (see (see (6.7)) with $p_0 = 1$ in particular) or simply multiplying (6.8) by t^{p-1} and using Proposition 6.1, we easily deduce

Proposition 6.12 *Let $f \in L^p(\mathbb{R}^n)$, $p > 1$. Then $Mf \in L^p(\mathbb{R}^n)$ and*

$$\|Mf\|_{L^p} \leq A(n,p)\|f\|_{L^p},$$

where $A(n,p) \sim \frac{1}{p-1}$ as $p \to 1$.

Also notice that

1. If $f_k \to f$ in L^p, then $M(|f_k - f|^p) \to 0$ in measure; in fact

$$\Big|\Big\{x \;\Big|\; M(|f_k - f|^p)(x) > \varepsilon\Big\}\Big| \leq \frac{c(n)}{\varepsilon^p}\|f_k - f\|_{L^p}.$$

2. $t^p\Big|\Big\{x \;\Big|\; Mf(x) > t\Big\}\Big| \to 0$ as $t \to \infty$.

Finally, from $|f(x)| \leq Mf(x)$ a.e. we also infer $\|f\|_{L^p} \leq \|Mf\|_{L^p}$, $p > 1$.

Lebesgue's differentiation theorem

In several instances we used that for a.e. x

$$\fint_{B_r(x)} f(y)dy \to f(x) \quad \text{as } r \to 0.$$

This can be in fact inferred using the maximal theorem.

Theorem 6.13 *Let $f \in L^p(\mathbb{R}^n)$, $p \geq 1$. For a.e. x we have $|f(x)| \leq Mf(x)$ and*

$$\lim_{r\to 0} \fint_{B_r(x)} |f(y) - f(x)|^p dy = 0.$$

Proof. Consider a sequence $\{f_k\} \subset C_c^\infty(\mathbb{R}^n)$ that converges to f in $L^p(\mathbb{R}^n)$. By Proposition 6.12 we may assume that

$$f_k(x) \to f(x) \text{ and } M(|f_k - f|^p)(x) \to 0 \text{ for a.e. } x \in \mathbb{R}^n. \tag{6.9}$$

Set $E := \{x \in \mathbb{R}^n \mid (6.9) \text{ holds}\}$. Then, as

$$f_k(x) = \lim_{\rho\to 0} \fint_{B_\rho(x)} f_k(y)dy$$

implies $|f_k(x)| \leq Mf_k(x)$, we see that $|f(x)| \leq Mf(x)$ for every $x \in E$.

The second part of the claim follows observing that

$$\begin{aligned}\fint_{B_r(x)} |f(y) - f(x)|^p dy &\leq c(p) \fint_{B_r(x)} \{[f(y) - f_k(y)]^p + [f(x) - f_k(x)]^p \\ &\qquad + [f_k(x) - f_k(y)]^p\} dy \\ &\leq c(p)\Big(\operatorname*{osc}_{B_r(x)} f_k\Big)^p + c(p)2M(|f - f_k|^p)(x).\end{aligned}$$

□

Exercise 6.14 Let $f \in L^1(\mathbb{R}^n)$. Deduce that for a.e. x

$$f(x) = \lim_{r\to 0} \fint_{B_r(x)} f(y)dy$$

and, indeed,

$$f(x) = \lim_{\substack{B \text{ ball} \\ |B|\to 0 \\ B\ni x}} \fint_B f(y)dy.$$

A theorem of F. Riesz

Here all cubes will have sides parallel to the axis. Let Q_0 be an n-dimensional cube in $\mathbb{R}^n$ and let $\mathcal{F}$ denote the family of all countable coverings of Q_0 by cubes with disjoint interiors. For $f \in L^1(Q_0)$ set

$$K_p(f) := \left[\sup_{\{Q_i\}\in\mathcal{F}} \sum_{i=1}^\infty |Q_i| \Big(\fint_{Q_i} |f(x)|dx\Big)^p\right]^{\frac{1}{p}}.$$

Theorem 6.15 (F. Riesz) *Given any $f \in L^1(Q_0)$, then $f \in L^p(Q_0)$ if and only if $K_p(f) < \infty$. Moreover*

$$\|f\|_{L^p(Q_0)} = K_p(f).$$

Proof. By Jensen's inequality $K_p(f) \le \|f\|_{L^p(Q_0)}$. Conversely, assume $K_p(f) < \infty$ and let $\{Q_{i,k}\}$ be the covering of Q_0 obtained dividing Q_0 into 2^{nk} cubes congruent to $2^{-k}Q_0$. Define

$$\varphi_k(x) = \fint_{Q_{i,k}} |f(y)|dy, \qquad \text{if } x \in Q_{i,k}.$$

We have $\varphi_k \in L^p(Q_0)$ and

$$\int_{Q_0} \varphi_k^p dx = \sum_{i=1}^{\infty} |Q_{i,k}|^{1-p} \Big(\int_{Q_{i,k}} |f|dx \Big)^p \le K_p^p(f) < \infty.$$

By the differentiation theorem $\varphi_k(x) \to |f(x)|$ as $k \to \infty$ for a.e. x; Fatou's lemma then yields

$$\int_{Q_0} |f|^p dx \le \liminf_{k\to\infty} \int_{Q_0} |\varphi_k|^p dx \le K_p^p(f).$$

□

6.2.2 Calderon-Zygmund decomposition argument

Here we present the *Calderon-Zygmund* or *stopping time argument* and its relations to the maximal and distribution functions. The conclusion of the argument states

Theorem 6.16 (Calderon-Zygmund decomposition) *Let Q be an n-dimensional cube in $\mathbb{R}^n$ and let f be a non-negative function in $L^1(Q)$. Fix a parameter $t > 0$ in such a way that*

$$\fint_Q f(x)dx \le t.$$

Then there exists a countable family $\{Q_i\}_{i\in I}$ of cubes in the dyadic decomposition of Q (as defined in the proof) such that

(i) $t < \fint_{Q_i} f dx \le 2^n t$ *for every* $i \in I$*;*

(ii) $f(x) \le t$ *for a.e.* $x \in Q \setminus \bigcup_{i\in I} Q_i$.

Proof. By bisection of the sides of Q, we subdivide Q into 2^n congruent subcubes. Those cubes P which satisfy

$$\fint_P f(x)dx > t$$

will belong to the family $\{Q_i\}$, while the others are similarly divided into dyadic subcubes and the process is repeated indefinitely (or finitely if at some step there is no such cube). Let $\mathcal{Q} := \{Q_i\}$ denote the family of subcubes so obtained for which

$$\fint_{Q_i} f(x)dx > t$$

and for each Q_i denote by $\widetilde{Q}_i$ the cube whose subdivision gave rise to Q_i. Since $|\widetilde{Q}_i| = 2^n |Q_i|$ we get immediatly (i) as

$$\fint_{\widetilde{Q}_i} f(x)dx \leq t.$$

If $x \in Q \setminus \bigcup_{i \in I} Q_i$ and is not on the boundary of some Q_i, then clearly it belongs to infinitely many cubes P in the successive subdivision with $|P| \to 0$. As Lebesgue differentiation theorem implies

$$f(x) = \lim_{\substack{P \ni x \\ |P| \to 0}} \fint_P f(y)dy \qquad a.e.\ x,$$

(ii) follows at once. □

Remark 6.17 If $f \in L^1(\mathbb{R}^n)$ and t is any positive constant, then the conclusion of Theorem 6.16 holds true, since we can first subdivide $\mathbb{R}^n$ into cubes for which we have

$$\fint_Q f(x)dx \leq t.$$

Remark 6.18 Let $\{Q_i^t\}$ and $\{Q_i^s\}$ be the families in the decompositions of Calderon-Zygmund corresponding to the parameters t and s, $s > t > 0$. Then each Q_i^s is contained in some Q_j^t.

We finally show that for any function $f \in L^1(Q)$

(a) the distibution function of the maximal function $Mf(x)$

(b) the function $\frac{1}{t} \int_{\{|f|>t\}} |f(y)|dy$

(c) the sum of the measures of the cubes Q_i^t of the Calderon-Zygmund decomposition relative to $|f|$ and t

are equivalent for large values of t. In fact we have

Proposition 6.19 *Let Q be a cube in $\mathbb{R}^n$, $f \in L^1(Q)$ and let*

$$\fint_Q |f| dx \leq t.$$

Denote by $\{Q_i^t\}_{i\in I}$ the Calderon-Zygmund cubes relative to $|f|$ and t. We have

$$\frac{2^{-n}}{t} \int_{\{x\in Q : |f(x)|>t\}} |f| dx \leq \sum_{i\in I} |Q_i^t| \leq \frac{2}{t} \int_{\{x\in Q : |f(x)|>t/2\}} |f| dx, \quad (6.10)$$

and, for constants $\gamma(n)$, $c_1(n)$ and $c_2(n)$,

$$\begin{aligned} \frac{1}{t} \int_{\{x\in Q : |f(x)|>t\}} |f| dx &\leq c_1(n) \big| A(Mf, \gamma(n)t) \big| \\ &\leq \frac{c_2(n)}{t} \int_{\{x\in Q : |f(x)|> \frac{\gamma(n)t}{2}\}} |f| dx. \end{aligned} \quad (6.11)$$

Proof. From

$$\int_{\{x\in Q : |f(x)|>t\}} |f| dx \leq \sum_{i\in I} \int_{Q_i^t} |f| dx \leq 2^n t \sum_{i\in I} |Q_i^t|$$

and

$$\begin{aligned} t|Q_i^t| &\leq \int_{Q_i^t} |f| dx = \int_{\{x\in Q_i^t : |f(x)|>t/2\}} |f| dx + \int_{\{x\in Q_i^t : |f(x)|\leq t/2\}} |f| dx \\ &\leq \int_{\{x\in Q_i^t : |f(x)|>t/2\}} |f| dx + \frac{t}{2} |Q_i^t| \end{aligned}$$

(6.10) follows. The inequality on the right-hand side of (6.11) is the maximal theorem. Finally, consider $x \in Q_i^t$; we have

$$\fint_{Q_i^t} |f(y)| dy > t,$$

and taking the smallest ball B centered at x and containing Q_i^t, we find

$$Mf(x) \geq \fint_B |f(y)| dy \geq \gamma(n) \fint_{Q_i^t} |f(y)| dy > \gamma(n) t.$$

Hence $A(Mf, \gamma(n)t) \supset \bigcup_{i\in I} Q_i^t$ and the inequality on the left-hand side of (6.11) follows from (6.10).

$\square$

6.3 *BMO*

The notion of functions of bounded mean oscillation was introduced and studied by F. John and L. Nirenberg in connection with the work of F. John on quasi-isometric maps and of J. Moser on Harnack inequality. It then proved to be extremely relevant in many different fields of real and complex analysis.

Definition 6.20 *Let Q_0 be an n-dimensional cube in $\mathbb{R}^n$. We say that a function $u \in L^1(Q_0)$ belongs to the space of functions with bounded mean oscillation $BMO(Q_0)$ if*

$$|u|_* := \sup \fint_Q |u - u_Q|\, dx < +\infty, \tag{6.12}$$

where the supremum is taken over all the n-cubes $Q \subset Q_0$ whose sides are parallel to those of Q_0, and being $u_Q := \fint_Q u\, dx$.

Commonly BMO is defined in the whole of $\mathbb{R}^n$ by requiring $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ and the supremum in (6.12) to be taken over all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axis (or even over all cubes in $\mathbb{R}^n$). But for future use we prefer to work in a cube given cube Q_0. It is easily seen that $BMO(Q_0) \cong \mathcal{L}^{1,n}(Q_0)$; we shall in fact see later that $BMO(Q_0) \cong \mathcal{L}^{p,n}(Q_0)$ for all p, $1 \le p < \infty$.

It is worth remarking that

(i) $u \in BMO(Q_0)$ if and only if for every $Q \subset Q_0$ there is a constant $c_{u,Q}$ such that

$$\sup_Q \fint_Q |u - c_{u,Q}| dx < \infty.$$

Indeed for $x \in Q$

$$\begin{aligned} |u(x) - u_Q| &\le |u(x) - c_{u,Q}| + |c_{u,Q} - u_Q| \\ &\le |u(x) - c_{u,Q}| + \fint_Q |u(y) - c_{u,Q}| dy, \end{aligned}$$

and averaging over Q we get

$$\fint_Q |u(x) - u_Q| dx \le 2 \fint_Q |u(x) - c_{u,Q}| dx.$$

(ii) In the definition we can replace cubes with balls, and in fact only balls with small radii are relevant.

(iii) $L^\infty(Q_0) \subset BMO(Q_0)$, but, for instance $\log|x| \in BMO([-1,1])$, and in fact to $\log|x| \in BMO(\mathbb{R})$; this can be seen observing that homotheties leave invariant $BMO(\mathbb{R})$ and that

$$\int_{x-1}^{x+1} \big|\log|y|\big|dy \qquad \text{and} \qquad \int_{x-1}^{x+1} \big|\log|y| - \log|x|\big|dy$$

are uniformly bounded with respect to x if $|x| \le 1$ and $|x| \ge 1$ respectively.

(iv) If ψ is a Lipschitz function (uniformly continuous suffices), the $u \in BMO$ implies $\psi(u) \in BMO$. Consequently $\max\{u,0\}$, $\min\{u,0\}$, $|u|$ are BMO function if u is a BMO function. In particular, if Ω is a bilipschitz tranform of a cube Q, then $BMO(\Omega) \cong BMO(Q)$.

(v) $W^{1,n}(Q_0) \subset BMO(Q_0)$. Indeed by Jensen's and Poincaré's inequalities, we see that

$$⨍_Q |u-u_Q|dx \le \bigg(⨍_Q |u-u_Q|^n dx\bigg)^{\frac{1}{n}} \le c\bigg(|Q| \int_Q |Du|^n dx\bigg)^{\frac{1}{n}}.$$

(vi) Finally, BMO enjoys a rigidity that is typical of smooth functions. For instance, if $u \in BMO(\mathbb{R}^n)$ and $\Omega \subset \mathbb{R}^n$ is a measurable set, then $u\chi_\Omega$ is not necessarily a BMO function. Indeed

$$v(x) := \chi_{(0,+\infty)} \log|x| \notin BMO(\mathbb{R})$$

while $\log|x| \in BMO(\mathbb{R})$.

6.3.1 John-Nirenberg lemma I

One of the important properties of BMO functions is the following weak estimate:

Theorem 6.21 (John-Nirenberg lemma I [63]) *There are constants $c_1, c_2 > 0$, depending only n, such that*

$$\Big|\big\{x \in Q \;\big|\; |u(x)-u_Q| > t\big\}\Big| \le c_1 \exp\Big(-c_2 \frac{t}{|u|_*}\Big) \cdot |Q| \tag{6.13}$$

for all cubes $Q \subset Q_0$ with sides parallel to those of Q_0, all $u \in BMO(Q_0)$ and all $t > 0$.

Proof. As $u \in BMO(Q_0) \Rightarrow u \in BMO(Q)$ and the sup in (6.12) can only decrease if we consider Q instead of Q_0, it is enough to prove (6.13) for $Q = Q_0$ only. Moreover, there is no loss of generality in assuming

$|u|_* = 1$: if it is not so, we can consider $\tilde{u} := u/|u|_*$, for which $|\tilde{u}|_* = 1$ and

$$\{x \in Q_0 : |u(x) - u_{Q_0}| > t\} = \left\{x \in Q_0 : |\tilde{u}(x) - \tilde{u}_{Q_0}| > \frac{t}{|u|_*}\right\}.$$

Take $\alpha > 1 = |u|_* \geq \fint_{Q_0} |u - u_{Q_0}|\, dx$; applying the Calderon-Zygmund argument with $f = |u - u_{Q_0}|$ and parameter α, we find a sequence $\{Q_k^1\}_{k \in K_1}$ such that

$$\alpha < \fint_{Q_k^1} |u - u_{Q_0}| dx \leq 2^n \alpha, \quad \text{for every } k \in K_1, \tag{6.14}$$

and

$$|u - u_{Q_0}| \leq \alpha \quad \text{a.e. on } Q \setminus \bigcup_{k \in K_1} Q_k^1. \tag{6.15}$$

Then by (6.14) we get

$$|u_{Q_k^1} - u_{Q_0}| = \left| \fint_{Q_k^1} (u - u_{Q_0})\, dx \right| \leq 2^n \alpha, \tag{6.16}$$

and

$$\sum_{k \in K_1} |Q_k^1| \leq \frac{1}{\alpha} \sum_{k \in K_1} \int_{Q_k^1} |u - u_{Q_0}| dx \leq \frac{1}{\alpha} \int_{Q_0} |u - u_{Q_0}| dx \leq \frac{1}{\alpha} |Q_0|. \tag{6.17}$$

Since $|u|_* = 1$, for all $k \in K_1$ we have $\fint_{Q_k^1} |u - u_{Q_k^1}|\, dx \leq 1 < \alpha$, so that we can apply again the Calderon-Zygmund argument with $Q = Q_k^1$, $f = |u - u_{Q_k^1}|$ and parameter α, finding a sequence of cubes $\{Q_{k,j}^1\}_{j \in J(k)}$ such that

$$\alpha < \frac{1}{\mu(Q_{k,j}^1)} \int_{Q_{k,j}^1} |u(x) - u_{Q_k^1}|\, dx \leq 2^n \alpha\,, \tag{6.18}$$

for all $j \in J(k)$ and

$$|u(x) - u_{Q_k^1}| \leq \alpha \qquad \text{a.e. on } Q_k^1 \setminus \bigcup_{j \in J(k)} Q_{k,j}^1. \tag{6.19}$$

As k ranges over K_1 we collect all cubes obtained and rename the denumerable family

$$\{Q_{k,j}\}_{j \in J(k), k \in K_1} = \{Q_k^2\}_{k \in K_2},$$

and claim that

$$|u - u_{Q_0}| \leq 2 \cdot 2^n \alpha, \quad \text{a.e. on } Q_0 \setminus \bigcup_{k \in K_2} Q_k^2. \tag{6.20}$$

Indeed we have

$$Q_0 \setminus \bigcup_{k\in K_2} Q_k^2 = \left(Q_0 \setminus \bigcup_{k\in K_1} Q_k^1\right) \cup \left(\bigcup_{k\in K_1} Q_k^1 \setminus \bigcup_{k\in K_2} Q_k^2\right).$$

On $Q_0 \setminus \bigcup_{k\in K_1} Q_k^1$ we have $|u-u_{Q_0}| \le \alpha \le 2\cdot 2^n\alpha$ a.e. by (6.15), while (6.14) and (6.16) imply that for a.e. $x \in \bigcup_{k\in K_1} Q_k^1 \setminus \bigcup_{k\in K_2} Q_k^2$ we have

$$|u(x)-u_{Q_0}| \le |u(x)-u_{Q_k^1}| + |u_{Q_k^1} - u_{Q_0}| \le \alpha + 2^n\alpha \le 2\cdot 2^n\alpha, \quad (6.21)$$

where $k = k(x) \in K_1$ is the (unique) index such that $x \in Q_k^1$. Then (6.20) is proven.

Moreover by (6.17) and the fact that $|u|_* = 1$,

$$\sum_{j\in K_2} |Q_k^2| \le \frac{1}{\alpha}\sum_{k\in K_1}\int_{Q_k^1} |u-u_{Q_k^1}|dx \le \frac{1}{\alpha}\sum_{k\in K_1}|Q_k^1| \le \frac{1}{\alpha^2}|Q_0|.$$

Repeating this procedure inductively, for every $k\in\mathbb{N}$ we can find a sequence of cubes $\{Q_k^i\}_{k\in K_i}$ such that

$$|u-u_{Q_0}| \le i2^n\alpha \quad \text{a.e. on } Q_0 \setminus \bigcup_{k\in K_i} Q_k^i \quad (6.22)$$

$$\sum_{k\in K_i} |Q_k^i| \le \frac{1}{\alpha^i}|Q_0|. \quad (6.23)$$

Indeed (6.23) follows simply as before. To get (6.22), write

$$Q_0 \setminus \bigcup_{k\in K_i} Q_k^i = \left(Q_0 \setminus \bigcup_{k\in K_1} Q_k^1\right) \cup \ldots \cup \left(\bigcup_{k\in K_{i-1}} Q_k^{i-1} \setminus \bigcup_{k\in K_i} Q_k^i\right).$$

Then for $x \in \left(\bigcup_{k\in K_{i-1}} Q_k^{i-1} \setminus \bigcup_{k\in K_i} Q_k^i\right)$, we have

$$\begin{aligned}|u(x)-u_{Q_0}| &\le \left|u(x)-u_{Q^{i-1}_{k_{i-1}}}\right| + \left|u_{Q^{i-1}_{k_{i-1}}} - u_{Q^{i-2}_{k_{i-2}}}\right| + \ldots + \left|u_{Q^1_{k_1}} - u_{Q_0}\right| \\ &\le \underbrace{\alpha}_{x\notin \cup_{k\in K_i} Q_k^i} + \underbrace{2^n\alpha + \ldots + 2^n\alpha}_{\text{by (6.16), (6.20), and analogs}} \\ &\le i2^n\alpha.\end{aligned}$$

To prove (6.13) take any $t>0$, and set $c_1 := \alpha$, $c_2 := \frac{\log\alpha}{2^n\alpha}$. If $t < 2^n\alpha$, we have $0 \le c_2(2^n\alpha - t) \Rightarrow 1 \le e^{c_2(2^n\alpha - t)} = c_1 e^{-c_2 t}$ and so

$$|\{x\in Q_0 : |u-u_{Q_0}| > t\}| \le |Q_0| \le c_1 e^{-c_2 t}|Q_0|.$$

If $t \geq 2^n\alpha$, choose $i \in \mathbb{N}$ in such a way that $i2^n\alpha \leq t < (i+1)2^n\alpha$; then finally

$$\begin{aligned} |\{x \in Q_0 : |u - u_{Q_0}| > t\}| &\leq |\{x \in Q_0 : |u - u_{Q_0}| > i2^n\alpha\}| \\ &\leq \sum_{k \in K_i} |Q_k^i| \leq \frac{1}{\alpha^i}|Q_0| \\ &\leq c_1 e^{-c_2 t}|Q_0|, \end{aligned}$$

where the last inequality follows from $-i \leq 1 - \frac{t}{2^n\alpha}$. □

Corollary 6.22 *Let $u \in BMO(Q_0)$; then $u \in L^p(Q_0)$ for all $1 \leq p < +\infty$ and there is a $C = C(n,p)$ such that*

$$\sup_{Q \subset Q_0} \left(\fint_Q |u - u_Q|^p \, dx \right)^{1/p} \leq C|u|_*. \tag{6.24}$$

Proof. Using (6.2) together with (6.13), we get

$$\begin{aligned} \int_Q |u - u_Q|^p dx &= p \int_0^{+\infty} t^{p-1} \big|\{x \in Q : |u(x) - u_Q| > t\}\big| dt \\ &\leq p \cdot c_1 \int_0^{+\infty} t^{p-1} \exp\Big(-\frac{c_2}{|u|_*} t\Big)|Q| dt \\ &= p \cdot c_1 \left(\frac{|u|_*}{c_2}\right)^p |Q| \int_0^{+\infty} s^{p-1} e^{-s} \, ds \\ &= C(n,p)\, |u|_*^p |Q|, \end{aligned}$$

and (6.24) follows. □

From Corollary 6.22 and Jensen's inequality we immediately get

Corollary 6.23 *For every $1 \leq p < +\infty$ the Campanato space $\mathcal{L}^{p,n}(Q_0)$ is isomorphic to $BMO(Q_0)$.*

Exercise 6.24 It is not true that if $u \in L^p(Q_0)$ for every $p \in [1, \infty)$, then $u \in BMO(Q_0)$. Let $u(x) := (\log |x|)^2$, $x \in [-1, 1]$. Show that $u \in L^p([-1, 1])$ for every $p \in [1, \infty)$, but is does not satisfy (6.13), hence $u \notin BMO([-1, 1])$.

In the following theorem we give some characterizations of BMO functions; in particular we show the converse of Theorem 6.21, so that (6.13) is in fact equivalent to u being a BMO function.

Theorem 6.25 *The following facts are equivalent:*

1. *$u \in BMO(Q_0)$;*
2. *there are c_1, c_2 such that for all $Q \subset Q_0, t > 0$*

$$|\{x \in Q : |u(x) - u_Q| > t\}| \leq c_1 e^{-c_2 t}|Q|;$$

3. *there are c_3, c_4 such that for all $Q \subset Q_0$*

$$\fint_Q \left(e^{c_4|u-u_Q|} - 1\right) dx \le c_3;$$

4. *there are c_5, c_6 such that for all $Q \subset Q_0$*

$$\left(\fint_Q e^{c_6 u} dx\right)\left(\fint_Q e^{-c_6 u} dx\right) \le c_5.$$

Moreover, we can choose c_1, c_3, c_5 depending only on n, while $c_2, c_4, c_6 = c_4$ can be chosen of the form $c(n)/|u|_*$.

Proof.
$(1 \Rightarrow 2)$ is John-Nirenberg lemma I with c_2 instead of $\frac{c_2}{|u|_*}$.

$(2 \Rightarrow 3)$ Set $c_4 := \frac{c_2}{2}$. Then using (6.2) and the change of variable $e^{c_4 t} = s$ we compute

$$\begin{aligned}\int_Q \left(e^{c_4|u-u_Q|} - 1\right) dx = & \int_1^\infty \left|\left\{x \in Q : e^{c_4|u(x)-u_Q|} > s\right\}\right| ds \\ = & \int_0^\infty c_4 e^{c_4 t}\left|\left\{x \in Q \mid |u(x) - u_Q| > t\right\}\right| dt \\ \le & c_1|Q| \int_0^\infty c_4 e^{c_4 t} e^{-c_2 t} dt \\ = & c_1|Q| \int_0^\infty \frac{c_2}{2} e^{-\frac{c_2}{2} t} dt = c_1|Q|.\end{aligned}$$

$(3 \Rightarrow 1)$ $t \le e^t - 1$, hence $|u - u_Q| \le \frac{1}{c_4} e^{c_4|u-u_Q|} - 1$, so that

$$\fint_Q |u - u_Q| dx \le \frac{1}{c_4} \fint_Q \left(e^{c_4|u-u_Q|} - 1\right) dx \le \frac{c_3}{c_4}.$$

$(3 \Rightarrow 4)$ We have

$$\begin{aligned}\fint_Q e^{c_4 u} dx \fint_Q e^{-c_4 u} dx = & \fint_Q e^{c_4(u-u_Q)} dx \fint_Q e^{-c_4(u-u_Q)} dx \\ \le & \left(\fint_Q e^{c_4|u-u_Q|} dx\right)^2.\end{aligned}$$

$(4 \Rightarrow 1)$ Set $w := \log v$, $v = e^{c_6 u}$; then

$$\fint_Q e^{w-w_Q} dx \fint_Q e^{-(w-w_Q)} dx = \fint_Q e^w dx \fint_Q e^{-w} dx \le c_5. \tag{6.25}$$

On the other hand, by Jensen's inequality

$$\begin{aligned}\fint_Q e^{w-w_Q}dx &\geq \exp \fint_Q (w-w_Q)dx = 1\\ \fint_Q e^{-(w-w_Q)}dx &\geq \exp \fint_Q (-(w-w_Q))dx = 1.\end{aligned}$$

Hence we conclude that both integrals in (6.25) are smaller than or equal to c_5. Finally, since

$$\fint_Q \exp|w-w_Q|dx \leq \fint_Q \exp(w-w_Q)dx + \fint_Q \exp(w_Q-w)dx \leq 2c_5,$$

using again Jensen's inequality we conclude

$$\exp \fint_Q |w-w_Q|dx \leq \fint_Q \exp|w-w_Q|dx \leq 2c_5.$$

Taking the supremum over all cubes $Q \subset Q_0$ we get $u \in BMO(Q_0)$. $\square$

Sobolev embedding in the limit case

If $u \in W^{1,n}(Q_0)$, $Q_0 \subset \mathbb{R}^n$, then by Sobolev embedding theorem $u \in L^p(Q_0)$ for all p, $1 \leq p < \infty$, but in general u is not bounded:

$$\log(-\log|x|) \in W^{1,2}(B_1(0)) \quad \text{if } n=2.$$

However, by Poincaré and Hölder's inequalities

$$\fint_Q |u-u_Q|dx \leq c(n)|Q|^{\frac{1-n}{n}} \int_Q |Du|dx, \tag{6.26}$$

i.e. $u \in BMO(Q_0)$. Since what matters here is (6.26), by John-Nirenberg lemma (or Theorem 6.25, 3), we can state

Proposition 6.26 *Let $u \in W^{1,1}(Q_0)$ and suppose that for any cube $Q_R \subset Q$ of side length R*

$$\int_{Q_R} |Du|dx \leq kR^{n-1}.$$

Then there are constants μ_1 and μ_2 depending only on n such that

$$\fint_{Q_0} \exp\Big(\frac{\mu_1}{k}|u-u_{Q_0}|\Big)dx \leq \mu_2.$$

6.3.2 John-Nirenberg lemma II

The next theorem may be regarded as a weak form of Riesz theorem 6.15.

Theorem 6.27 *Let $u \in L^1(Q_0)$ and suppose that for some $p \in [1,\infty]$ we have*

$$K_p(u) := \left(\sup_{\Delta \in \{\Delta\}} \sum_{Q_i \in \Delta} |Q_i| \left(\fint_{Q_i} |u - u_{Q_i}| \right)^p \right)^{\frac{1}{p}} < \infty, \qquad (6.27)$$

where $\{\Delta\}$ denotes the collection of all finite decompositions Δ of the cube Q_0 into subcubes Q_i with sides parallel to the axes. Then the function $u - u_{Q_0}$ (hence also u) belongs to $L^p_w(Q_0)$ and for all $t > 0$

$$\left|\{x \in Q_0 : |u(x) - u_{Q_0}| > t\}\right| \le c(n,p) \left(\frac{K_p(u)}{t} \right)^p.$$

Proof. Let $q := \frac{p}{p-1}$ be the conjugate exponent of p and

$$\begin{aligned} \lambda_j &:= 1 + \frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^j} = \frac{1 - q^{-j-1}}{1 - q^{-1}} = p(1 - q^{-j-1}) \\ \tau_j &:= \frac{1}{2^{n+j(n+1)} q^j \lambda_j} < 1. \end{aligned}$$

Observe that $q^j \lambda_j = q^j \lambda_{j-1} + 1$. Fix $\sigma > 0$ and

$$\lambda(\sigma) := |\{x \in Q_0 : |u(x) - u_{Q_0}| > \sigma\}|.$$

We first show by induction on j that if

$$\begin{cases} u \in L^1(Q_0) \text{ is such that } K_p(u) < \infty \text{ and} \\ \tau_j \sigma \ge K_p(u) |Q_0|^{-\frac{1}{p}} \end{cases} \qquad (A)_j$$

then

$$\begin{gathered} \lambda(\sigma) \le A_j \left(\frac{\lambda_j K_p(u)}{\sigma} \right)^{\lambda_j} \left(\frac{1}{K_p(u)} \int_{Q_0} |u - u_{Q_0}| dx \right)^{\frac{1}{q^j}}, \\ \text{where } A_0 := 1, \ \ A_j = \prod_{i=1}^{j} (q^i 2^{n+i(n+1)})^{\frac{1}{q^i}}. \end{gathered} \qquad (B)_j$$

Since obviously

$$\lambda(\sigma) \le \frac{1}{\sigma} \int_{Q_0} |u - u_{Q_0}| dx,$$

$(B)_0$ trivially holds, hence also the implication $(A)_0 \Rightarrow (B)_0$. Let us now assume that

$$(A)_{j-1} \Rightarrow (B)_{j-1}$$

for all $f \in L^1(Q_0)$ with $K_p(f) < \infty$ and that σ satisfies $(A)_j$. Taking $\Delta = \{Q_0\}$ in (6.27) we get

$$\tau_j \sigma \geq \frac{K_p(u)}{|Q_0|^{\frac{1}{p}}} \geq \fint_{Q_0} |u - u_{Q_0}| dx$$

and we can apply the Calderon-Zygmund argument to $|u - u_{Q_0}|$ with parameter $t = \tau_j \sigma$ to obtain a sequence of cubes $\{Q_k\}$, $Q_k \subset Q_0$ such that

$$\begin{cases} \tau_j \sigma < \fint_{Q_k} |u - u_{Q_0}| dx \leq 2^n \tau_j \sigma \quad \forall k \\ |u - u_{Q_0}| \leq \sigma \tau_j \qquad \text{a.e. in } Q_0 \backslash \bigcup_k Q_k. \end{cases} \tag{6.28}$$

Let

$$v(x) := \begin{cases} u(x) - u_{Q_k} & \text{for } x \in Q_k \\ 0 & \text{for } x \in Q_0 \backslash \bigcup_k Q_k. \end{cases}$$

Then

(i) $v_{Q_0} = 0$

(ii) $K_p(v) \leq K_p(u)$

(iii) because of (6.28)

$$\begin{aligned} \int_{Q_0} |v| dx &= \sum_k |Q_k|^{\frac{1}{q}} |Q_k|^{\frac{1}{p}} \fint_{Q_k} |u - u_{Q_k}| dx \\ &\leq \Big(\sum_k |Q_k| \Big(\fint_{Q_k} |u - u_{Q_k}| dx \Big)^p \Big)^{\frac{1}{p}} \Big(\sum_k |Q_k| \Big)^{\frac{1}{q}} \\ &\leq K_p(v) \Big(\frac{1}{\tau_j \sigma} \int_{Q_0} |u - u_{Q_0}| dx \Big)^{\frac{1}{q}}. \end{aligned}$$

Since

$$|u - u_{Q_k}| \geq |u - u_{Q_0}| - |u_{Q_0} - u_{Q_k}| \geq |u - u_{Q_0}| - \int_{Q_k} |u - u_{Q_0}| dx,$$

using (6.28) we have

$$\{x \in Q_0 : |u(x) - u_{Q_0}| > \sigma\} \subset \{x \in Q_0 : |v(x)| > \sigma(1 - 2^n \tau_j)\}.$$

From

$$1 - 2^n \tau_j > 1 - \frac{1}{q^j \lambda_j} = \frac{\lambda_{j-1}}{\lambda_j} \tag{6.29}$$

we infer

$$2^{n+1}\tau_j = \frac{2^{-n-(j-1)(n+1)}}{q^{j-1}\lambda_{j-1}}\frac{\lambda_{j-1}}{q\lambda_j} < \tau_{j-1}(1-2^n\tau_j).$$

Therefore v satisfies $(A)_{j-1}$ with $\widetilde{\sigma} := (1-2^n\tau_j)\sigma$.

From the induction argument, and using (6.29), we then get

$$\begin{aligned}\lambda(\sigma) &\le \left|\left\{x \in Q_0 : |v(x)| > \sigma(1-2^n\tau_j)\right\}\right| \\ &\le A_{j-1}\left(\frac{\lambda_j K_p(v)}{\sigma}\right)^{\lambda_{j-1}}\left(\frac{1}{K_p(v)}\int_{Q_0}|v|dx\right)^{q^{1-j}},\end{aligned}$$

and using (ii) and (iii) we conclude

$$\lambda(\sigma) \le A_{j-1}\lambda_j^{\lambda_{j-1}}\tau_j^{-q^{-j}}\left(\frac{K_p(u)}{\sigma}\right)^{\lambda_j}\left[\frac{1}{K_p(u)}\int_{Q_0}|u-u_{Q_0}|dx\right]^{q^{-j}}.$$

To get $(B)_j$, we observe that $\tau_j^{-1} = q^j\lambda_j 2^{n+j(n+1)}$, whence

$$\lambda_j^{j-1}\tau_j^{-q^{-j}} \le \lambda_j^{\lambda_j}[q^j 2^{n+j(n+1)}]^{\frac{1}{q^j}}.$$

This concludes the proof of the induction.

Let us assume that $(A)_j$ holds for a given $\sigma > 0$. By the trivial estimate

$$|Q_0|^{\frac{1}{p}} \fint_{Q_0}|u-u_{Q_0}|dx \le K_p(u)$$

and $(B)_j$ we deduce the existence of a constant $c(n,p)$ such that

$$\begin{aligned}\lambda(\sigma) &\le c(n,p)\left(\frac{K_p(u)}{\sigma}\right)^{p\left(1-\frac{1}{q^{j+1}}\right)}|Q_0|^{\frac{1}{q^{j+1}}} \\ &= c(n,p)\left(\frac{K_p(u)}{\sigma}\right)^{p}\left(\frac{\sigma|Q_0|^{\frac{1}{p}}}{K_p(u)}\right)^{\frac{p}{q^{j+1}}}.\end{aligned} \tag{6.30}$$

We assume now $K_p(u)|Q_0|^{-\frac{1}{p}} < 2^{-n}\sigma$ and we choose the greatest integer j for which $(A)_j$ holds. Then we have

$$\sigma\tau_{j+1} < |Q_0|^{-\frac{1}{p}}K_p(u) \le \sigma\tau_j. \tag{6.31}$$

Inserting (6.31) into (6.30) we conclude

$$\lambda(\sigma) \le c_1(n,p)\left(\frac{K_p(u)}{\sigma}\right)^p\tau_{j+1}^{-pq^{-j-1}} \le c_2(n,p)\left(\frac{K_p(u)}{\sigma}\right)^p, \tag{6.32}$$

where we used that $\tau_{j+1}^{-pq^{-j-1}}$ is bounded. Finally, since $\lambda(\sigma) \leq |Q_0|$ we see that (6.32) holds also for $0 < \sigma < 2^n K_p(u)|Q_0|^{-\frac{1}{p}}$ with a different constant.

To see that also $u \in L^p_w(Q_0)$, write $|u| \leq |u - u_{Q_0}| + |u_{Q_0}|$ and notice that

$$\left|\left\{x \in Q_0 : |u(x)| > t\right\}\right| \leq \left|\left\{x \in Q_0 : |u(x) - u_{Q_0}| > t/2\right\}\right| + \begin{cases} 0 & \text{if } |u_{Q_0}| < \frac{t}{2} \\ |Q_0| \leq \frac{2^p |u_{Q_0}|^p}{t^p} & \text{if } \frac{t}{2} \leq |u_{Q_0}|. \end{cases}$$

□

Proposition 6.28 *A function $u \in L^1(Q_0)$ belongs to $BMO(Q_0)$ if and only if*

$$\liminf_{p\to\infty} K_p(u) < \infty.$$

In this case we have

$$|u|_* = \lim_{p\to\infty} K_p(u).$$

Proof. We may assume $|u|_* > 0$. Then for any $M \in (0, |u|_*)$ we can find a cube $Q \subset Q_0$ such that

$$\fint_Q |u - u_Q| dx > M,$$

hence

$$K_p(u) \geq |Q|^{\frac{1}{p}} M,$$

concluding

$$M \leq \liminf_{p\to\infty} K_p(u) \leq \limsup_{p\to\infty} K_p(u) \leq |u|_*.$$

□

6.3.3 Interpolation between L^p and BMO

Following G. Stampacchia and S. Campanato we now prove

Theorem 6.29 *Let $1 \leq p < \infty$ and let T be a linear operator of strong type (p,p) and bounded from L^∞ into BMO, i.e.*

$$\|Tu\|_{L^p} \leq c_1 \|u\|_{L^p}, \quad \textit{for every } u \in L^p(Q_0)$$

and

$$\|Tu\|_* \leq c_2 \|u\|_{L^\infty}, \quad \textit{for every } u \in BMO(Q_0).$$

Then T maps continuously $L^q(Q_0)$ into $L^q(Q_0)$ for all $q \in (p, \infty)$.

Proof. Let $\Delta = \{Q_i\}$ be a fixed subdivision of Q_0. Given $u : Q_0 \to \mathbb{R}$ define

$$(T_\Delta u)(x) := \fint_{Q_i} |Tu - (Tu)_{Q_i}|dx, \qquad \text{for } x \in Q_i.$$

Then T_Δ is of strong-(p,p) type, since

$$\begin{aligned}
\|T_\Delta u\|^p_{L^p(Q_0)} &= \sum_{Q_i\in\Delta} |Q_i| \Big(\fint_{Q_i} |Tu - (Tu)_{Q_i}|dx \Big)^p \\
&\le \sum_{Q_i\in\Delta} \int_{Q_i} |Tu - (Tu)_{Q_i}|^p dx \\
&\le 2^{p-1} \sum_{Q_i\in\Delta} \int_{Q_i} [|Tu|^p + |(Tu)_{Q_i}|^p] dx \\
&\le 2^p \sum_{Q_i\in\Delta} \int_{Q_i} |Tu|^p dx \\
&= 2^p \|Tu\|^p_{L^p(Q_0)} \le c_1 \|u\|^p_{L^p(Q)}.
\end{aligned}$$

Moreover T_Δ is also of strong-(∞,∞) type: indeed for all $u \in L^\infty(Q)$ we have

$$\|T_\Delta u\|_{L^\infty(Q_0)} \le |Tu|_* \le c_2 \|u\|_{L^\infty(Q_0)}.$$

Finally T_Δ is clearly quasi-linear.

Marcinkiewicz's theorem then implies that T_Δ is a bounded operator between $L^r(Q)$ and $L^r(Q_0)$ for all $r \in (p,\infty)$; moreover, the (r,r)-operator norm of T_Δ can be estimated with a constant that depends only on p, r, c_1 and c_2. In particular, there is a constant $c > 0$ not depending on Δ, such that

$$\|T_\Delta u\|_{L^r(Q_0)} \le c\|u\|_{L^r(Q)}.$$

On the other hand

$$K_r(Tu) = \sup_{\Delta\in\{\Delta\}} \|T_\Delta u\|_r \le C\|u\|_r < \infty;$$

therefore, thanks to John-Nirenberg's theorem, we have that $Tu \in L^r_w(Q_0)$ and T is of weak (r,r)-type for each $r \in (p,\infty)$. Again by Marcinkiewicz's theorem, T is of strong (q,q)-type for every $q \in (p,r)$, hence for every $q \in (p,\infty)$. □

6.3.4 Sharp function and interpolation $L^p - BMO$

The *sharp function* of $u \in L^1(Q_0)$ can be defined according to Fefferman and Stein as follows:

$$u^\#(x) := \sup_{x\in Q\subset Q_0} \fint_Q |u(y) - u_Q|dy.$$

The *centered sharp function* is defined as

$$\widetilde{u}(x) := \sup_{Q\subset Q_0} \fint_Q |u(y) - u_Q|dy.$$

where the supremum is taken among all cubes centered at x. We have

$$\widetilde{u}(x) \le u^{\#}(x) \le 2^n \widetilde{u}(x), \quad |u|_* = \|\widetilde{u}\|_{L^\infty},$$

and, extending u to $L^1(\mathbb{R}^n)$ by setting $u = 0$ on $\mathbb{R}^n \setminus Q_0$, we can define $Mu(x)$ and verify

$$u(x) \le c(n)Mu(x),$$

hence if $u \in L^p(Q_0)$ for some $p > 1$, then $u^{\#} \in L^p(Q_0)$. Conversely we have:

Theorem 6.30 (Fefferman-Stein) *Consider $u \in L^1(Q_0)$, and suppose that $u^{\#} \in L^p(Q_0)$ for some $p > 1$. Then $u \in L^p(Q_0)$ and*

$$\left(\fint_{Q_0} |u|^p dx\right)^{\frac{1}{p}} \le c(n,p)\left[\left(\fint_{Q_0} |u^{\#}|^p dx\right)^{\frac{1}{p}} + \fint_{Q_0} |u| dx\right].$$

Set

$$\mu(t) := \sum_{j\in J} |Q_j^t|, \tag{6.33}$$

where $\{Q_j^t\}_{j\in J}$ is the Calderón-Zygmund family of cubes corresponding to $|u|$ and t. The proof of Theorem 6.30 uses the following weak estimate, known as *good-λ-inequality* (we use t as parameter for λ).

Proposition 6.31 *We have*

$$\mu((2^n+1)t) \le \left|\left\{x \in Q_0 \;\middle|\; u^{\#}(x) > \beta t\right\}\right| + \beta\mu(t) \tag{6.34}$$

for any $\beta \in (0,1)$ and any t such that

$$t > \fint_{Q_0} |u| dx.$$

Proof. Set $s := (2^n + 1)t$. Let $\{Q_j^t\}_{j\in J}$ and $\{Q_i^s\}_{i\in I}$ be the Calderón-Zygmund family of cubes corresponding to the function $|u|$ and the parameters t and s respectively. We can write

$$\mu(s) = \sum_{j\in J} \sum_{i\in I: Q_i^s \subset Q_j^t} |Q_i^s|.$$

Fix $j \in J$; then we have two possibilities:

1. $Q_j^t \subset \{x \in Q_0 : u^\#(x) > \beta t\}$. In this case
$$\sum_{i \in I : Q_i^s \subset Q_j^t} |Q_i^s| \leq \big|\{x \in Q_j^t \mid u^\#(x) > \beta t\}\big|.$$

2. There is $y \in Q_j^t$ such that $u^\#(y) \leq \beta t$, thus
$$\fint_{Q_j^t} |u - u_{Q_j^t}| dx \leq \beta t$$
and
$$\fint_{Q_i^s} |u - u_{Q_j^t}| dx \geq \fint_{Q_i^s} |u| dx - \fint_{Q_j^t} |u| dx \geq s - 2^n t = t.$$

Therefore, in this second case,
$$t \sum_{i \in I : Q_i^s \subset Q_j^t} |Q_i^s| \leq \sum_{i \in I : Q_i^s \subset Q_j^t} \int_{Q_i^s} |u - u_{Q_j^t}| dx \leq \int_{Q_j^t} |u - u_{Q_j^t}| \leq \beta t |Q_j^t|$$
i.e.
$$\sum_{i \in I : Q_i^s \subset Q_j^t} |Q_i^s| \leq \beta |Q_j^t|.$$
In both cases summing on j we deduce (6.34). □

Proof of Theorem 6.30. We rewrite (6.34) as
$$\mu(t) \leq \big|\{x \in Q_0 \mid u^\#(x) > \beta (2^n+1)^{-1} t\}\big| + \beta \mu\big((2^n+1)^{-1} t\big)$$
for $t > (2^n+1) \fint_{Q_0} |u| dx =: t_0$, and consider
$$I(\tau) := p \int_{t_0}^{\tau} t^{p-1} \mu(t) dt.$$
We have
$$\begin{aligned}
I(\tau) &\leq p \int_{t_0}^{\tau} t^{p-1} \big|\{x \in Q_0 : Mu(x) > t\}\big| dt \\
&\leq p \int_{t_0}^{\tau} t^{p-1} \big|\{x \in Q_0 : u^\#(x) > \beta (2^n+1)^{-1} t\}\big| dt \\
&\quad + \beta p \int_{t_0}^{\tau} t^{p-1} \mu((2^n+1)^{-1} t) dt \\
&\leq \left(\frac{2^n+1}{\beta}\right)^p p \int_{\beta (2^n+1)^{-1} t_0}^{\infty} s^{p-1} \big|\{x \in Q_0 : u^\#(x) > s\}\big| ds \\
&\quad + \beta (2^n+1)^p p \int_{(2^n+1)^{-1} t_0}^{(2^n+1)^{-1}\tau} r^{p-1} \mu(r) dr \\
&\leq \left(\frac{2^n+1}{\beta}\right)^p \|u^\#\|_{L^p(Q_0)}^p + \beta (2^n+1)^p I(\tau) + \beta (2^n+1)^p t_0^p |Q_0|
\end{aligned}$$

i.e., if we choose $\beta = \frac{1}{2}(2^n+1)^{-p}$,

$$I(\tau) \leq 2^{p+1}(2^n+1)^{p(p+1)}\|u^\#\|^p_{L^p(Q_0)} + t_0^p|Q_0|,$$

and the result follows at once for (6.2), since $|A(u,t)| \leq \mu(t)$. □

In terms of the sharp function we can now give a more transparent

Proof of the interpolation $L^p - BMO$. Consider the map

$$\mathcal{T}(u) := (Tu)^\#.$$

$\mathcal{T}$ is sublinear and

1. is of type (p,p) if $p > 1$ since

$$\begin{aligned}\|\mathcal{T}u\|_{L^p(Q_0)} &\leq c(n)\|M(Tu)\|_{L^p(Q_0)}\\ &\leq c(n,p)\|Tu\|_{L^P(Q_0)}\\ &\leq c(n,p)A_p\|u\|_{L^p(Q)};\end{aligned}$$

2. is of type weak-$(1,1)$ if $p = 1$ since

$$\begin{aligned}\big|\{x \in Q_0 : (Tu)^\#(x) > t\}\big| &\leq \big|\{x \in Q_0 : M(Tu)(x) > t/c(n)\}\big|\\ &\leq \frac{c'(n)\|Tu\|_{L^1}}{t} \leq \frac{c'(n)A_1\|u\|_{L^1}}{t};\end{aligned}$$

3. is of type (∞,∞) since

$$\|\mathcal{T}(u)\|_{L^\infty} \leq 2^n|Tu|_* \leq 2^n A_\infty\|u\|_{L^\infty}.$$

Marcinkiewicz interpolation theorem implies that $\mathcal{T}$ is of strong type (q,q) for all $q \in (p,\infty)$, and Theorem 6.30 finally yields the result, since by Jensen's inequality we get

$$\begin{aligned}\int_{Q_0}|Tu|^q dx &\leq c_1\int_{Q_0}|\mathcal{T}u|^q dx + c_2|Q_0|\left(\fint_{Q_0}|Tu|dx\right)^q\\ &\leq c_3\int_{Q_0}|u|^q dx + c_2|Q_0|\left(\fint_{Q_0}|Tu|^p dx\right)^{\frac{q}{p}}\\ &\leq c_3\int_{Q_0}|u|^q dx + c_4|Q_0|\left(\fint_{Q_0}|u|^p dx\right)^{\frac{q}{p}}\\ &\leq c_5\int_{Q_0}|u|^q dx.\end{aligned}$$

□

6.4 The Hardy space $\mathcal{H}^1$

The Hardy space was introduced by E. Stein and G. Weiss [104] and can be characterized as

$$\mathcal{H}^1(\mathbb{R}^n) := \Big\{ f \in L^1(\mathbb{R}^n) : \sup_{t>0} |\phi_t * f| \in L^1(\mathbb{R}^n) \Big\},$$

where

$$h_t(x) := \frac{1}{t^n} h\Big(\frac{x}{t}\Big),$$

for a given function $h \in C_c^\infty(\mathbb{R}^n)$ with

$$\mathrm{supp}(h) \subset B_1(0), \quad \int_{B_1(0)} h dx \neq 0.$$

The definition is independent of the choice of h (see Fefferman and Stein [34]).

Exercise 6.32 Prove that if $f \in \mathcal{H}^1(\mathbb{R}^n)$ is non-negative, then $f \equiv 0$.

As Exercise 6.32 suggest, the Hardy space is a strict subspace of L^1, i.e. $\mathcal{H}^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$. In fact it turns out that

$$\int_{\mathbb{R}^n} f dx := \lim_{R\to\infty} \int_{B_R(0)} f dx = 0$$

for every $f \in \mathcal{H}^1(\mathbb{R}^n)$.

The Hardy space is a good replacement of L^1 in the theory of partial differential equations. For instance, we shall see that L^1-estimates for the Laplace equation do not hold, in the sense that

$$\Delta u = f \text{ in } B_1, \quad u = 0 \text{ on } \partial B_1, \quad \text{with } f \in L^1(B_1)$$

does not imply $D^2 u \in L^1(B_1)$ (see Example 7.5). On the other hand if we replace the assumption $f \in L^1(B_1)$ by $f \in \mathcal{H}^1(\mathbb{R}^n)$, then $D^2 u \in L^1(B_1)$ and in fact

$$\|D^2 u\|_{L^1} \le c \|f\|_{\mathcal{H}^1}.$$

A classical example of a function belonging to $\mathcal{H}^1(\mathbb{R}^n)$ is the Jacobian Ju of a function $u \in W^{1,n}(\mathbb{R}^n)$. It is clear that $Ju \in L^1(\mathbb{R}^n)$, but as we shall seee the special Jacobian structure makes Ju slightly "better" than an arbitrary integrable function. This is part of the following theorem.

Theorem 6.33 (Coifman-Lions-Meyer-Semmes [22]) *1) Let u satisfy*

$$u \in L^q_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n) \text{ for all } q \in [1,\infty), \quad \nabla u \in L^n(\mathbb{R}^n). \tag{6.35}$$

Then $Ju := \det(\nabla u) \in \mathcal{H}^1(\mathbb{R}^n)$ *and*

$$\|Ju\|_{\mathcal{H}^1} \le C\|\nabla u\|_{L^n}.$$

2) Let E, B *satisfy*

$$E \in L^p(\mathbb{R}^n, \mathbb{R}^n), \quad B \in L^{p'}(\mathbb{R}^n, \mathbb{R}^n), \quad \text{with } p \in (1, \infty), \quad \frac{1}{p} + \frac{1}{p'} = 1 \tag{6.36}$$

and

$$\operatorname{div} E = 0, \quad \operatorname{curl} B = 0,^{3} \tag{6.37}$$

in the sense of distributions. Then $E \cdot B \in \mathcal{H}^1(\mathbb{R}^n)$, *and*

$$\|E \cdot B\|_{\mathcal{H}^1} \le \|E\|_{L^p} \|B\|_{L^{p'}}. \tag{6.38}$$

The proof of Theorem 6.33 is based on the following lemma.

Lemma 6.34 *Let* E, B *satisfy* (6.36) *and* (6.37). *Then for every* α *and* β *satisfying*

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1 + \frac{1}{n}, \quad 1 \le \alpha \le p, \quad 1 \le \beta \le p',$$

there is a constant $C = C(h, \alpha, \beta)$ *such that*

$$|h_t * (E \cdot B)(x)| \le C \left(\fint_{B_t(x)} |E(y)|^\alpha dy \right)^{\frac{1}{\alpha}} \left(\fint_{B_t(x)} |B(y)|^\beta dy \right)^{\frac{1}{\beta}},$$

for every $x \in \mathbb{R}^n$, $t > 0$, *where* h_t *is as in the definition of the Hardy space.*

Proof. Since $\operatorname{curl} B = 0$, by the Poincaré lemma (see Corollary 10.71) we can find a function π such that $\nabla \pi = B$, where $\pi \in L^{(p')^*}_{\text{loc}}(\mathbb{R}^n)$ if $p' < n$ or a function $\pi \in L^q_{\text{loc}}(\mathbb{R}^n)$ for every $q \in [1, \infty)$ if $p' \ge n$, where

$$(p')^* = \frac{np'}{n - p'}$$

is the Sobolev exponent. We have

$$E \cdot B = \operatorname{div}(E\pi)$$

in the sense of distributions. This is obvious if E and B are smooth, since $\operatorname{div}(E\pi) = \operatorname{div}(E)\pi + E \cdot \nabla \pi$, and in the general case it follows by

[3] $\operatorname{curl} B = 0$ means that weakly $\frac{\partial B_i}{\partial x_j} - \frac{\partial B_j}{\partial x_i} = 0$ for $1 \le i, j \le n$.

mollifying E and B. Then we can write

$$\begin{aligned} h_t * (E \cdot B)(x) &= \int_{\mathbb{R}^n} \frac{1}{t^n} h\Big(\frac{x-y}{t}\Big) \operatorname{div}(E(y)\pi(y))dy \\ &= \int_{\mathbb{R}^n} \frac{1}{t^{n+1}} \nabla h\Big(\frac{x-y}{t}\Big) \cdot E(y)\pi(y)dy \\ &= \int_{\mathbb{R}^n} \frac{1}{t^{n+1}} \nabla h\Big(\frac{x-y}{t}\Big) \cdot E(y)\big(\pi(y) - \pi_{x,t}\big)dy, \end{aligned}$$

with

$$\pi_{x,t} := \fint_{B_t(x)} \pi(\xi)d\xi.$$

Then with Hölder's inequality, and bounding

$$\frac{1}{t^n} \nabla h\Big(\frac{x-y}{t}\Big) \le \frac{C}{t^n} \chi_{B_t(x)}(y),$$

we deduce

$$|h_t * (E \cdot B)(x)| \le C \Big(\fint_{B_t(x)} |E(y)|^\beta dy \Big)^{\frac{1}{\beta}} \Big(\fint_{B_t(x)} \Big| \frac{\pi(y) - \pi_{x,t}}{t} \Big|^{\beta'} dy \Big)^{\frac{1}{\beta'}}.$$

By the Sobolev-Poincaré inequality, Proposition 3.27, we then infer

$$\begin{aligned} \Big(\fint_{B_t(x)} \Big| \frac{\pi(y) - \pi_{x,t}}{t} \Big|^{\beta'} dy \Big)^{\frac{1}{\beta'}} &\le C \Big(\fint_{B_t(x)} |\nabla \pi|^\alpha \Big)^{\frac{1}{\alpha}} \\ &= C \Big(\fint_{B_t(x)} |B|^\alpha \Big)^{\frac{1}{\alpha}}, \end{aligned}$$

since

$$\frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{1}{n} = 1 - \frac{1}{\beta} = \frac{1}{\beta'},$$

and the proof is complete. $\square$

Proof of Theorem 6.33. We first prove part 2). Apply Lemma 6.34 with some α, β as in the lemma satisfying $\alpha \in (1,p)$ and $\beta \in (1,p')$. Then, since p/α, $p'/\beta > 1$,

$$\begin{aligned} \sup_{t>0} |h_t * (E \cdot B)(x)| &\le C \sup_{t>0} \Big(\fint_{B_t(x)} |E(y)|^\alpha dy \Big)^{\frac{1}{\alpha}} \Big(\fint_{B_t(x)} |B(y)|^\beta dy \Big)^{\frac{1}{\beta}} \\ &\le C \Big(\sup_{t>0} \fint_{B_t(x)} |E(y)|^\alpha dy \Big)^{\frac{1}{\alpha}} \Big(\sup_{t>0} \fint_{B_t(x)} |B(y)|^\beta dy \Big)^{\frac{1}{\beta}} \\ &= C(M(|E|^\alpha)(x))^{\frac{1}{\alpha}} C(M(|B|^\beta)(x))^{\frac{1}{\beta}}, \end{aligned}$$

and we conclude by noticing that, thanks to the maximal theorem (Proposition 6.12),

$$\|(M(|E|^\alpha))^{\frac{1}{\alpha}}\|_{L^p} = \|M(|E|^\alpha)\|_{L^{p/\alpha}}^{\frac{1}{\alpha}} \le C\||E|^\alpha\|_{L^{p/\alpha}}^{\frac{1}{\alpha}} = C\|E\|_{L^p},$$

and similarly

$$\|(M(|B|^\beta))^{\frac{1}{\beta}}\|_{L^{p'}} \le C\|B\|_{L^{p'}},$$

so that

$$\|E\cdot B\|_{\mathcal{H}^1} = \int_{\mathbb{R}^n} \sup_{t>0} |h_t * (E\cdot B)(x)|dx \le C\|E\|_{L^p}\|B\|_{L^{p'}},$$

as claimed.

We now see how part 2) of Theorem 6.33 implies part 1). Indeed we can write $u = (u^1, \dots, u^n)$ and

$$Ju = \det(\nabla u) = \nabla u^1 \cdot \sigma,$$

where

$$\operatorname{div}\sigma = 0 \text{ as distribution}, \quad |\sigma| \le \prod_{j=2}^n |\nabla u^j| \text{ a.e.},$$

and apply part 2) with

$$E = \sigma \in L^{n/(n-1)}(\mathbb{R}^n, \mathbb{R}^n), \quad B = \nabla u^1 \in L^n(\mathbb{R}^n, \mathbb{R}^n)$$

to get

$$\|Ju\|_{\mathcal{H}^1} \le \|\nabla u^1\|_{L^n}\|\sigma\|_{L^{n/(n-1)}} \le \|\nabla u\|_{L^n}^n,$$

as claimed. □

6.4.1 The duality between $\mathcal{H}^1$ and BMO

It was proven by Fefferman and Stein that $BMO(\mathbb{R}^n)$ is dual to the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$. In particular every continuous and linear functional L on $\mathcal{H}^1(\mathbb{R}^n)$ can be written as

$$L(g) = \int_{\mathbb{R}^n} f(x)g(x)dx, \tag{6.39}$$

for some $f \in BMO(\mathbb{R}^n)$, where the integral has to be intended in the sense that we shall now explain. Indeed notice that for $g \in \mathcal{H}^1(\mathbb{R}^n)$ and $f \in BMO(\mathbb{R}^n)$ the integral in (6.39) need not be absolutely convergent, since f might be unbounded (for instance $f(x) = \chi_{B_1(0)}\log|x|$) and $|g|$ might be only L^1. However, if we consider the linear space

$$\mathcal{H}^1_a(\mathbb{R}^n) = \Big\{g \in L^\infty(\mathbb{R}^n) \,:\, \mathrm{supp}(g) \text{ compact, and } \int_{\mathbb{R}^n} g(x)\,dx = 0\Big\},$$

which is a dense subspace of $\mathcal{H}^1(\mathbb{R}^n)$ (the proof of this fact is not elementary), then the integral in (6.39) converges whenever $g \in \mathcal{H}^1_a(\mathbb{R}^n)$. Moreover, the fact that the average of g is zero ensures that (6.39) is well-defined, although elements of $BMO(\mathbb{R}^n)$ are defined up to additive constants.

We can now state the duality theorem:

Theorem 6.35 (Fefferman-Stein [34]) *Let $f \in BMO(\mathbb{R}^n)$ and consider the linear functional $L_f : \mathcal{H}^1_a(\mathbb{R}^n) \to \mathbb{R}$ given by*

$$L_f(g) = \int_{\mathbb{R}^n} f(x)g(x)\,dx.$$

Then L_f is bounded and its unique bounded linear extension to $\mathcal{H}^1(\mathbb{R}^n)$ satisfies

$$\|L_f\| := \sup_{g\in\mathcal{H}^1, \|g\|_{\mathcal{H}^1}=1} L_f(g) \leq C|f|_*.$$

Conversely, for every continuous and linear functional $L \in (\mathcal{H}^1(\mathbb{R}^n))^*$, there exists a unique[4] *function $f \in BMO(\mathbb{R}^n)$ such that $L = L_f$, where L_f is as in the first part of the theorem. Moreover, we have that*

$$|f|_* \leq C'\|L\|.$$

6.5 Reverse Hölder inequalities

The last tools we want to discuss here are the *reverse Hölder inequalities.* These are inequalities such as

$$\left(\fint_Q f^p dx \right)^{\frac{1}{p}} \leq b\left(\fint_Q f^q dx \right)^{\frac{1}{q}} \qquad \forall Q \text{ congruent to } Q_0, \tag{6.40}$$

where $p > q$ and $b > 1$. There are several occurences of inequalities of this type, for instance: Harnack inequality for harmonic functions

$$\sup_Q u \leq c(n) \inf_Q u$$

can be seen as (6.40) with $p = +\infty$, $q = -\infty$. In Theorem 6.25 we saw that $e^{\alpha f}$ satisfies (6.40) with $p = 1$, $q = -1$ for some α if $f \in BMO(Q_0)$. Reverse Hölder inequalities also appear in the theory of weights in harmonic analysis, and in the theory of quasi-conformal mappings.

However, inequalities (6.40) are too stringent; for instance they imply that $f \equiv 0$ if $f = 0$ in some open subset of Q_0 (principle of unique

[4] Remember that functions in BMO are defined up to constants, hence also uniqueness is intended up to constants.

continuation), and this is too strong for solutions of elliptic systems. More suited for the last case are *reverse Hölder inequalities with increasing support*:

$$\left(\fint_{B_r(x_0)} f^p dx \right)^{\frac{1}{p}} \le b \left(\fint_{B_{2r}(x_0)} f^q dx \right)^{\frac{1}{q}} \qquad \forall x \in \Omega,\ r < r_0, \tag{6.41}$$

which were introduced and studied by M. Giaquinta and G. Modica [46]. The key property is that, whenever (6.40) or (6.41) hold with exponents (p, q), they also hold with exponent $(p+\varepsilon, p)$ for some positive ε, providing, this way, higher integrability of f.

6.5.1 Gehring's lemma

The following result is due to F. W. Gehring [35].

Theorem 6.36 *Let $f \in L^q(Q_0)$ for some $q > 1$. Suppose that there exists $b > 1$ such that for all congruent*[5] *subcubes Q of Q_0 we have*

$$\left(\fint_Q |f|^q dx \right)^{\frac{1}{q}} \le b \fint_Q |f| dx.$$

Then there exists $p > q$ and a constant $c(n, p, q, b)$ such that $f \in L^p(Q_0)$ and

$$\left(\fint_Q |f|^p dx \right)^{\frac{1}{q}} \le c \left(\fint_Q |f|^q dx \right)^{\frac{1}{q}}, \qquad \forall Q \subset Q_0. \tag{6.42}$$

Proof. Clearly it is enough to prove (6.42) for $Q = Q_0$. Given $t \ge \fint_{Q_0} |f| dx$, denote by $\{Q_i\}_{i \in I}$ the Calderón-Zygmund cubes relative to t and $|f|$. Also set $A_s = \{x \in Q_0 : |f(x)| > s\}$. According to Proposition 6.19

$$\int_{A_t} |f|^q dx \le 2^n t^q \sum_{i \in I} |Q_i|. \tag{6.43}$$

Moreover for every $i \in I$

$$t^q \le \fint_{Q_i} |f|^q dx.$$

The last inequality transforms, using the assumption, in

$$t|Q_i| \le b \int_{Q_i} |f| dx \le b \int_{Q_i \cap A_{\beta t}} |f| dx + b\beta t |Q_i|$$

[5] As usual all cubes have sides parallel to the axes.

for every $\beta \in (0, 1/b)$. If we choose $\beta = \frac{1}{2b}$ we infer

$$t|Q_i| \leq c_1 \int_{Q_i \cap A_{\beta t}} |f| dx, \tag{6.44}$$

with c_1 depending only on b. From (6.43) and (6.44) we then deduce

$$\int_{A_t} |f|^q dx \leq c_2 t^{q-1} \int_{A_{\beta t}} |f| dx.$$

Since

$$\int_{A_{\beta t} \setminus A_t} |f|^q dx \leq t^{q-1} \int_{A_{\beta t}} |f| dx,$$

we also have

$$\int_{A_{\beta t}} |f|^q dx \leq (c_2 + 1) t^{q-1} \int_{A_{\beta t}} |f| dx,$$

and, if we set

$$h(t) := \int_{A_t} |f| dx$$

and observe that

$$\int_{A_t} |f|^q dx = -\int_t^{\infty} s^{q-1} dh(s),$$

we conclude

$$-\int_{\tau}^{\infty} s^{q-1} dh(s) \leq a \tau^{q-1} h(t),$$

for every $\tau \geq \beta t = \frac{t}{2b}$, with $c_3 = c_3(b, q, n)$, and the result follows from the lemma below. □

Lemma 6.37 *Let $h : [t_0, +\infty) \to [0, +\infty)$ be a non increasing function with $\lim_{t \to +\infty} h(t) = 0$. Suppose that for every $t \geq t_0$ and for some constant $a > 1$ we have*

$$-\int_t^{+\infty} s^{q-1} dh(s) \leq a t^{q-1} h(t). \tag{6.45}$$

Then, for every $p \in [q, q + \frac{q-1}{a-1})$, we have

$$-\int_{t_0}^{+\infty} s^{p-1} dh(s) \leq -c t_0^{p-q} \int_{t_0}^{+\infty} s^{q-1} dh(s), \tag{6.46}$$

where $c = c(a, p, q)$.

Proof. Up to rescaling, we may assume $t_0 = 1$ and, by an approximation argument, we may assume that, for some $k > 0$, $h(s) = 0$ if $s \geq k$. For any $r > 0$ set

$$I(r) := -\int_1^{+\infty} s^r\, dh(s) = -\int_1^k s^r\, dh(s).$$

Then integration by parts yields

$$\begin{aligned} I(p-1) &= -\int_1^k s^{p-q} s^{q-1} dh(s) \\ &= I(q-1) + (p-q) \underbrace{\int_1^k s^{p-q-1}\Big(-\int_s^k t^{q-1} dh(t)\Big) ds}_{=:J}. \end{aligned} \tag{6.47}$$

By (6.45) we have, again integrating by parts,

$$\begin{aligned} J \leq a \int_1^k s^{p-2} h(s) ds &= -\frac{a}{p-1} h(1) - \frac{a}{p-1} \int_1^k s^{p-1} dh(s) \\ &\leq -\frac{1}{p-1} I(q-1) - \frac{a}{p-1} \int_1^k s^{p-1} dh(s). \end{aligned}$$

Inserting this into (6.47) we get

$$\Big(1 - a\frac{p-q}{p-1}\Big) I(p-1) \leq \frac{q-1}{p-1} I(q-1),$$

that is (6.46), up to rescaling. □

6.5.2 Reverse Hölder inequalities with increasing support

The following result was proved by M. Giaquinta and G. Modica [46].

Theorem 6.38 *Let $f \in L^q_{\mathrm{loc}}(\Omega)$, $q > 1$, be a non negative function. Suppose that for some constants $b > 0, R_0 > 0$*

$$\Big(⨍_{B_R(x_0)} f^q dx \Big)^{\frac{1}{q}} \leq b ⨍_{B_{2R}(x_0)} f dx \tag{6.48}$$

for all $x_0 \in \Omega$, $0 < R < \min(R_0, \frac{\mathrm{dist}(x_0, \partial\Omega)}{2})$. Then $f \in L^p_{\mathrm{loc}}(\Omega)$ for some $p > q$ and there is a constant $c = c(n, q, p, b)$ such that

$$\Big(⨍_{B_R(x_0)} f^p dx \Big)^{\frac{1}{p}} \leq c \Big(⨍_{B_{2R}(x_0)} f^q dx \Big)^{\frac{1}{q}}.$$

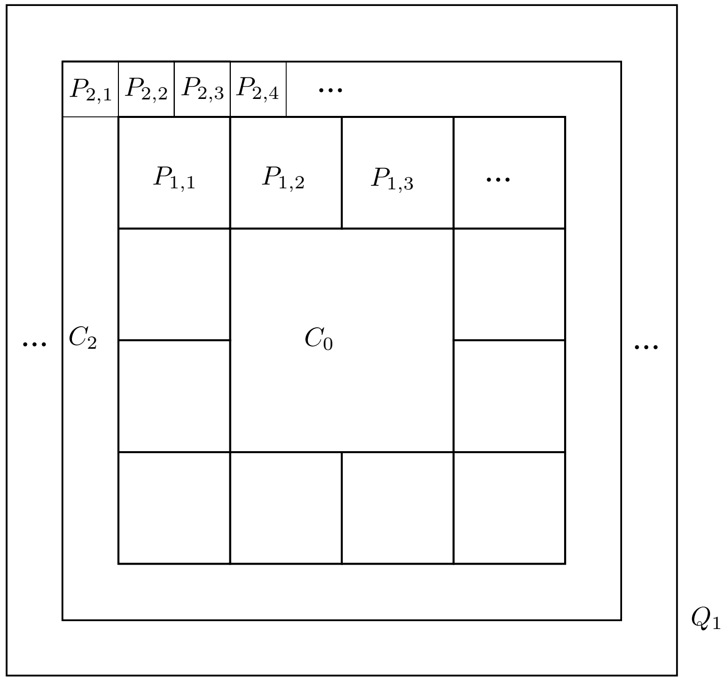

Figure 6.1: Decomposition of $Q_1 = C_0 \cup (\bigcup_{k=1}^{\infty} C_k)$ and $C_k = \bigcup P_{k,j}$.

Proof. Since the theorem is scale and translation invariant, and it is local, we may work in the cube

$$Q_1 := \left\{x \in \mathbb{R}^n : |x_i| < \frac{3}{2}, i = 1, \dots, n\right\}$$

and assume $g \equiv 0$ in $\mathbb{R}^n \backslash Q_1$. Define also

$$\begin{aligned} C_0 &:= \left\{x \in \mathbb{R}^n : |x_i| \le \frac{1}{2}, i = 1, \dots, n\right\}, \\ C_k &:= \left\{x \in Q_1 : \frac{1}{2^k} \le \operatorname{dist}(x, \partial Q_1) \le \frac{1}{2^{k-1}}\right\},\ k \ge 1, \end{aligned}$$

so that $Q_1 = \bigcup_{k=0}^{+\infty} C_k$. Decompose each C_k into a union of nonoverlapping cubes $\{P_{k,j}\}$ of side length $\frac{1}{2^k}$, in the obvious way (see Figure 6.1) and define the function

$$\phi(x) := \big(\operatorname{dist}(x, \partial Q_1)\big)^n, \quad x \in Q_1.$$

Let $\sigma = \sigma(n) > 0$ be a constant such that

$$\frac{1}{\sigma}|P_{k,j}| \le \phi(x) \le \sigma|P_{k,j}|, \quad \forall x \in C_{k-1} \cup C_k \cup C_{k+1}. \tag{6.49}$$

Choose $t > 0$ satisfying

$$t \geq \gamma_0 \|f\|_{L^q(Q_1)}, \tag{6.50}$$

for some $\gamma_0 \geq 1$ to be determined. By (6.49) and $|P_{k,j}| \leq 1$, we have

$$\fint_{P_{k,j}} (f\phi)^q dx \leq \sigma^q |P_{k,j}|^q \fint_{P_{k,j}} f^q dx \leq \sigma^q \|f\|^q_{L^q(Q_1)};$$

therefore we may choose $\gamma_0 > \sigma$, so that (6.50) implies

$$t^q > \fint_{P_{k,j}} (f\phi)^q dx.$$

Now we apply Calderón-Zygmund argument to each $P_{k,j}$, with parameter t^q and function $(f\phi)^q$, obtaining non overlapping cubes $Q^l_{k,j} \subset P_{k,j}$ satisfying:

$$t^q < \fint_{Q^l_{k,j}} (f\phi)^q dx \leq 2^n t^q, \qquad \text{for every } j, k, l, \tag{6.51}$$

and

$$f(x)\phi(x) \leq t, \qquad x \in \bigcup_{k,j} \Big(P_{k,j} \backslash \bigcup_l Q^l_{k,j} \Big) = Q_1 \backslash \bigcup_{k,j,l} Q^l_{k,j},$$

i.e., setting $A_s = A_{f\phi}(s) = \big\{ x \in Q_1 : f(x)\phi(x) > s \big\}$ we have

$$A_t \subset \bigcup_{k,j,l} Q^l_{k,j}.$$

This, together with (6.51), implies

$$\begin{aligned} \int_{A_t} (f\phi)^q dx &\leq \sum_{k,j,l} |Q^l_{j,k}| \fint_{Q^l_{k,j}} (f\phi)^q dx \\ &\leq 2^n t^q \sum_{k,j,l} |Q^l_{j,k}|. \end{aligned} \tag{6.52}$$

By (6.51) and (6.49) we have that, for any $x \in Q^l_{k,j}$,

$$\begin{aligned} t &< \Big(\fint_{Q^l_{k,j}} (f\phi)^q dx \Big)^{\frac{1}{q}} \\ &\leq \sigma |P_{k,j}| \Big(\fint_{Q^l_{k,j}} f^q dx \Big)^{\frac{1}{q}} \\ &\leq \sigma |P_{k,j}| \fint_{Q^{l,(2)}_{k,j}} f dx \\ &\leq c_1 \fint_{Q^{l,(2)}_{k,j}} f\phi dx, \end{aligned} \tag{6.53}$$

where $Q^{l,(2)}_{k,j} \subset Q_1$ is the cube having the same center as $Q^l_{k,j}$, but twice the side length. We deduce

$$t|Q^{l,(2)}_{k,j}| \le c_1 \fint_{Q^{l,(2)}_{k,j} \cap A_{\beta t}} f\phi dx + c_1 \beta t |Q^{l,(2)}_{k,j}|, \quad \forall \beta > 0.$$

Fixing $\beta = \frac{1}{2c_1}$, this becomes

$$t|Q^{l,(2)}_{k,j}| \le 2c_1 \fint_{Q^{l,(2)}_{k,j} \cap A_{\beta t}} f\phi dx. \tag{6.54}$$

The $Q^{l,(2)}_{k,j}$ form an open covering of $\cup_{j,k,l} Q^l_{k,j}$, and we can choose countably many pairwise disjoint $Q^{l,(2)}_{k,j}$, which we rename as $Q^{(2)}_m$ such that

$$\bigcup_{j,k,l} Q^l_{j,k} \subset \bigcup_m Q^{(5)}_m,$$

where $Q^{(5)}_m$ is the cube having the same center as $Q^{(2)}_m$, but 5 times the side length. Returning to (6.52), we can conclude

$$\int_{A_t} (f\phi)^q dx \le c_2 t^{q-1} \int_{A_{\beta t}} f\phi dx, \quad \forall t > \gamma_0 \|f\|_{L^q(Q_1)}.$$

Observing that

$$\int_{A_{\beta t} \setminus A_t} (f\phi)^q dx \le t^{q-1} \int_{A_{\beta t}} f\phi dx,$$

we obtain

$$\int_{A_\tau} (f\phi)^q dx \le c_3 \tau^{q-1} \int_{A_\tau} f\phi dx, \quad \forall \tau = \beta t > \frac{\gamma_0}{2c_1} \|f\|_{L^q(Q_1)},$$

and if we set

$$h(t) := \int_{A_t} |f\phi| dx$$

and observe that

$$\int_{A_t} |f\phi|^q dx = -\int_t^\infty s^{q-1} dh(s),$$

we conclude

$$-\int_\tau^\infty s^{q-1} dh(s) \le a\tau^{q-1} h(t),$$

for every $\tau \ge \frac{\gamma_0}{2c_1} \|f\|_{L^q(Q_1)}$, and the result follows from Lemma 6.37. $\square$

Chapter 7
L^p-theory

In the first section of this chapter we establish L^p-estimates for solutions of elliptic systems both in divergence and non-divergence form as consequence of Stampacchia's interpolation theorem, see [103] [19] therefore without using potential theory. The rest of the chapter is dedicated to a short introduction to singular integrals.

7.1 L^p-estimates

The L^p-estimates of the gradient of weak solutions to elliptic systems may be obtained by interpolating L^2 and BMO estimates.

7.1.1 Constant coefficients

Let Ω be a domain that is bilipschitz equivalent to the unit cube. Consider in the weak form the Dirichlet problem

$$\begin{cases} D_\alpha\big(A_{ij}^{\alpha\beta} D_\beta u^j\big) = D_\alpha F_i^\alpha & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.1}$$

where $A_{ij}^{\alpha\beta}$ are constant coefficients satisfying the Legendre-Hadamard condition

$$A_{ij}^{\alpha\beta}\xi_\alpha\xi_\beta\eta^i\eta^j \geq \lambda|\xi|^2|\eta|^2, \quad \text{for some } \lambda > 0.$$

From Gårding's inequality we know that the linear operator

$$T : L^2(\Omega) \to L^2(\Omega)$$

that to each $F \in L^2(\Omega)$ associates the gradient of the weak $W_0^{1,2}$- solution to problem (7.1) is continuous

$$\|Du\|_{L^2(\Omega)} \leq c\|F\|_{L^2(\Omega)},$$

since

$$\lambda \int_\Omega |Du|^2 dx \le \int_\Omega A_{ij}^{\alpha\beta} D_\beta u^j D_\alpha u^i dx = \int_\Omega F_i^\alpha D_\alpha u^i dx$$
$$\le \Big(\int_\Omega |F|^2 dx \Big)^{\frac{1}{2}} \Big(\int_\Omega |Du|^2 dx \Big)^{\frac{1}{2}}.$$

From the (interior plus boundary) regularity theory, we know that T maps continuously $L^\infty(\Omega)$ into $BMO(\Omega)$, compare Theorem 5.14 and Corollary 6.23:

$$|Du|_* \le c_1 [Du]_{\mathcal{L}^{2,n}} \le c_2 \big(\|Du\|_{L^2} + \|F\|_{\mathcal{L}^{2,n}} \big)$$
$$\le c_3 \|F\|_{\mathcal{L}^{2,n}} \le c_4 \|F\|_{L^\infty}.$$

Stampacchia's interpolation theorem then yields at once

Theorem 7.1 *Let $u \in W^{1,2}(\Omega, \mathbb{R}^m)$ be a weak solution of the Dirichlet problem*

$$\begin{cases} D_\alpha \big(A_{ij}^{\alpha\beta} D_\beta u^i \big) = D_\alpha F_i^\alpha \\ u - g \in W_0^{1,2}(\Omega), \end{cases}$$

where the constant coefficients $A_{ij}^{\alpha\beta}$ satisfy the Legendre-Hadamard condition (3.17)*, and $F_i^\alpha \in L^p(\Omega)$ and $g \in W^{1,p}(\Omega, \mathbb{R}^m)$ for some $p \ge 2$. Then $Du \in L^p(\Omega)$ and*

$$\|Du\|_{L^p(\Omega)} \le c \big(\|Dg\|_{L^p(\Omega)} + \|F\|_{L^p(\Omega)} \big), \tag{7.2}$$

for some constant $c(\Omega, p, \lambda, |A|)$

7.1.2 Variable coefficients: divergence and non-divergence case

Consider a weak solution $u \in W^{1,2}(\Omega, \mathbb{R}^m)$ of the system

$$D_\alpha \big(A_{ij}^{\alpha\beta} D_\beta u^j \big) = g_i + D_\alpha f_i^\alpha \tag{7.3}$$

where the coefficients $A_{ij}^{\alpha\beta}(x)$ are uniformly continuous and satisfy the Legendre-Hadamard condition (3.17).

Fix $\eta \in C_c^\infty(B_R(x_0))$, where $B_R(x_0) \Subset \Omega$. One easily computes that $u\eta$ is a weak solution in $B_R(x_0)$ of

$$D_\alpha \big(A_{ij}^{\alpha\beta}(x_0) D_\beta(u^j \eta) \big) = D_\alpha \big([A_{ij}^{\alpha\beta}(x_0) - A_{ij}^{\alpha\beta}(x)] D_\beta(u^j \eta) \big)$$
$$+ G_i + D_\alpha F_i^\alpha,$$

where

$$\begin{aligned} G_i &:= g_i\eta - f_i^\alpha D_\alpha\eta + A_{ij}^{\alpha\beta}(x)D_\beta u^j D_\alpha\eta \\ F_i^\alpha &:= f_i^\alpha\eta + A_{ij}^{\alpha\beta}(x)u^j D_\beta\eta. \end{aligned}$$

Let now $w \in W_0^{1,2}(B_R(x_0), \mathbb{R}^m)$ be the weak solution of

$$\Delta w^i = G_i.$$

We conclude that $u\eta$ is a weak solution of

$$D_\alpha\big(A_{ij}^{\alpha\beta}(x_0)D_\beta(u^j\eta)\big) = D_\alpha\big([A_{ij}^{\alpha\beta}(x_0) - A_{ij}^{\alpha\beta}(x)]D_\beta(u^j\eta) + \widetilde{F}_i^\alpha\big),$$

where

$$\widetilde{F}_i^\alpha = F_i^\alpha + D_\alpha w^i.$$

Suppose now that for some $p \in (2, \infty)$

$$\begin{aligned} &f_i^\alpha \in L^p(\Omega) \\ &g_i \in L^{p_*}(\Omega), \qquad p_* := \frac{np}{n+p} \\ &Du \in L^m(\Omega), \qquad \text{for some } m \in [2, p]. \end{aligned}$$

Then $G_i \in L^{\min(m,p_*)}$ and $F_i^\alpha \in L^{\min(p,m^*)}$. Here $m^* = \frac{nm}{n-m}$ is the Sobolev exponent for $m < n$ and $m^* := \infty$ for $m \geq n$. On account of the L^2 theory $D^2 w \in L^2(\Omega)$ and

$$\Delta(D_\alpha w^i) = D_\alpha G_i;$$

Theorem 7.1 then yields

$$Dw \in L^{r^*}(\Omega), \quad r^* = \min(m^*, (p_*)^*) = \min(m^*, p),$$

and in conclusion

$$\widetilde{F}_i^\alpha \in L^{\min(m^*,p)}(\Omega).$$

Now for $s = \min(m^*, p)$ fix $V \in W_0^{1,s}(B_R(x_0))$ and let $v \in W_0^{1,s}(B_R(x_0))$ be the weak solution of

$$D_\alpha\big(A_{ij}^{\alpha\beta}(x_0)D_\beta v^j\big) = D_\alpha\Big\{\big[A_{ij}^{\alpha\beta}(x_0) - A_{ij}^{\alpha\beta}(x)\big]D_\beta V^j + \widetilde{F}_i^\alpha\Big\}. \tag{7.4}$$

By Theorem 7.1 we have

$$\|Dv\|_{L^{\min(m^*,p)}} \leq c\big|A(x_0) - A(x)\big|\,\|DV\|_{L^{\min(m^*,p)}} + c\|\widetilde{F}\|_{L^{\min(m^*,p)}},$$

and the map

$$T : V \in W_0^{1,s}(B_R(x_0)) \mapsto v \in W_0^{1,s}(B_R(x_0))$$

satisfies

$$\|D(TV_1 - TV_2)\|_{L^{\min(m^*,p)}} \le c\omega(R) \|(DV_1 - DV_2)\|_{L^{\min(m^*,p)}},$$

where $\omega(R)$ is the modulus of continuity of $(A_{ij}^{\alpha\beta})$, hence it is a contraction for R sufficiently small. Then, provided $Du \in L^m(B_R(x_0))$, T has a unique fixed point in $W^{1,\min(m^*,p)}(B_R(x_0))$ that agrees with $u\eta$. Since $Du \in L^2(B_R(x_0))$ we infer that $D(u\eta) \in L^{\min(2^*,p)}$, and choosing $\eta \equiv 1$ in $B_{R/2}(x_0)$ we infer $Du \in L^{\min(2^*,p)}(B_{R/2}(x_0))$. Noticing that

$$2^{*\cdots*(k-\text{times})} \ge p \quad \text{for } k \text{ large enough},$$

a bootstrap argument yields

Theorem 7.2 *Let $u \in W^{1,2}(\Omega, \mathbb{R}^m)$ be a weak solution of (7.3). Assume the coefficients $A_{ij}^{\alpha\beta}$ uniformly continuous with modulus of continuity ω and satisfying the Legendre-Hadamard condition (3.17) with ellipticity λ. Suppose moreover that $f_i^\alpha \in L^p(\Omega)$ and $g_i \in L^{\frac{np}{n+p}}(\Omega)$ for some $p \in (2,\infty)$. Then $Du \in L^p_{\mathrm{loc}}(\Omega)$ and for any open set $\Omega_0 \Subset \Omega$ we have*

$$\|Du\|_{L^p(\Omega_0)} \le c\Big[\|f\|_{L^p(\Omega)} + \|g\|_{L^{\frac{np}{n+p}}(\Omega)} + \|Du\|_{L^2(\Omega)}\Big],$$

with $c = c(p, n, \lambda, \omega, |A|, \mathrm{dist}(\Omega_0, \partial\Omega))$.

Similarly one can show

Theorem 7.3 *Suppose that $u \in W^{2,2}(\Omega, \mathbb{R}^m)$ is a solution of*

$$A_{ij}^{\alpha\beta} D_{\alpha\beta} u^j = f_i$$

with $A_{ij}^{\alpha\beta}$ as in Theorem 7.2 and $f_i \in L^p(\Omega)$ for some $p \in (2,\infty)$. Then $D^2u \in L^p_{\mathrm{loc}}(\Omega)$ and for any open set $\Omega_0 \Subset \Omega$

$$\|D^2u\|_{L^p(\Omega_0)} \le c(p, n, \lambda, \omega, |A|, \mathrm{dist}(\Omega_0, \partial\Omega))\Big[\|f\|_{L^p(\Omega)} + \|D^2u\|_{L^2(\Omega)}\Big].$$

Of course one can also prove global estimates, but we shall not deal with that.

7.1.3 The cases $p = 1$ and $p = \infty$

The L^p-estimates of the previous section actually extend to the case $p \in (1,2)$, hence they hold for every $p \in (1,\infty)$. For instance we state without proof:

Theorem 7.4 *Let $u \in W^{1,2}(\Omega, \mathbb{R}^m)$ solve*

$$\Delta u = f \ \text{in}\ \Omega, \quad u - g \in W_0^{1,2}(\Omega, \mathbb{R}^m),$$

for some $f \in L^p(\Omega, \mathbb{R}^m)$, $g \in W^{2,p}(\Omega, \mathbb{R}^m)$, where $p \in (1, \infty)$. Then we have $u \in W^{2,p}(\Omega, \mathbb{R}^m)$ and

$$\|D^2 u\|_{L^p(\Omega)} \leq c(\|f\|_{L^p(\Omega)} + \|D^2 g\|_{L^p(\Omega)}).$$

One might wonder whether Theorem 7.4 extends to the cases $p = 1$ and $p = \infty$. This is not the case, as we shall now show.

Example 7.5 (Failure of the L^1-estimates) Let

$$D^2 := \{x \in \mathbb{R}^2 : |x| < 1\},$$

and consider the function $u \in W_0^{1,2}(D^2)$ given by

$$u(x) = \log\log(e|x|^{-1}).$$

Then

$$\Delta u(x) = -\frac{1}{|x|^2 \log^2(e|x|^{-1})},$$

hence owing to

$$\int_0^1 \frac{1}{r \log^2(er^{-1})} dr < \infty,$$

we immediately infer

$$\Delta u \in L^1(D^2).$$

On the other hand, $u \notin W^{2,1}(D^2)$. For instance, writing $|x| = r$, one can easily verify that

$$|D^2 u| \geq \frac{\partial^2 u}{\partial r^2} = \frac{\log(er^{-1}) - 1}{r^2 \log^2(er^{-1})} \geq \frac{1}{2r^2 \log(er^{-1})} \quad \text{for } r \text{ sufficiently small,}$$

and since

$$\int_0^\varepsilon \frac{1}{r \log(er^{-1})} dr = \infty, \quad \text{for every } \varepsilon \in (0, 1],$$

we have

$$\int_{D^2} |D^2 u| dx = \infty.$$

Example 7.6 (Failure of the L^∞-estimates) Let $u : D^2 \to \mathbb{C}$ be defined in polar coordinates by

$$u(r, \theta) = r^2 \log(r) e^{2i\theta}.$$

We can easily compute

$$\begin{aligned} u_r(r,\theta) &= (2r\log r + r)e^{2i\theta}, \\ u_{rr}(r,\theta) &= (2\log r + 3)e^{2i\theta}, \\ u_{\theta\theta}(r,\theta) &= -4r^2\log(r)e^{2i\theta}, \end{aligned}$$

hence, since in polar coordinates

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

we easily see that

$$\Delta u = 4e^{2i\theta} \in L^\infty(\Omega),$$

while $\nabla^2 u \notin L^\infty(\Omega)$.

Exercise 7.7 Let u be as in Example 7.6. Show that $D^2u \in BMO(D^2)$ (compare Theorem 5.20).

7.1.4 Wente's result

Consider again $\Omega = D^2 := \{x \in \mathbb{R}^2 : |x| < 1\}$. Although $\Delta u \in L^1(D^2)$ does not imply $D^2u \in L^1(D^2)$, nor $\nabla u \in L^2$ or $u \in L^\infty$ in general, these facts are true if Δu presents a special structure, like in Wente's theorem below (compare also Remark 7.9, and see also [15] and [113]).

Theorem 7.8 (Wente [112]) *Consider two functions $a, b \in W^{1,2}(D^2)$ and let $u \in W_0^{1,1}(D^2)$ solve*

$$\begin{cases} \Delta u = \nabla^\perp a \cdot \nabla b & \text{in } D^2 \\ u = 0 & \text{on } \partial D^2, \end{cases} \tag{7.5}$$

where

$$\nabla^\perp a \cdot \nabla b = \frac{\partial a}{\partial x_1}\frac{\partial b}{\partial x_2} - \frac{\partial a}{\partial x_2}\frac{\partial b}{\partial x_1}.$$

Then $u \in C^0(\overline{D^2}) \cap W^{1,2}(D^2)$ and

$$\|u\|_{L^\infty} + \|\nabla u\|_{L^2} \le C\|\nabla a\|_{L^2}\|\nabla b\|_{L^2}, \tag{7.6}$$

where C is a fixed constant, not depending on a or b.

Proof. First consider the case when $a, b \in C^\infty(\overline{D^2})$. Let us notice that if we can bound

$$\|u\|_{L^\infty} \le C\|\nabla a\|_{L^2}\|\nabla b\|_{L^2}, \tag{7.7}$$

then integration by parts and Hölder's inequality yield

$$\int_{D^2} |\nabla u|^2 dx = -\int_{D^2} u\Delta u dx \le 2\|u\|_{L^\infty}\|\nabla a\|_{L^2}\|\nabla b\|_{L^2},$$

hence, using the inequality $2ts \le t^2 + s^2$,

$$\|\nabla u\|_{L^2}^2 \le \|u\|_{L^\infty}^2 + \|\nabla a\|_{L^2}^2 \|\nabla b\|_{L^2}^2 \le (C+1)\|\nabla a\|_{L^2}^2 \|\nabla b\|_{L^2}^2$$

and (7.6) follows.

Let us now prove (7.7). Extend a and b to smooth functions $\tilde{a}$ and $\tilde{b}$ defined on $\mathbb{R}^2$ and with compact support. This can be done in such a way that

$$\|\nabla \tilde{a}\|_{L^2(\mathbb{R}^2)} \le C\|\nabla a\|_{L^2(D^2)}, \quad \|\nabla \tilde{b}\|_{L^2(\mathbb{R}^2)} \le C\|\nabla b\|_{L^2(D^2)}, \tag{7.8}$$

with C not depending on a or b. Next define

$$\tilde{u} := \psi * (\nabla^\perp \tilde{a} \cdot \nabla \tilde{b}),$$

where

$$\psi(x) := \frac{1}{2\pi} \log \frac{1}{|x|}$$

is a fundamental solution of the Laplacian, i.e. $\Delta \psi = \delta_0$. In particular one can easily verify that

$$\Delta \tilde{u} = \Delta u \quad \text{in } D^2.$$

Now notice that in polar coordinates (r, θ)

$$\nabla^\perp \tilde{a} \cdot \nabla \tilde{b} = -\frac{1}{r}\frac{\partial \tilde{a}}{\partial \theta}\frac{\partial \tilde{b}}{\partial r} + \frac{1}{r}\frac{\partial \tilde{b}}{\partial \theta}\frac{\partial \tilde{a}}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(\tilde{a}\frac{\partial \tilde{b}}{\partial \theta}\right) - \frac{1}{r}\frac{\partial}{\partial \theta}\left(\tilde{a}\frac{\partial \tilde{b}}{\partial r}\right).$$

It follows

$$\begin{aligned}
\tilde{u}(0) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \log r \left[\frac{1}{r}\frac{\partial}{\partial r}\left(\tilde{a}\frac{\partial \tilde{b}}{\partial \theta}\right) - \frac{1}{r}\frac{\partial}{\partial \theta}\left(\tilde{a}\frac{\partial \tilde{b}}{\partial r}\right)\right] dx \\
&= \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} \log r \left[\frac{\partial}{\partial r}\left(\tilde{a}\frac{\partial \tilde{b}}{\partial \theta}\right) - \frac{\partial}{\partial \theta}\left(\tilde{a}\frac{\partial \tilde{b}}{\partial r}\right)\right] d\theta dr \\
&= \frac{1}{2\pi}\int_0^{2\pi} \int_0^\infty \log r \frac{\partial}{\partial r}\left(\tilde{a}\frac{\partial \tilde{b}}{\partial \theta}\right) dr d\theta \\
&= -\frac{1}{2\pi}\int_0^{2\pi} \int_0^\infty \frac{1}{r}\left(\tilde{a}\frac{\partial \tilde{b}}{\partial \theta}\right) dr d\theta.
\end{aligned} \tag{7.9}$$

Now let

$$\tilde{a}_r := \fint_{\partial D_r} \tilde{a} d\theta$$

denote the average of $\tilde{a}$ over the circle of radius r. Since

$$\int_0^{2\pi} \frac{\partial \tilde{b}(r,\theta)}{\partial \theta} d\sigma = 0$$

for every $r>0$, we can also write (7.9) as

$$\tilde{u}(0) \le -\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}\frac{1}{r}\left((\tilde{a}-\tilde{a}_r)\frac{\partial \tilde{b}}{\partial\theta}\right) dr d\theta.$$

Then with the Cauchy-Schwarz and Poincaré inequalities we bound

$$\begin{aligned}
\tilde{u}(0) &\le \frac{1}{2\pi}\int_0^{\infty}\frac{1}{r}\left(\int_0^{2\pi}|\tilde{a}-\tilde{a}_r|^2 d\theta\right)^{\frac{1}{2}}\left(\int_0^{2\pi}\left|\frac{\partial\tilde{b}}{\partial\theta}\right|^2 d\theta\right)^{\frac{1}{2}} dr\\
&\le \frac{1}{2\pi}\int_0^{\infty}\left(\int_0^{2\pi}\left|\frac{\partial\tilde{a}}{\partial\theta}\right|^2 d\theta\right)^{\frac{1}{2}}\left(\int_0^{2\pi}\left|\frac{\partial\tilde{b}}{\partial\theta}\right|^2 d\theta\right)^{\frac{1}{2}}\frac{dr}{r}\\
&\le \frac{1}{2\pi}\left(\int_0^{\infty}\int_0^{2\pi}\left|\frac{\partial\tilde{a}}{\partial\theta}\right|^2 d\theta\frac{dr}{r}\right)^{\frac{1}{2}}\left(\int_0^{\infty}\int_0^{2\pi}\left|\frac{\partial\tilde{b}}{\partial\theta}\right|^2 d\theta\frac{dr}{r}\right)^{\frac{1}{2}}\\
&\le \frac{1}{2\pi}\|\nabla\tilde{a}\|_{L^2}\|\nabla\tilde{b}\|_{L^2}\\
&\le C_0\|\nabla a\|_{L^2}\|\nabla b\|_{L^2},
\end{aligned} \tag{7.10}$$

where, taking into account (7.8) we chose $C_0=\frac{C^2}{2\pi}$.

By translation invariance (7.10) actually implies

$$\|\tilde{u}\|_{L^\infty}\le C_0\|\nabla a\|_{L^2}\|\nabla b\|_{L^2}.$$

Now observe that $v:=\tilde{u}-u$ is harmonic, hence by the maximum principle

$$\sup_{D^2}|\tilde{u}-u|\le\sup_{\partial D^2}|\tilde{u}|\le\|\tilde{u}\|_{L^\infty},$$

and by the triangle inequality

$$\sup_{D^2}|u|\le\sup_{D^2}|\tilde{u}-u|+\sup_{D^2}|\tilde{u}|\le 2\|\tilde{u}\|_{L^\infty}\le 2C_0\|\nabla a\|_{L^2}\|\nabla b\|_{L^2}.$$

The general case is obtained by approximation. Indeed, if

$$a_n\in C^\infty(\overline{D^2})\cap H^1(D^2),\quad a_n\to a \text{ in } H^1(D^2)$$
$$b_n\in C^\infty(\overline{D^2})\cap H^1(D^2),\quad b_n\to b \text{ in } H^1(D^2)$$

and u_n is the solution of (7.5) with a_n and b_n instead of a and b, then by (7.6) (u_n) is a Cauchy sequence in $C^0(\overline{D^2})\cap H^1(D^2)$, hence $u_n\to v$ in $C^0(\overline{D^2})\cap H^1(D^2)$ for some function v. On the other hand, since $a_nb_n\to ab$ in $L^1(D^2)$, by L^p estimates (in fact a version we have not proven in the previous sections), we also have that $u_n\to u$ in $W^{1,p}(D^2)$ for every $p\in[1,2)$, where u solves (7.5). It follows that $u=v$ hence u satisfies (7.6). □

Remark 7.9 In the same hypothesis of Theorem 7.8 Coifman, Lions, Meyer and Semmes [22] later proved that $D^2u \in L^1(D^2)$. It is well-known that this also implies $u \in C^0(D^2)$ and $Du \in L^2(D^2)$.

With a similar proof, one can also prove estimates for the Neumann problem:

Theorem 7.10 *Consider two functions $a, b \in W^{1,2}(D^2)$ and let $u \in W_0^{1,1}(D^2)$ solve*

$$\begin{cases} \Delta u = \nabla^\perp a \cdot \nabla b & \text{in } D^2 \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial D^2, \\ \displaystyle\int_{D^2} u dx = 0. \end{cases}$$

Then $u \in C^0(\overline{D^2}) \cap W^{1,2}(D^2)$ and

$$\|u\|_{L^\infty} + \|\nabla u\|_{L^2} \le C \|\nabla a\|_{L^2} \|\nabla b\|_{L^2}, \tag{7.11}$$

where C is a fixed constant, not depending on a or b.

7.2 Singular integrals

Given a function $k \in L^1(\mathbb{R}^n)$ the *convolution product* of k with a function $f \in L^1(\mathbb{R}^n)$

$$k * f(x) := \int_{\mathbb{R}^n} k(x-y) f(y) dy \tag{7.12}$$

defines a map from $L^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, see Section 6.1.2. In this and in the next section we shall be concerned with *singular* and *fractional integrals*. This amounts to studying $k*f$ for functions $k(x)$ (called kernels) which are *positively homogeneous* of degree $-\alpha$ for some $\alpha > 0$

$$k(y) = |y|^{-\alpha} k\Big(\frac{y}{|y|}\Big) = \frac{\omega(y)}{|y|^\alpha}, \qquad \omega(y) := k\Big(\frac{y}{|y|}\Big),$$

where ω is *homogeneous of degree zero.* Notice that k is not integrable in $\mathbb{R}^n$, unless it is 0. Two situations, corresponding respectively to *fractional* and *singular integrals*, are of particular interest:

$$\text{(a)}\ \ 0 < \alpha < n, \qquad\qquad \text{(b)}\ \ \alpha = n.$$

Case (a) is of course the simplest. If, for example, we assume ω bounded and $f \in L^1(\mathbb{R}^n)$, then

$$\begin{aligned} k * f(x) &= \int_{\mathbb{R}^n} \omega(y) \frac{f(x-y)}{|y|^\alpha} dy \\ &= \int_{B_1(0)} \omega(y) \frac{f(x-y)}{|y|^\alpha} dy + \int_{\mathbb{R}^n \setminus B_1(0)} \omega(y) \frac{f(x-y)}{|y|^\alpha} dy. \end{aligned} \tag{7.13}$$

The first term, being the convolution of f with an integrable function, the function $k_1 = k\chi_{B_1(0)}$, exists for almost every $x \in \mathbb{R}^n$ and is integrable in $\mathbb{R}^n$, while for the second integral we have

$$\left| \int_{\mathbb{R}^n \setminus B_1(0)} \frac{\omega(y)}{|y|^\alpha} f(x-y)dy \right| \le \max_{\mathbb{R}^n \setminus \{0\}} |\omega| \int_{\mathbb{R}^n \setminus B_1(0)} |f(x-y)|dy,$$

therefore it represents a bounded function.

Case (b) is much more complicated and in fact it will be our main concern. Assuming ω bounded, say $\omega \equiv 1$, we have

$$k * f(x) = \int_{B_1(0)} \frac{f(x-y)}{|y|^n} dy + \int_{\mathbb{R}^n \setminus B_1(0)} \frac{f(x-y)}{|y|^n} dy \tag{7.14}$$

and the second integral again represents a bounded function. But it is clear that the first integral needs not exist as a Lebesgue integral at any point x. For example, if f is non-zero in a neighborhood of a point x_0, say a non-zero constant, then the integral diverges for x in a neighborhood of x_0. Therefore, if we want to consider those integrals, we need to redefine their meaning.

Example 7.11 (Newtonian potential) Consider the potential of a density of mass $f(x)$ in $\mathbb{R}^3$

$$V(x) := -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy.$$

If, say, f is smooth and with compact support, differentiation under the sign of integral formally leads to

$$\frac{\partial^2 V}{\partial x^i \partial x^j}(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} k_{ij}(x-y) f(y) dy \tag{7.15}$$

where

$$k_{ij}(y) = \frac{1}{4\pi |y|^3} \Big(\delta_{ij} - 3 \frac{y^i y^j}{|y|^2} \Big). \tag{7.16}$$

A classical result by Hölder states that if $f \in C^{0,\alpha}_c(\mathbb{R}^3)$ then $V \in C^{2,\alpha}(\mathbb{R}^3)$ and $\Delta V = f$, so that (7.15) amounts also to

$$\frac{\partial^2 V}{\partial x^i \partial x^j} = K_{ij} \Delta g, \qquad K_{ij} f(x) := \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} k_{ij}(x-y) f(y) dy,$$

which expresses the monomial differential operator $\frac{\partial^2}{\partial x^i \partial x^j}$ in terms of the Laplacian and of a singular integral.

The integral in (7.15) is of course of the same nature of the integral in (7.13), and both are singular integrals. Notice that k_{ij} has mean value zero on the unit sphere $|x| = 1$. This is actually, as we shall see, the reason why the limit exists.

In the following subsection we shall study boundedness of singular integrals on Hölder spaces, extending Hölder's result on the continuity of the second derivatives of the Newtonian potential of a Hölder continuous distribution of mass, and on L^p spaces, proving the nowadays classical Calderón-Zygmund theorem. Finally, in the last subsection we shall discuss fractional integral operators proving, as a consequence, the classical Sobolev inequalities.

7.2.1 The cancellation property and the Cauchy principal value

Let $k(y) = \frac{\omega(y)}{|y|^n}$ be a continuous function from $\mathbb{R}^n \setminus \{0\}$ into $\mathbb{R}$ which is positively homogeneous of degree $-n$, so that ω is continuous and positively homogeneous of degree zero. For every $\varepsilon > 0$ and for every $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, the integral

$$T_\varepsilon f(x) := \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} k(x-y) f(y) dy = \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} k(y) f(x-y) dy$$

is absolutely convergent, since

$$|k(y)| \le |y|^{-n} \sup_{\partial B_1(0)} |\omega(y)|.$$

Notice that in order for $T_\varepsilon f(x)$ to be well defined we only need ω to be bounded (not necessarily continuous) and that $T_\varepsilon f(x) = f * k_\varepsilon(x)$, where $k_\varepsilon := k\chi_{\mathbb{R}^n \setminus B_\varepsilon(0)}$.

Motived by the previous example we would like to define

$$Tf(x) = \int k(x-y) f(y) dy$$

as *Cauchy principal value*, i.e. as

$$Tf(x) = \int k(x-y) f(y) dy := \lim_{\varepsilon \to 0} T_\varepsilon f(x).$$

However, as the following example shows, this is not always possible without further assumptions on $\omega(y)$.

Example 7.12 Let $n = 1$, $\omega \equiv 1$ and $f(t) = \chi_{[-1,1]}$. For

$$T_\varepsilon f(x) = \int_{\mathbb{R} \setminus B_\varepsilon(x)} \frac{f(t)}{|x-t|} dt$$

we have

$$\lim_{\varepsilon \to 0} T_\varepsilon f(x) = \begin{cases} 0 & \text{if } x \notin [-1,1] \\ +\infty & \text{if } x \in [-1,1]. \end{cases}$$

Definition 7.13 (Cauchy principal value) *Let $\Omega \subset \mathbb{R}^n$ be an open set containing 0 and $g : \Omega\backslash\{0\} \to \mathbb{R}$ be a measurable function. We say that g is* integrable in the Cauchy principal sense with respect to a neighborhood U of 0 *if g is integrable on each $\Omega\backslash\varepsilon U$ and*

$$\lim_{\varepsilon\to 0}\int_{\Omega\backslash\varepsilon U} g dx$$

exists (of course $\varepsilon U := \{\varepsilon x : x \in U\}$). Such a limit, which in general depends on U, is called Cauchy principal value of the integral with respect to U.

Proposition 7.14 *Let $k(y) = |y|^{-n}\omega(y/|y|)$ be as above, with ω measurable and homogeneous of degree 0. We have:*

(i) *The Cauchy principal value of the integral of $k(y)$ with respect to a neighborhood U of 0 exists if and only if it exists for any other neighborhood V of 0.*

(ii) *The principal value of the singular integral $\int k(y)dy$ exists for all U if and only if the following* cancellation property *holds:*

$$\int_{\partial B_1(0)} k(y)dy = 0.$$

Proof. Let U and V be two neighborhood of 0. For ε small we have $\varepsilon U \cap \varepsilon V \subset \Omega$ and

$$\begin{aligned}\int_{\Omega\backslash\varepsilon U} k(x)dx - \int_{\Omega\backslash\varepsilon V} k(x)dx &= \int_{\varepsilon(V\backslash U)} k(x)dx - \int_{\varepsilon(U\backslash V)} k(x)dx \\ &= \int_{V\backslash U} k(\varepsilon x)\varepsilon^n dx - \int_{U\backslash V} k(\varepsilon x)\varepsilon^n dx \\ &= \int_{V\backslash U} k(x)dx - \int_{U\backslash V} k(x)dx,\end{aligned}$$

which implies (i). By (i) we can take $U = B_1(0)$ in (ii). For $0 < \varepsilon < \eta$, η small, we have

$$\begin{aligned}\int_{\Omega\backslash B_\eta(0)} k(x)dx - \int_{\Omega\backslash B_\varepsilon(0)} k(x)dx &= \int_\varepsilon^\eta \rho^{n-1}\Big(\int_{\partial B_1(0)} k(\rho\theta)d\mathcal{H}^{n-1}(\theta)\Big)d\rho \\ &= \int_\varepsilon^\eta \frac{d\rho}{\rho}\int_{\partial B_1(0)} \omega(\theta)d\mathcal{H}^{n-1}(\theta) \\ &= \log\left(\frac{\eta}{\varepsilon}\right)\int_{\partial B_1(0)} \omega(\theta)d\mathcal{H}^{n-1}(\theta),\end{aligned}$$

which proves (ii). □

Motivated by the previous considerations we now set

Definition 7.15 *We say that a function $k : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ is a* singular kernel *or a* Calderón-Zygmund kernel *if*

(i) *k is positively homogeneous of degree $-n$, i.e.*

$$k(x) = \frac{\omega(x)}{|x|^n}, \qquad \forall x \in \mathbb{R}^n \backslash \{0\},$$

where $\omega(x)$ is a zero-homogeneous function;

(ii) $k\big|_{\partial B_1(0)} = \omega \in L^\infty$

(iii) $\int_{\partial B_1(0)} k d\mathcal{H}^{n-1} = \int_{\partial B_1(0)} \omega d\mathcal{H}^{n-1} = 0.$

Examples of singular kernels are $k(x) = \frac{1}{\pi x}$ for $n = 1$ and $k_{ij}(x)$ as defined in (7.16). More generally, if $F \in C^1(\mathbb{R}^n \backslash \{0\})$ is homogeneous of degree $1 - n$, then $D_j F$, $j = 1, \dots, n$ is a singular kernel. In fact (i) and (ii) are easily verified, and as for (iii) we have by (i) and the divergence theorem

$$\begin{aligned} 0 &= \frac{1}{R-r}\left(\int_{\partial B_R(0)} \frac{x_j}{|x|} F(x) d\mathcal{H}^{n-1} - \int_{\partial B_r(0)} \frac{x_j}{|x|} F(x) d\mathcal{H}^{n-1}\right) \\ &= \frac{1}{R-r} \int_{B_R(0)\backslash B_r(0)} D_j F dx \\ &\to \int_{\partial B_R(0)} D_j F d\mathcal{H}^{n-1}, \quad \text{as } r \to R. \end{aligned}$$

From now on a *singular integral* will be an integral of the type

$$Tf(x) = \int k(x-y) f(y) dy := \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \backslash B_\varepsilon(x)} k(x-y) f(y) dy,$$

where k is a singular kernel.

7.2.2 Hölder-Korn-Lichtenstein-Giraud theorem

In this section we discuss singular integrals as operators on the space of Hölder continuous functions with compact support. Let $k(x)$ be a Calderón-Zygmund kernel and let

$$T_\varepsilon f(x) := \int_{\mathbb{R}^n \backslash B_\varepsilon(x)} k(x-y) f(y) dy.$$

As a consequence of the cancellation property of $k(x)$ we get

Proposition 7.16 *Let $f \in C_c^{0,\alpha}(\mathbb{R}^n)$, $0 < \alpha < 1$. Then $T_\varepsilon f$ converges uniformly to Tf in $\mathbb{R}^n$. In particular $Tf(x)$ is a continuous function of x.*

Proof. For $x \in \mathbb{R}^n$ and $0 < \delta < \varepsilon$, the cancellation property of k yields

$$|T_\varepsilon f(x) - T_\delta f(x)| = \int_{B_\varepsilon(x)\setminus B_\delta(x)} k(x-y)f(y)dy$$
$$= \int_{B_\varepsilon(x)\setminus B_\delta(x)} k(x-y)\big[f(y)-f(x)\big]dy,$$

hence

$$|T_\varepsilon f(x) - T_\delta f(x)| \le c(n)\|\omega\|_{L^\infty}[f]_{0,\alpha}\int_\delta^\varepsilon t^{\alpha-1}dt$$
$$= \frac{c(n)}{\alpha}\|\omega\|_{L^\infty}[f]_{0,\alpha}\big(\varepsilon^\alpha - \delta^\alpha\big).$$

This shows that $T_\varepsilon f(x)$ is a Cauchy sequence for the uniform convergence. □

We now prove that actually Tf is Hölder continuous if $f \in C^{0,\alpha}_c(\mathbb{R}^n)$ and in fact T is a *bounded* operator from $C^{0,\alpha}_c(\mathbb{R}^n)$ into $C^{0,\alpha}(\mathbb{R}^n)$. In order to do that we need however some regularity on the kernel k. We shall assume that its trace ω on $\Sigma_1 := \partial B_1(0)$ is Lipschitz continuous, though less would suffice. We then have

Theorem 7.17 (Hölder-Korn-Lichtenstein-Giraud) *Assume that k is a Calderón-Zygmund kernel. Suppose that $k\big|_{\Sigma_1} = \omega\big|_{\Sigma_1}$ is Lipschitz continuous. Then for every $f \in C^{0,\alpha}_c(\mathbb{R}^n)$, $0 < \alpha < 1$, we have $Tf \in C^{0,\alpha}(\mathbb{R}^n)$ and*

$$[Tf]_{0,\alpha} \le c(n, \alpha, \|\omega\|_{C^{0,1}(\Sigma_1)})[f]_{0,\alpha},$$

where

$$\|\omega\|_{C^{0,1}(\Sigma_1)} = \|\omega\|_{L^\infty(\Sigma_1)} + \sup_{x,y\in\Sigma_1,\, x\neq y}\frac{\omega(x)-\omega(y)}{|x-y|}.$$

Proof. As we have seen in the proof of Proposition 7.16, for $0 < \delta < \varepsilon$ we have

$$|T_\varepsilon f(x) - T_\delta f(x)| \le c(n,\alpha)\|\omega\|_{L^\infty}[f]_{0,\alpha}(\varepsilon^\alpha - \delta^\alpha).$$

Letting $\delta \to 0$ we infer

$$|T_\varepsilon f(x) - Tf(x)| \le c[f]_{0,\alpha}\varepsilon^\alpha, \qquad \forall x \in \mathbb{R}^n.$$

Now we fix $x, z \in \mathbb{R}^n$. Since

$$|Tf(x)-Tf(z)| \le |Tf(x)-T_\varepsilon f(x)|+|T_\varepsilon f(x)-T_\varepsilon f(z)|+|T_\varepsilon f(z)-Tf(z)|.$$

In order to prove the theorem it clearly suffices to show that

$$|T_\varepsilon f(x) - T_\varepsilon f(z)| \le c[f]_{0,\alpha}\varepsilon^\alpha \qquad \text{for } \varepsilon = 2|x-z|. \tag{7.17}$$

In the next two lemmas we state separately the two simple estimates which lead to (7.17).

Lemma 7.18 *There is a constant $c = c\big(n, \|\omega\|_{C^{0,1}(\Sigma_1)}\big)$ such that*

$$|k(x-y) - k(x)| \le c\frac{|y|}{|x|^{n+1}}, \qquad \text{if } |x| \ge 2|y|.$$

Proof. We write

$$\begin{aligned} k(x-y) - k(x) &= \omega(x-y)\left[\frac{1}{|x-y|^n} - \frac{1}{|x|^n}\right] + \frac{1}{|x|^n}\big[\omega(x-y) - \omega(x)\big] \\ &=: (\mathrm{I}) + (\mathrm{II}). \end{aligned}$$

First observe that, if $|x| \ge 2|y|$, then $\frac{|x|}{2} \le |x-y| \le \frac{3}{2}|x|$. Moreover, if ξ is a point in the segment with end points x and $x-y$, we have

$$\frac{|x|}{2} \le |\xi| \le \frac{3}{2}|x|.$$

In fact, for $t \in (0,1)$, $\xi = tx + (1-t)(x-y)$ we have

$$\begin{aligned} |\xi| &\le t|x| + (1-t)|x-y| \le t|x| + \frac{3}{2}(1-t)|x| \le \frac{3}{2}|x| \\ |\xi| &= |x - (1-t)y| \ge |x| - (1-t)|y| \ge \frac{1}{2}|x|. \end{aligned}$$

Using Lagrange's mean value theorem with the function

$$t \mapsto \frac{1}{|tx + (1-t)(x-y)|}, \quad t \in [0,1],$$

we estimate (I) for some ξ in the segment with end points x and $x-y$ by

$$|(\mathrm{I})| \le n\|\omega\|_{L^\infty}|\xi|^{-n-1}|y| \le n2^{n+1}\|\omega\|_{L^\infty}\frac{|y|}{|x|^{n+1}}.$$

On the other hand, by the regularity of ω we have

$$|(\mathrm{II})| \le \frac{\|\omega\|_{C^{0,1}(\Sigma_1)}}{|x|^n}\left|\frac{x-y}{|x-y|} - \frac{x}{|x|}\right| \le \frac{c\|\omega\|_{C^{0,1}(\Sigma_1)}}{|x|^n}\frac{|y|}{|x|},$$

where we applied the mean value theorem to the function

$$t \mapsto \frac{tx + (1-t)(x-y)}{|tx + (1-t)(x-y)|}, \quad t \in [0,1],$$

hence the proof is complete. □

Lemma 7.19 *There exists a constant $c = c\big(n, \|\omega\|_{C^{0,1}(\Sigma_1)}\big)$ such that for any $\varepsilon \ge 2|x-z|$ we have*

$$|T_\varepsilon f(x) - T_\varepsilon f(z)| \le c\left\{|z-x|\int_{\mathbb{R}^n\setminus B_\varepsilon(z)}\frac{|f(y)|}{|z-y|^{n+1}}dy + \varepsilon^{-n}\int_{B_{2\varepsilon}(z)}|f(y)|dy\right\}.$$

Proof. We write

$$\begin{aligned} T_\varepsilon f(x) - T_\varepsilon f(z) = &\int_{\mathbb{R}^n \setminus B_\varepsilon(z)} \big[k(x-y) - k(z-y)\big] f(y)dy \\ &+ \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} k(x-y)f(y)dy \\ &- \int_{\mathbb{R}^n \setminus B_\varepsilon(z)} k(x-y)f(y)dy. \end{aligned}$$

Applying Lemma 7.18 with x replaced by $z-y$ and y by $z-x$ we infer

$$\left| \int_{\mathbb{R}^n \setminus B_\varepsilon(z)} \big[k(x-y) - k(z-y)\big] f(y)dy \right| \leq c|z-x| \int_{\mathbb{R}^n \setminus B_\varepsilon(z)} \frac{|f(y)|}{|z-y|^{n+1}} dy,$$

with $c = c\big(n, \|\omega\|_{C^{0,1}(\Sigma_1)}\big)$. On the other hand

$$\left| \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} k(x-y)f(y)dy - \int_{\mathbb{R}^n \setminus B_\varepsilon(z)} k(x-y)f(y)dy \right| \leq \frac{c}{\varepsilon^n} \int_{B_{2\varepsilon}(z)} |f(y)|dy,$$

as $\big(\mathbb{R}^n \setminus B_\varepsilon(x)\big) \Delta \big(\mathbb{R}^n \setminus B_\varepsilon(z)\big) \subset B_{2\varepsilon}(z)$, and the proof is completed. □

Completion of the proof of Theorem 7.17. On account of the cancellation property of k we have

$$T_\varepsilon f(x) = T_\varepsilon(f - \lambda)(x), \qquad \forall \lambda \in \mathbb{R}.$$

We can therefore apply Lemma 7.19 to the function $x \mapsto f(x) - f(z)$ to get

$$\begin{aligned} |T_\varepsilon f(x) - T_\varepsilon f(z)| &\leq c_1 \bigg\{ |z-x| \int_{\mathbb{R}^n \setminus B_\varepsilon(z)} \frac{|f(y) - f(z)|}{|z-y|^{n+1}} dy \\ &\qquad + \varepsilon^{-n} \int_{B_{2\varepsilon}(z)} |f(y) - f(z)|dy \bigg\} \\ &\leq c_2 \bigg\{ \int_{\mathbb{R}^n \setminus B_\varepsilon(z)} \frac{[f]_{0,\alpha} |y-z|^\alpha}{|y-z|^n} dy + [f]_{0,\alpha} \varepsilon^\alpha \bigg\} \\ &\leq c_3 \bigg\{ \int_\varepsilon^\infty t^{\alpha-1} dt + \varepsilon^\alpha \bigg\} [f]_{0,\alpha} \leq c_4 [f]_{0,\alpha} \varepsilon^\alpha, \end{aligned}$$

which is (7.17). □

7.2.3 L^2-theory

In this section we begin the study of the action of a singular integral on L^p spaces, and more precisely here we restrict ourselves to the study of the behaviour of $T_\varepsilon f$ and Tf when $f \in L^2(\mathbb{R}^n)$. Since $T_\varepsilon f$ is a convolution, a natural tool in the L^2 setting is of course the Fourier transform.

Theorem 7.20 *Let $k(x)$ be a Calderón-Zygmund kernel and let $f \in L^2(\mathbb{R}^n)$. Then we have*

(i) *For every $\varepsilon > 0$, $T_\varepsilon f$ belongs to $L^2(\mathbb{R}^n)$ and*

$$\|T_\varepsilon f\|_{L^2(\mathbb{R}^n)} \leq A_2 \|f\|_{L^2(\mathbb{R}^n)}$$

with constant A_2 independent of ε.

(ii) *The limit Tf of $T_\varepsilon f$ as $\varepsilon \to 0$ exists in the sense of $L^2(\mathbb{R}^n)$, and*

$$\|Tf\|_{L^2(\mathbb{R}^n)} \leq A_2 \|f\|_{L^2(\mathbb{R}^n)}.$$

(iii) *We have*

$$\widehat{Tf}(\xi) = m(\xi)\widehat{f}(\xi),$$

where $m(\xi)$ is a homogeneous function of degree zero, and more precisely

$$m(\xi) = \int_{\partial B_1(0)} k(x')\Big[\log \frac{1}{|x' \cdot \xi|} - i\frac{\pi}{2}\operatorname{sign}(x' \cdot \xi)\Big] d\mathcal{H}^{n-1}(x'), \quad |\xi| = 1.$$

Proof. Set

$$k_\varepsilon(x) := \begin{cases} k(x) & \text{if } |x| > \varepsilon \\ 0 & \text{if } |x| \leq \varepsilon. \end{cases}$$

Obviously $k_\varepsilon \in L^2(\mathbb{R}^n)$ and $T_\varepsilon f = k_\varepsilon * f$. Then by Parseval theorem

$$\begin{aligned} \|T_\varepsilon f\|_{L^2(\mathbb{R}^n)} &= \|\widehat{k_\varepsilon * f}\|_{L^2(\mathbb{R}^n)} \\ &= \|\widehat{k}_\varepsilon \widehat{f}\|_{L^2(\mathbb{R}^n)} \\ &\leq \|\widehat{k}_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \|\widehat{f}\|_{L^2(\mathbb{R}^n)} \\ &= \|\widehat{k}_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)}, \end{aligned}$$

provided $\widehat{k}_\varepsilon$ is bounded.

In order to prove boundedness of $\widehat{k}_\varepsilon$ we introduce the polar coordinates

$$\xi = R\xi', \qquad \xi' = \frac{\xi}{|\xi|}; \qquad x = rx', \qquad x' = \frac{x}{|x|}.$$

Then for $\xi \neq 0$, i.e., $R > 0$, and using the cancellation property of k,

$$\begin{aligned} \widehat{k}_\varepsilon(\xi) &= \lim_{\eta \to \infty} \int_{B_\eta(0)} e^{-2\pi i x \cdot \xi} k_\varepsilon(x) dx \\ &= \lim_{\eta \to \infty} \int_{\partial B_1(0)} \omega(x') \Big(\int_\varepsilon^\eta \Big[e^{-2\pi i R r x' \cdot \xi'} - \cos(2\pi r R) \Big] \frac{dr}{r} \Big) d\sigma(x'). \end{aligned}$$

Consider the integral

$$\int_\varepsilon^\eta \left[e^{-2\pi iRrx'\cdot\xi'} - \cos(2\pi rR)\right]\frac{dr}{r} = \int_{R\varepsilon}^{R\eta} \left[e^{-2\pi irx'\cdot\xi'} - \cos(2\pi r)\right]\frac{dr}{r}. \tag{7.18}$$

We claim that its imaginary part

$$-\int_{R\varepsilon}^{R\eta} \frac{\sin(2\pi rx'\cdot\xi')}{r}dr$$

is uniformly bounded and converges for $\eta \to \infty$ and then $\varepsilon \to 0$ to

$$-\int_0^\infty \frac{\sin t}{t}dt\cdot \operatorname{sign}(x'\cdot\xi') = -\frac{\pi}{2}\operatorname{sign}(x'\cdot\xi'). \tag{7.19}$$

In fact, without loss of generality, we may assume $x'\cdot\xi' > 0$. Then

$$\int_{R\varepsilon}^{R\eta} \frac{\sin(2\pi rx'\cdot\xi')}{r}dr = \int_{2\pi Rx'\cdot\xi'\varepsilon}^{2\pi Rx'\cdot\xi'\eta} \frac{\sin r}{r}dr =: I$$

and we distinguish three cases:

(a) $2\pi Rx'\cdot\xi'\varepsilon \le 1 \le 2\pi Rx'\cdot\xi'\eta$. Then since $\sin r \le r$ for $r > 0$

$$\begin{aligned} I &= \int_{2\pi Rx'\cdot\xi'\varepsilon}^{1} \frac{\sin r}{r}dr + \int_1^{2\pi Rx'\cdot\xi'\eta} \frac{\sin r}{r}dr \le 1 + \int_1^{2\pi Rx'\cdot\xi'\eta} \frac{\sin r}{r}dr \\ &= 1 - \frac{\cos r}{r}\Big|_1^{2\pi Rx'\cdot\xi'\eta} - \int_1^{2\pi Rx'\cdot\xi'\eta} \frac{\cos r}{r^2}dr \\ &\le 3 + \int_1^{2\pi Rx'\cdot\xi'\eta} \frac{1}{r^2}dr \le 4. \end{aligned}$$

(b) $2\pi Rx'\cdot\xi'\eta \le 1$. In this case the interval $(2\pi Rx'\cdot\xi'\varepsilon, 2\pi Rx'\cdot\xi'\eta)$ is contained in $(0,1)$ and since $\frac{\sin r}{r} \le 1$ the claim is trivial.

(c) $2\pi Rx'\cdot\xi'\varepsilon \ge 1$. In this case one proceeds as in (a).

Similarly one can show that the real part of (7.18)

$$\int_\varepsilon^\eta \left[\cos(2\pi rRx'\cdot\xi') - \cos(2\pi rR)\right]\frac{dr}{r}$$

is equibounded. Finally let us show that

$$\lim_{\substack{\varepsilon\to 0\\ \eta\to\infty}} \int_\varepsilon^\eta \left[\cos(2\pi rRx'\cdot\xi') - \cos(2\pi rR)\right]\frac{dr}{r} = \log\frac{1}{|x'\cdot\xi'|}. \tag{7.20}$$

To prove this, we observe that if

$$F(\lambda) := \int_\varepsilon^\eta \frac{h(\lambda r) - h(r)}{r}dr, \qquad \text{then} \qquad F'(\lambda) = \frac{h(\lambda\eta) - h(\lambda\varepsilon)}{\lambda},$$

hence, being $F(1) = 0$,

$$F(\lambda) = \int_1^\lambda \frac{h(t\eta) - h(t\varepsilon)}{t} dt = \int_\eta^{\lambda\eta} \frac{h(t)}{t} dt - \int_\varepsilon^{\lambda\varepsilon} \frac{h(t)}{t} dt.$$

Now, if $\frac{h(t)}{t}$ is integrable in $(1, +\infty)$ and if $h(t) = h(0)+$ bounded function of t times t near zero, letting $\eta \to +\infty$ and $\varepsilon \to 0$ we infer

$$\lim_{\substack{\eta\to\infty\\ \varepsilon\to 0}} \int_\varepsilon^\eta \frac{h(\lambda r) - h(r)}{r} dr = -h(0) \log \lambda. \tag{7.21}$$

Applying (7.21) with $h(r) := \cos(2\pi R r)$, $\lambda := |x' \cdot \xi'|$, we infer at once (7.20). This completes the proof of (i).

For any $\sigma > 0$ we now split f as $f = g + b$ where $g \in C^1_c(\mathbb{R}^n)$ and $b := f - g \in L^2(\mathbb{R}^n)$ with $\|b\|_{L^2(\mathbb{R}^n)} \le \sigma$. From (i) we then deduce

$$\|T_\varepsilon b\|_{L^2} \le A_2 \|b\|_{L^2} \le A_2 \sigma.$$

On the other hand $T_\varepsilon g \to Tg$ uniformly in $\mathbb{R}^n$ by Proposition 7.16 and $T_\varepsilon g(x) = Tg(x)$ for $x \notin \operatorname{supp}(g)$ and $\varepsilon < \operatorname{dist}(x, \operatorname{supp}(g))$. Consequently $T_\varepsilon g \to Tg$ in L^2. If follows that $T_\varepsilon f$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$, which proves (ii).

Finally, thanks to (7.19), (7.20) and the dominated convergence, we have

$$\widehat{k}_\varepsilon(\xi) \to m(\xi), \quad \text{for every } \xi \neq 0.$$

Then since $T_\varepsilon f \to Tf$ in L^2 as $\varepsilon \to 0$, we have $\widehat{T_\varepsilon f} = \widehat{k}_\varepsilon \widehat{f} \to \widehat{Tf}$ in L^2, and since $\widehat{k}_\varepsilon \widehat{f} \to m\widehat{f}$ pointwise a.e. as $\varepsilon \to 0$, (iii) follows at once. $\square$

Example 7.21 Returning to the Newtonian potential of a distribution $f \in L^2(\mathbb{R}^n)$

$$V(x) := \int_{\mathbb{R}^n} \Gamma(x - y) f(y) dy,$$

where

$$\Gamma(x-y) := \begin{cases} \dfrac{1}{n(2-n)\omega_n} |x-y|^{2-n} & \text{if } n > 2 \\ \dfrac{1}{2\pi} \log|x-y| & \text{if } n = 2 \end{cases}$$

for which we have $\Delta V = f$, a consequence of Theorem 7.20 is that V has second derivatives bounded in L^2 by $c\|f\|_{L^2}$.

7.2.4 Calderón-Zygmund theorem

In this section we study the action of singular integrals on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. We shall show that Calderón-Zygmund operators are of weak-$(1,1)$ type; this, together with the L^2-theory in section 7.2.3 and Marcinkiewicz interpolation theorem, will allow us to prove the celebrated Calderón-Zygmund inequality

$$\|Tf\|_{L^p(\mathbb{R}^n)} \leq A_p \|f\|_{L^p(\mathbb{R}^n)} \tag{7.22}$$

in the range $1 < p \leq 2$, and finally, by duality, in the full range $1 < p < \infty$. Classical examples show that (7.22) does not hold for $p = 1$ or $p = \infty$, even in the case of the second derivatives of a Newtonian potential.

More precisely we shall prove

Theorem 7.22 *Let $k(x)$ be a Calderón-Zygmund kernel with Lipschitz continuous restriction on $\Sigma_1 := \partial B_1(0)$. Then we have*

(i) *Suppose $f \in L^1(\mathbb{R}^n)$. Then for all $\varepsilon > 0$, T_ε is of weak type $(1,1)$ uniformly in ε, i.e.*

$$\left|\left\{x \in \mathbb{R}^n : |T_\varepsilon f(x)| > t\right\}\right| \leq \frac{c}{t}\|f\|_{L^1(\mathbb{R}^n)}, \quad \forall t > 0, \tag{7.23}$$

where c is a constant independent of ε and f.

(ii) *Suppose $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$. Then $T_\varepsilon f \in L^p(\mathbb{R}^n)$ and*

$$\|T_\varepsilon f\|_{L^p(\mathbb{R}^n)} \leq A_p \|f\|_{L^p(\mathbb{R}^n)}. \tag{7.24}$$

(iii) *If $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, then the limit of $T_\varepsilon f$ as $\varepsilon \to 0$ exists in the sense of $L^p(\mathbb{R}^n)$, and*

$$\|Tf\|_{L^p(\mathbb{R}^n)} \leq A_p \|f\|_{L^p(\mathbb{R}^n)}, \tag{7.25}$$

where A_p is independent of f.

In fact it is possible to prove pointwise convergence of $T_\varepsilon f(x)$ to $Tf(x)$ for almost every x, suitably controlling the *maximal singular integral*

$$T^* f(x) := \sup_{\varepsilon > 0} |T_\varepsilon f(x)|,$$

but we shall not do that. Under the same assumptions of Theorem 7.22 the following holds.

Theorem 7.23 *We have*

(i) *For* $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$

$$T_\varepsilon f(x) \to Tf(x) \qquad \text{as } \varepsilon \to \infty$$

pointwise for a.e. x.

(ii) *If* $f \in L^1(\mathbb{R}^n)$, *then the mapping* $f \mapsto T^* f$ *is of weak type (1,1).*

(iii) *If* $1 < p < \infty$, *then*

$$\|T^* f\|_{L^p(\mathbb{R}^n)} \le A_p \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof of Theorem 7.22. The key point is the weak estimate (7.23) and the key idea is to split $f \in L^1(\mathbb{R}^n)$ in a good part g and in a bad remainder b, $f = g + b$, where g is obtained as follows. Given $t > 0$ cover $\mathbb{R}^n$ with congruent disjoint cubes Q_i^0, $i \in \mathbb{N}$, such that $⨍_{Q_i^0} |f|dx < t$ for every i (this is possible if the cubes Q_i^0 are chosen to be large enough). Applying the Calderón-Zygmund argument to every such cube, we can find a denumerable family of dyadic cubes $\{Q_j : j \in J\}$ with interiors mutually disjoint such that

$$\mathbb{R}^n = F \cup \bigcup_{j \in J} Q_j, \qquad |f(x)| \le t \quad \forall x \in F,$$

$$\sum_{j \in J} |Q_j| \le \frac{1}{t} \int_{\mathbb{R}^n} |f(x)|dx, \qquad ⨍_{Q_j} |f|dx \le 2^n t, \quad \text{for } j \in J.$$

We then set

$$g(x) := \begin{cases} f(x) & \text{if } x \in F \\ ⨍_{Q_j} |f|dx & \text{if } x \in Q_j \text{ for some } j \in J \end{cases}$$

and $b := f - g$. Trivially

$$b(x) = 0 \quad \text{for } x \in F, \quad \text{and} \quad \int_{Q_j} b(x)dx = 0.$$

Now since $T_\varepsilon f = T_\varepsilon g + T_\varepsilon b$, it follows that

$$\begin{aligned} \left|\{x \in \mathbb{R}^n : |T_\varepsilon f(x)| > t\}\right| \le & \left|\{x \in \mathbb{R}^n : |T_\varepsilon g(x)| > t/2\}\right| \\ & + \left|\{x \in \mathbb{R}^n : |T_\varepsilon b(x)| > t/2\}\right|, \end{aligned}$$

and it suffices to establish separately for both terms of the right side inequalities like (7.23).

Estimate for $T_\varepsilon g$. Trivially $g \in L^2(\mathbb{R}^n)$ and

$$\begin{aligned}\int_{\mathbb{R}^n} |g|^2 dx = \int_F |g|^2 dx + \int_{\cup_{j\in J} Q_j} |g|^2 dx \quad &\leq \quad t\int_F |g|dx + c_1^2 t^2 \sum_{j\in J} |Q_j| \\ &\leq \quad c_2 t \int_{\mathbb{R}^n} |f|dx.\end{aligned}$$

The L^2-theory then yields

$$\|T_\varepsilon g\|_{L^2} \leq A_2 \|g\|_{L^2},$$

consequently

$$\big|\big\{x \in \mathbb{R}^n : |T_\varepsilon g(x)| > t\big\}\big| \leq \frac{A_2^2}{t^2}\int_{\mathbb{R}^n} |g|^2 dx \leq \frac{c_3}{t}\int_{\mathbb{R}^n} |f|dx. \tag{7.26}$$

Estimate for $T_\varepsilon b$. For each cube Q_j we consider the cube Q_j^* which has the same center $y^{(j)}$, but which is expanded $2\sqrt{n}$ times. Set

$$F^* := \mathbb{R}^n \setminus \bigcup_{j\in J} Q_j^*, \qquad \Omega^* := \bigcup_{j\in J} Q_j^*.$$

Of course $F^* \subset F$, and

$$|\Omega^*| \leq (2\sqrt{n})^n \sum_{j\in J} |Q_j| \leq \frac{c_5}{t}\int_{\mathbb{R}^n} |f|dx; \tag{7.27}$$

notice also that

$$|x - y^{(j)}| \geq \sqrt{n}\,\mathrm{side}(Q_j) \geq |y - y^{(j)}| \qquad \text{if } x \notin Q_j^*,\ y \in Q_j. \tag{7.28}$$

Thanks to (7.27), we only need to estimate $\big|\big\{x \in F^* : |T_\varepsilon b(x)| > t\big\}\big|$ in order to complete the proof of (7.23). As a first step in this direction we claim that for all $\varepsilon > 0$ and $x \in F^*$ we have

$$|T_\varepsilon b(x)| \leq \sum_{j\in J} \int_{Q_j} \big|k(x-y) - k(x - y^{(j)})\big||b(y)|dy + cMb(x), \tag{7.29}$$

where Mb is the maximal function of b. In fact we have

$$T_\varepsilon b(x) = \sum_{j\in J} \int_{Q_j} k_\varepsilon(x-y) b(y) dy.$$

Fix $x \in F^*$ and $\varepsilon > 0$; then the cubes Q_j fall into the following three classes:

(a) $Q_j \subset B_\varepsilon(x)$. In this case $k_\varepsilon(x-y) = 0$ for every $y \in Q_j$, hence

$$\int_{Q_j} k_\varepsilon(x-y) b(y) dy = 0.$$

(b) $Q_j \subset \mathbb{R}^n \backslash B_\varepsilon(x)$. In this case

$$\int_{Q_j} k_\varepsilon(x-y)b(y)dy = \int_{Q_j} \big[k(x-y) - k(x-y^{(j)})\big]b(y)dy,$$

being $\int_{Q_j} b(y)dy = 0$. This term is bounded by

$$\int_{Q_j} \big|k(x-y) - k(x-y^{(j)})\big||b(y)|dy,$$

which is the expression appearing in (7.29).

(c) Q_j has a non-empty intersection with $B_\varepsilon(x)$ and $\mathbb{R}^n \backslash B_\varepsilon(x)$. In this case $Q_j \subset B_{2\varepsilon}(x)$ and

$$\begin{aligned}\left|\int_{Q_j} k_\varepsilon(x-y)b(y)dy\right| &= \left|\int_{Q_j \cap (\mathbb{R}^n \backslash B_\varepsilon(x))} k(x-y)b(y)dy\right| \\ &\le \frac{c_6}{\varepsilon^n} \int_{B_{2\varepsilon}(x) \cap Q_j} |b(y)|dy.\end{aligned}$$

If we add over all cubes Q_j, we then get

$$|T_\varepsilon b(x)| \le \sum_{j \in J} \int_{Q_j} \big|k(x-y) - k(x-y^{(j)})\big||b(y)|dy + c \fint_{B_{2\varepsilon}(x)} |b(y)|dy,$$

which yields at once (7.29). Inequality (7.29) can be written as

$$|T_\varepsilon b(x)| \le \Sigma(x) + cMb(x), \qquad x \in F^*,$$

hence

$$\begin{aligned}\big|\{x \in F^* : |T_\varepsilon b(x)| > t/2\}\big| \le \big|&\{x \in F^* : |\Sigma(x)| > t/4\} \\ &+ \big|\{x \in F^* : cMb(x) > t/4\}\big|.\end{aligned} \tag{7.30}$$

The maximal theorem then yields

$$\begin{aligned}\big|\{x \in F^* : cMb(x) > t/4\}\big| \;&\le\; \frac{c_7}{t}\int_{\mathbb{R}^n} |b(y)|dy \\ &\le\; \frac{c_7}{t}\int_{\mathbb{R}^n} |f|dx + \frac{c_7}{t}\int_{\mathbb{R}^n} |g|dx \\ &\le\; \frac{c_8}{t}\int_{\mathbb{R}^n} |f|dx.\end{aligned}$$

To estimate the first term on the right-hand side of (7.30) we integrate $\Sigma(x)$ over F^* to get

$$\int_{F^*} |\Sigma(x)|dx \le \sum_{j \in J} \int_{x \notin Q_j^*} \int_{y \in Q_j} \big|k(x-y) - k(x-y^{(j)})\big||b(y)|dydx,$$

but on account of (7.28) and by Lemma 7.18, for $y \in Q_j$ and writing $x' = x - y^{(j)}$, $y' = y - y^{(j)}$,

$$\begin{aligned}\int_{x \notin Q_j^*} \big|k(x-y) - k(x-y^{(j)})\big| dx &\le \int_{|x'| \ge 2|y'|} \big|k(x'-y') - k(x')\big| dx' \\ &\le c_9 \int_{|x'| \ge 2|y'|} \frac{|y'|}{|x'|^{n+1}} dx' \le c_{10};\end{aligned}$$

therefore with Fubini's theorem

$$\int_{F^*} |\Sigma(x)| dx \le c_{11} \sum_{j \in J} \int_{Q_j} |b(y)| dy \le c_{12} \|f\|_{L^1},$$

which yields

$$\big|\{x \in F^* : |\Sigma(x)| > t/4\}\big| \le \frac{c}{t} \int_{\mathbb{R}^n} |f| dx.$$

This concludes the proof of (i).

The L^p-inequalities. Being T_ε of weak type $(1,1)$ and $(2,2)$ for all $\varepsilon > 0$ with bounds independent of ε, Marcinkiewicz's interpolation theorem implies that T_ε is of strong type (p,p) for $1 < p < 2$ and

$$\|T_\varepsilon f\|_{L^p(\mathbb{R}^n)} \le A_p \|f\|_{L^p(\mathbb{R}^n)}$$

with A_p independent of ε. To conclude the proof of Theorem 7.22 (ii) it remains to consider the case $2 < p < \infty$.
Let $f \in L^p(\mathbb{R}^n)$, $2 < p < \infty$, and actually $f \in L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, which suffices by density. Consider for any smooth map φ with $\|\varphi\|_{L^{p'}} \le 1$ the integral

$$\begin{aligned}\int_{\mathbb{R}^n} (T_\varepsilon f)\varphi dx &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k_\varepsilon(x-y) f(y)\varphi(x) dy dx \\ &= \int_{\mathbb{R}^n} f(y) \Big(\int_{\mathbb{R}^n} k_\varepsilon(x-y)\varphi(x) dx\Big) dy.\end{aligned}$$

The integral $\int_{\mathbb{R}^n} k_\varepsilon(x-y)\varphi(x)dx$ is the ε-approximation of a singular integral with kernel $k(-x)$, and $\varphi \in L^{p'}$, $1 < p' < 2$, therefore it belongs to $L^{p'}$, and its $L^{p'}$ norm is bounded by $A_{p'}\|\varphi\|_{p'} \le A_{p'}$ (notice that $A_{p'}$ is the same for $k(x)$ and $k(-x)$.) Hölder's inequality then gives

$$\Big|\int (T_\varepsilon f)\varphi dx\Big| \le A_{p'} \|f\|_{L^p}$$

and, taking the supremum on all φ indicated above, we get

$$\|Tf\|_{L^p} \le A_{p'} \|f\|_{L^p}, \qquad 2 < p < \infty.$$

This concludes the proof of Theorem 7.22 (ii).

The convergence in L^p follows as in the case $p = 2$, compare Theorem 7.20 (ii). □

7.3 Fractional integrals and Sobolev inequalities

In this section we shortly discuss fractional integrals

$$\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy, \qquad 0<\alpha<n.$$

It is common to normalize such integrals as

$$I_\alpha(f)(x) := \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy,$$

where

$$\gamma(\alpha) := \frac{\pi^{\frac{n}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n}{2}-\frac{\alpha}{2}\right)},$$

Γ being the Euler function (9.38), and call $I_\alpha(f)$ the *Riesz potential of order α of f.* Of course the Riesz potential is the convolution

$$I_\alpha(f)(x) = I_\alpha * f(x)$$

where I_α denotes the kernel

$$I_\alpha(x) := \gamma(\alpha)^{-1}|x|^{-n+\alpha}.$$

Proposition 7.24 *Let $0<\alpha<n$ and let $f \in L^p(\mathbb{R}^n)$, $1 \le p < \frac{n}{\alpha}$. Then the Riesz potential $I_\alpha(f)(x)$ is a.e. well defined.*

Proof. Write $k(x) = |x|^{-n+\alpha}$; it suffices to consider $k*f$. We decompose k as $k = k_1 + k_\infty$ where

$$k_1(x) = \begin{cases} k(x) & \text{if } |x| \le \mu \\ 0 & \text{if } |x| > \mu \end{cases} \qquad k_\infty(x) = \begin{cases} 0 & \text{if } |x| \le \mu \\ k(x) & \text{if } |x| > \mu. \end{cases}$$

where μ is any positive constant. Trivially $k*f = k_1*f + k_\infty*f$. The integral $k_1*f(x)$ is well defined for *almost every* x, since it is the convolution of the function $k_1 \in L^1(\mathbb{R}^n)$ with the function $f \in L^p(\mathbb{R}^n)$, while the integral $k_\infty * f$ is well defined for *all* x as it is the convolution of the function $f \in L^p(\mathbb{R}^n)$ with the function k_∞ which is easily seen to belong to $L^{p'}(\mathbb{R}^n)$ on account of the condition $p < \frac{n}{\alpha}$. □

Next we ask for what good pairs (p,q) is the operator $f \mapsto I_\alpha(f)$ bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$, i.e. we have the inequality

$$\|I_\alpha(f)\|_{L^q} \le A\|f\|_{L^p} \quad \text{for every } f \in L^p(\mathbb{R}^n). \tag{7.31}$$

Set $f_\delta(x) := f(\delta x)$. Then we have

$$\begin{aligned}\|f_\delta\|_{L^p} &= \delta^{-\frac{n}{p}}\|f\|_{L^p},\\ I_\alpha(f_\delta) &= \delta^{-\alpha}(I_\alpha(f))_\delta,\\ \|I_\alpha(f_\delta)\|_{L^q} &= \delta^{-\alpha-\frac{n}{q}}\|I_\alpha(f)\|_{L^q},\end{aligned}$$

hence it follows at once that (7.31) is possible only if

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}.$$

We shall see below that this condition is also sufficient, except for $p = 1$. In this case inequality (7.31), i.e.,

$$\|I_\alpha(f)\|_{L^{\frac{n}{n-\alpha}}} \le A\|f\|_{L^1} \tag{7.32}$$

cannot hold. In fact applying (7.32) to f_k, where

$$f_k \ge 0, \quad \int_{\mathbb{R}^n} f_k dx = 1, \quad \operatorname{supp}(f_k) \subset B_{1/k}(0), \quad f_k \rightharpoonup \delta_0 \text{ as measures},$$

and passing to the limit as $k \to \infty$, we would infer

$$\left\| \frac{1}{\gamma(\alpha)}|x|^{\alpha-n} \right\|_{L^{\frac{n}{n-\alpha}}} \le A,$$

i.e.,

$$\int_{\mathbb{R}^n} |x|^{-n} dx < \infty,$$

and this is a contradiction.

Theorem 7.25 (Hardy-Littlewood-Sobolev inequality) *Consider* $0 < \alpha < n$, $1 \le p < \frac{n}{\alpha}$, *and let* $q := \frac{np}{n-\alpha p}$.

(i) *If* $p > 1$, *then*

$$\|I_\alpha(f)\|_{L^q} \le A_{p,q}\|f\|_{L^p}.$$

(ii) *If* $p = 1$, *then the mapping* $f \to I_\alpha(f)$ *is of weak type* $(1, q)$, *i.e.*

$$\left|\{x \in \mathbb{R}^n : |I_\alpha(f)(x)| > t\}\right| \le \left(\frac{A\|f\|_{L^1}}{t}\right)^q \qquad \forall t > 0.$$

Proof. With the same notations as in the proof of Proposition 7.24, we show that the mapping $f \mapsto k * f$ is of weak type (p, q), i.e.

$$\left|\{x \in \mathbb{R}^n : |k * f(x)| > t\}\right\| \le \left(A_{p,q}\frac{\|f\|_{L^p}}{t}\right)^q, \qquad \forall f \in L^p(\mathbb{R}^n),\ t > 0. \tag{7.33}$$

For that we may assume $\|f\|_p = 1$ and estimate

$$\begin{aligned}\Big|\Big\{x \in \mathbb{R}^n : |k * f(x)| > 2t\Big\}\Big| \quad &\leq \quad \Big|\Big\{x \in \mathbb{R}^n : |k_1 * f(x)| > t\Big\}\Big| \\ &\quad + \Big|\Big\{x \in \mathbb{R}^n : |k_\infty * f(x)| > t\Big\}\Big|.\end{aligned}$$

Since $k_1 \in L^1(\mathbb{R}^n)$, $f \in L^p(\mathbb{R}^n)$, we have $k_1 * f \in L^p(\mathbb{R}^n)$ by Section 6.1.2, hence

$$\Big|\Big\{x \in \mathbb{R}^n : |k_1 * f(x)| > t\Big\}\Big| \leq \frac{\|k_1 * f\|_p^p}{t^p} \leq \frac{\|k_1\|_1^p \|f\|_p^p}{t^p} = \frac{\|k_1\|_1^p}{t^p},$$

and

$$\|k\|_1 = \int_{B_\mu(0)} |x|^{-n+\alpha} dx = c_1 \mu^\alpha.$$

On the other hand $\|k_\infty * f\|_\infty \leq \|k_\infty\|_{p'} \|f\|_p = \|k_\infty\|_{p'}$ and

$$\|k_\infty\|_{p'} = \left(\int_{\mathbb{R}^n \setminus B_\mu(0)} \left(|x|^{-n+\alpha}\right)^{p'} dx\right)^{\frac{1}{p'}} = c_2 \mu^{-\frac{n}{q}}.$$

Consequently

$$\|k_\infty\|_{p'} = t \quad \text{if} \quad c_2 \mu^{-\frac{n}{q}} = t \quad \text{or} \qquad \mu = c_3 t^{-\frac{q}{n}}.$$

For this value of μ we then have $\|k_\infty * f\|_\infty \leq t$ so that

$$\Big|\Big\{x \in \mathbb{R}^n : |k_\infty * f(x)| > t\Big\}\Big| = 0,$$

and we conclude

$$\Big|\Big\{x \in \mathbb{R}^n : |k * f(x)| > 2t\Big\}\Big| \leq \left(c_1 \frac{\mu^\alpha}{t}\right)^p \leq c_4 t^{-q} = c_4 \left(\frac{\|f\|_{L^p}}{t}\right)^q,$$

since $\|f\|_{L^p} = 1$. This of course gives (7.33). The special case for $p = 1$ now gives part (ii) of the theorem, and part (i) follows by Marcinkiewicz interpolation, Theorem 6.8. □

It is worth noticing that Riesz's kernels behave quite badly at infinity, compared with their behaviour near zero. This fact is responsible for the restrictions $p < \frac{n}{\alpha}$ in Propostion 7.24 and Theorem 7.25. For functions with compact support we can however complement the results above with the following

Proposition 7.26 *Let $0 < \alpha < n$ and let $f \in L^p(\mathbb{R}^n)$ be a function with compact support in $B_R(0)$. Then*

(i) *For $p > \frac{n}{\alpha}$ we have*

$$\|I_\alpha(f)\|_{L^\infty} \leq A_p R^{\alpha - \frac{n}{p}} \|f\|_{L^p}.$$

(ii) *For* $p = \frac{n}{\alpha}$ *we have for all* $q < \infty$

$$\|I_\alpha(f)\|_{L^q} \le A_{p,q}|B_R(0)|^{\frac{1}{q}}\|f\|_{L^p}.$$

Proof. By Hölder's inequality

$$\begin{aligned}
|I_\alpha(f)(x)| &\le \gamma(\alpha)^{-1}\left|\int_{B_R(0)}|x-y|^{-n+\alpha}f(y)dy\right| \\
&\le \gamma(\alpha)^{-1}\left(\int_{B_R(0)}|x-y|^{-(n-\alpha)p'}dx\right)^{\frac{1}{p'}}\|f\|_{L^p} \\
&\le \gamma(\alpha)^{-1}\left(\int_{B_R(0)}|y|^{-(n-\alpha)p'}dx\right)^{\frac{1}{p'}}\|f\|_{L^p} \\
&= c(n,\alpha,p)R^{\alpha-\frac{n}{p}}\|f\|_{L^p}
\end{aligned}$$

which gives (i).

For any $q < \infty$ set $r := \frac{nq}{n+\alpha q} < \frac{n}{\alpha}$. Theorem 7.25 (i) and Hölder's inequality then yield

$$\|I_\alpha(f)\|_{L^q} \le A_{r,q}\|f\|_{L^r} \le A_{r,q}\|f\|_{L^p}\|\chi_{B_R(0)}\|^{\frac{1}{r}}_{L^{\frac{p}{p-r}}} = A_{r,q}\|f\|_{L^p}|B_R(0)|^{\frac{1}{q}}.$$

□

Actually in the case $p = \frac{n}{\alpha}$ not only we have $I_\alpha(f) \in L^q(B_R(0))$ for all $q < \infty$, but also

Proposition 7.27 *There are constants* c_1 *and* c_2 *depending only on* n *and* p *such that*

$$⨍_{B_R(0)} \exp\left(c_1\frac{|I_{\frac{n}{p}}(f)(x)|^{p'}}{\|f\|_{L^p}^{p'}}\right)dx \le c_2$$

for all $f \in L^p(\mathbb{R}^n)$ *with compact support in* $B_R(0)$.

Proof. We may assume that $\|f\|_p = 1$. We have for any $\delta > 0$

$$I_\alpha(f)(x) = \int_{B_\delta(x)} f(y)|x-y|^{\alpha-n}dy + \int_{B_R(0)\setminus B_\delta(x)} f(y)|x-y|^{\alpha-n}dy.$$

We estimate the first integral on the right as

$$\begin{aligned}
\int_{B_\delta(x)}|f(y)||x-y|^{\alpha-n}dy &= \sum_{k=0}^\infty \int_{B(x,2^{-k}\delta)\setminus B(x,2^{-k-1}\delta)}|f(y)||x-y|^{\alpha-n}dy \\
&\le \sum_{k=0}^\infty \left(\frac{\delta}{2^{k+1}}\right)^{\alpha-n}\int_{B(x,2^{-k}\delta)}|f(y)|dy \\
&\le c(n)\sum_{k=0}^\infty \left(\frac{1}{2}\right)^{\alpha-n}\left(\frac{\delta}{2^k}\right)^{\alpha} ⨍_{B(x,2^{-k}\delta)}|f(y)|dy \\
&\le c\delta^\alpha Mf(x),
\end{aligned}$$

and the second integral by Hölder's inequality and the fact that $\|f\|_p = 1$:

$$\begin{aligned}\int_{B_R(0)\setminus B_\delta(x)} |f(y)||x-y|^{\alpha-n}dy &\leq \|f\|_{L^p}\left(\int_{B_R(0)\setminus B_\delta(x)} |x-y|^{(\alpha-n)p'}dy\right)^{\frac{1}{p'}} \\ &\leq \left(|\Sigma_1|\int_\delta^{2R} r^{(\alpha-n)p'+n-1}dr\right)^{\frac{1}{p'}} \\ &\leq \left(|\Sigma_1|\log\frac{2R}{\delta}\right)^{\frac{1}{p'}},\end{aligned}$$

where $\Sigma_1 := \partial B_1(0)$. Thus

$$|I_\alpha(f)(x)| \leq c\delta^\alpha Mf(x) + \left(|\Sigma_1|\log\frac{2R}{\delta}\right)^{\frac{1}{p'}}.$$

If we choose

$$\delta^\alpha := \min\left\{\frac{\varepsilon}{cMf(x)}, (2R)^\alpha\right\},$$

then

$$\begin{aligned}|I_\alpha(f)(x)| &\leq \varepsilon + \left[|\Sigma_1|\max\left(0, \log\left(2R\varepsilon^{-\frac{1}{\alpha}}c^{\frac{1}{\alpha}}(Mf(x))^{\frac{1}{\alpha}}\right)\right)\right]^{\frac{1}{p'}} \\ &= \varepsilon + \left[\frac{|\Sigma_1|}{n}\max\left(0, \log\left((2R)^n\varepsilon^{-p}c^p(Mf(x))^p\right)\right)\right]^{\frac{1}{p'}}\end{aligned}$$

since $\alpha p = n$. It follows, for any $\beta > 1$,

$$|I_\alpha(f)(x)|^{p'} \leq c(\beta)\varepsilon^{p'} + \beta\frac{|\Sigma_1|}{n}\max\left(0, \log\left((2R)^n\varepsilon^{-p}c^pMf(x)^p\right)\right),$$

i.e.

$$\exp\left(\frac{1}{\beta}\frac{n}{|\Sigma_1|}\left|\frac{I_\alpha(f)(x)}{\|f\|_p}\right|^{p'}\right) \leq \exp\left(c\varepsilon^{p'}\right)\max\left(1, (2R)^n\varepsilon^{-p}c^pMf(x)^p\right),$$

with $c = c(\beta, n, p)$. Integrating over $B_R(0)$ the result then easily follows, since

$$\|Mf\|_p \leq c\|f\|_p = c$$

by the maximal theorem. □

Lemma 7.28 *For every function $u \in W_0^{1,1}(\mathbb{R}^n)$ (or equivalently $u \in W_0^{1,1}(\Omega)$ for an open set $\Omega \subset \mathbb{R}^n$) we have*

$$u(x) = \frac{1}{n\omega_n}\int_{\mathbb{R}^n}\frac{x-y}{|x-y|^n}\cdot\nabla u dy,$$

i.e. u can be represented as a Riesz potential of order 1 of its derivatives.

Proof. By density of $C_c^\infty(\mathbb{R}^n)$ in $W_0^{1,1}(\mathbb{R}^n)$ we can assume that $u \in C_c^\infty(\mathbb{R}^n)$. Write

$$\begin{aligned}\frac{1}{n\omega_n}\int_{\mathbb{R}^n}\frac{x-y}{|x-y|^n}\cdot\nabla u dy &= \frac{1}{n\omega_n}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)}\frac{x-y}{|x-y|^n}\cdot\nabla u dy\\ &\quad+\frac{1}{n\omega_n}\int_{B_\varepsilon(x)}\frac{x-y}{|x-y|^n}\cdot\nabla u dy\\ &=: (I)_\varepsilon + (II)_\varepsilon.\end{aligned}$$

Since $\frac{x-y}{|x-y|^n}$ is integrable, we have

$$\lim_{\varepsilon\to 0}(II)_\varepsilon = 0.$$

Since

$$\operatorname{div}\left(\frac{x-y}{|x-y|^n}\right) = 0,$$

integration by part gives

$$(I)_\varepsilon = \fint_{\partial B_\varepsilon(x)} u d\mathcal{H}^{n-1},$$

hence $\lim_{\varepsilon\to 0}(I)_\varepsilon = u(x)$. □

The above results then yield at once

Theorem 7.29 (Sobolev) *Let $u \in W_0^{1,p}(B_R(0))$, where $B_R(0) \subset \mathbb{R}^n$. Then there are universal constants c, c_1, c_2 and c_3, depending on n, such that*

$$\begin{aligned}&\|u\|_{L^{p^*}} \le c\|Du\|_{L^p} && \text{if } 1<p<n, \quad p^* := \tfrac{np}{n-p}\\ &\fint_{B_R(0)} \exp\left(c_1\frac{|u|}{\|Du\|_{L^n}}\right)^{\frac{n}{n-1}} dx \le c_2 && \text{if } p = n\\ &\|u\|_{L^\infty} \le c_3 R^{1-\frac{n}{p}}\|Du\|_{L^p} && \text{if } p > n.\end{aligned}$$

Actually, the case $p = n$ is due to N. Trudinger [107] and the inequality is often referred to as *Moser-Trudinger's inequality*, since the optimal constant c_1 was found by Moser [81].

Chapter 8
The regularity problem in the scalar case

In this chapter we address the problem of the regularity of minimizers of variational integrals and quasilinear elliptic equations in the scalar case. The key result is the celebrated theorem of De Giorgi [24], also known as De Giorgi-Nash theorem or De Giorgi-Nash-Moser theorem.

8.1 Existence of minimizers by direct methods

Consider the functional

$$\mathcal{F}(u) := \int_\Omega F(x, u, Du)dx,$$

where $\Omega \Subset \mathbb{R}^n$ is bounded and $F : \Omega \times \mathbb{R}^m \times \mathbb{R}^{n\times m}$ is a smooth function satisfying

(i) $F(x, u, p) \geq 0$;

(ii) F and $F_{p^i_\alpha} = \frac{\partial F}{\partial p^i_\alpha}$ are continuous,

(iii) $F(x, u, p)$ is convex with respect to p.

Then we have

Theorem 8.1 (Semicontinuity) *If*

$$F(x, u, Du) \leq \Lambda(1 + |Du|^q) \quad \textit{for some } q \in [1, \infty) \textit{ and } \Lambda > 0,$$

then the functional $\mathcal{F}$ *is* weakly lower semicontinuous *in* $W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$, *i.e. if* $u_k \rightharpoonup u$ *in* $W^{1,q}(\Omega_0,\mathbb{R}^m)$ *for any compact subset* $\Omega_0 \Subset \Omega$, *then*

$$\mathcal{F}(u) \le \liminf_{k\to\infty} \mathcal{F}(u_k).$$

Proof. Since the convergence in $W^{1,q}_{\mathrm{loc}}$ implies the convergence in $W^{1,1}_{\mathrm{loc}}$, we can consider the case $q=1$. Up to extracting a subsequence, we may assume that $u_k \to u$ a.e. and $u_k \to u$ in $L^1_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$ (Rellich's theorem).

Fix $\Omega_0 \Subset \Omega$ and $\varepsilon > 0$. Then, by Egorov's and Lusin's theorems, and by the absolute continuity of Lebesgue integral, there exists a compact subset $K \subset \Omega_0$ such that $\mathrm{meas}(\Omega_0 \backslash K) < \varepsilon$ and

1. $u_k \to u$ uniformly in K,
2. $u\big|_K$ and $Du\big|_K$ are continuous,
3. $\int_K F(x,u,Du)dx \ge \int_{\Omega_0} F(x,u,Du)dx - \varepsilon$.

By convexity of F in p, setting $\mathcal{F}(u_k,K) := \int_K F(x,u_k,Du_k)dx$, we find

$$\begin{aligned}
\mathcal{F}(u_k,K) &\ge \int_K F(x,u_k,Du)dx + \int_K F_{p^i_\alpha}(x,u_k,Du)(D_\alpha u^i_k - D_\alpha u^i)dx \\
&= \int_K F(x,u_k,Du)dx + \int_K F_{p^i_\alpha}(x,u,Du)(D_\alpha u^i_k - D_\alpha u^i)dx \\
&\quad + \int_K \big[F_{p^i_\alpha}(x,u_k,Du) - F_{p^i_\alpha}(x,u,Du)\big](D_\alpha u^i_k - D_\alpha u^i)dx.
\end{aligned}$$

Taking the limit as $k \to \infty$, the second and the third integrals on the right-hand side vanish: the former because $F_{p^\alpha_i}(x,u,Du)$ is bounded on K and $D_\alpha u^i_k - D_\alpha u^i \rightharpoonup 0$ in $L^1(K)$, the latter because $F_{p^i_\alpha}(x,u_k,Du_k) - F_{p^i_\alpha}(x,u,Du_k) \to 0$ uniformly on K and $D_\alpha u^i_k - D_\alpha u^i$ is equibounded in $L^1(K)$. Finally

$$\liminf_{k\to\infty} \int_K F(x,u_k,Du)dx \ge \int_K F(x,u,Du)dx,$$

by Fatou's lemma. Therefore

$$\begin{aligned}
\liminf_{k\to\infty} \int_{\Omega_0} F(x,u_k,Du_k)dx &\ge \int_K F(x,u,Du)dx \\
&\ge \int_{\Omega_0} F(x,u,Du)dx - \varepsilon.
\end{aligned}$$

Since this is true for every $\varepsilon > 0$, we conclude

$$\liminf_{k\to\infty} \int_{\Omega_0} F(x,u_k,Du_k)dx \ge \int_{\Omega_0} F(x,u,Du)dx.$$

On the other hand, since $F(x,u(x),Du(x)) \in L^1(\Omega)$ we have

$$\int_{\Omega\setminus\Omega_0} F(x,u,Du)dx = o(1)$$

with $o(1) \to 0$ as $|\Omega \setminus \Omega_0| \to 0$, and we conclude. □

By direct methods we have

Theorem 8.2 (Existence) *In addition to the hypothesis of Theorem 8.1, assume that F is a smooth function of growth $q > 1$, i.e. there exists $A, \Lambda > 0$ such that*

$$A(|p|^q - 1) \leq F(x,u,p) \leq \Lambda(|p|^q + 1). \tag{8.1}$$

Then for every $g \in W^{1,q}(\Omega, \mathbb{R}^m)$, $\mathcal{F}$ has a minimizer in the set

$$\mathcal{A} := \big\{ u \in W^{1,q}(\Omega, \mathbb{R}^m) \bigm| u - g \in W_0^{1,q}(\Omega, \mathbb{R}^m) \big\}.$$

Proof. Take a minimizing sequence u_k. By (8.1)

$$\int_\Omega |Du_k|^q dx \leq \frac{\mathcal{F}(u_k)}{A} + \mathrm{meas}(\Omega) \leq c_1. \tag{8.2}$$

By Poincaré's inequality and (8.2)

$$\begin{aligned}
\int_\Omega |u_k|^q dx &\leq c_2 \int_\Omega |u_k - g|^q dx + c_2 \int_\Omega |g|^q dx \\
&\leq c_3 \int_\Omega |D(u_k - g)|^q dx + c_2 \int_\Omega |g|^q dx \\
&\leq c_4 \int_\Omega |Du_k|^q dx + \int_\Omega |Dg|^q dx + c_2 \int_\Omega |g|^q dx \\
&\leq c_5,
\end{aligned}$$

hence the sequence u_k is bounded in $W^{1,q}(\Omega, \mathbb{R}^m)$, which is reflexive. Therefore there exists subsequence $u_{k'}$ weakly converging to some function u in $W^{1,q}(\Omega, \mathbb{R}^m)$. By Theorem 8.1

$$\mathcal{F}(u) \leq \liminf_{k' \to \infty} \mathcal{F}(u_{k'}),$$

hence u is a minimizer. □

Let us make a few remarks concerning the convexity of F with respect to p that go back to Morrey.

We say that $u_k \to u$ in the *Lipschitz convergence* if

(i) $u_k \to u$ uniformly

(ii) the u_k's have equibounded Lipschitz norms.

Proposition 8.3 *Suppose that*

$$\mathcal{F}(u) = \int_\Omega F(x, u, Du)dx$$

(with F smooth) is lower semicontinuous with respect to the Lipschitz convergence. Then for all $D \Subset \Omega$, all $(x_0, u_0, p_0) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm}$ and all $\phi \in C^1_c(D, \mathbb{R}^m)$, we have

$$\int_D F(x_0, u_0, p_0 + D\phi)dx \geq \int_D F(x_0, u_0, p_0)dx. \tag{8.3}$$

Proof. For the sake of simplicity we let F depend only on p and D be the unit cube Q centered at 0 in $\mathbb{R}^n$. Every $\phi \in C^1_c(Q)$ extends to a periodic function in $\mathbb{R}^n$. Define $\phi_\nu(x) := \nu^{-1}\phi(\nu x)$, $\nu \in \mathbb{N}$,

$$u(x) := u_0 + p_0 \cdot x, \qquad u_\nu(x) := u_0 + p_0 \cdot x + \phi_\nu(x).$$

Then $u_\nu \to u$ in the Lipschitz convergence, thus, by the assumption,

$$|Q|F(p_0) \leq \liminf_{\nu\to\infty} \int_Q F(p_0 + D\phi_\nu)dx.$$

Now $D\phi_\nu(x) = D\phi(\nu x)$ and, by the change of variable $\nu x = y$ and using the periodicity of ϕ,

$$\begin{aligned} |Q|F(p_0) &\leq \liminf_{\nu\to\infty} \int_{\nu Q} F(p_0 + D\phi(y))\frac{dy}{\nu^n} \\ &= \liminf_{\nu\to\infty} \nu^n \int_Q F(p_0 + D\phi(y))\frac{dy}{\nu^n} \\ &= \int_Q F(p_0 + D\phi)dy, \end{aligned}$$

i.e. (8.3). □

If (8.3) holds for every $D \Subset \Omega$, $(x_0, u_0, p_0) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm}$ and $\phi \in C^\infty_c(D, \mathbb{R}^m)$, one says that F is *quasi-convex*. In presence of additional assumptions, it turns out that quasi-convexity of F is equivalent to semicontinuity of $\mathcal{F}$, as appears from the following two theorems, due respectively to Morrey and Meyers, and Acerbi-Fusco. We state them without proof.

Theorem 8.4 *Suppose that $F \geq 0$ and for some $s \geq 1$*

$$|F(p) - F(q)| \leq k(1 + |p|^{s-1} + |q|^{s-1})|p - q|.$$

Then $\mathcal{F}$ is weakly lower semicontinuous in $W^{1,s}$ if and only if F is quasi-convex.

Theorem 8.5 *Suppose that $F(x,u,p)$ is measurable in x and continuous in (u,p), and for some $s \geq 1$*

$$0 \leq F(x,u,p) \leq \lambda(1 + |u|^s + |p|^s).$$

Then $\mathcal{F}$ is weakly lower semicontinuous in $W^{1,s}$ if and only if F is quasi-convex.

Of course, by Jensen's inequality, convexity of F with respect to p implies quasi-convexity, and in fact it is equivalent to quasi-convexity in the scalar case. In turn, quasi-convexity implies *rank-one convexity*, i.e.

$$F(p + \xi \otimes \eta) \geq F(p) + A_i^\alpha \xi^i \eta_\alpha,$$

where $A_i^\alpha = F_{p_\alpha^i}$ if F is of class C^1, the converse being false. Classical examples of quasi-convex integrands are *poly-convex* integrands , i.e., convex functions of the determinant minors of the matrix Du. We shall not dwell any further with these topics; for a first reading we refer e.g. to [49] and, for further information, to the wide recent literature.

8.2 Regularity of critical points of variational integrals

Consider the variational integral defined on $W^{1,2}(\Omega, \mathbb{R}^m)$

$$\mathcal{F}(u) = \int_\Omega F(Du)dx, \tag{8.4}$$

where F is smooth, $|F(p)| \leq L|p|^2$ for some $L > 0$, and $A_i^\alpha := D_{p_\alpha^i} F$ satisfy the *growth and ellipticity conditions*

$$\begin{cases} |A_i^\alpha(p)| \leq c|p|, & |D_{p_\beta^j} A_i^\alpha(p)| \leq M \\ D_{p_\beta^j} A_i^\alpha(p) \xi_\alpha^i \xi_\beta^j \geq \lambda|\xi|^2, & \forall \xi \in \mathbb{R}^{n\times m}, \end{cases} \tag{8.5}$$

for some $\lambda, M > 0$.

Any minimizer u of $\mathcal{F}$ with respect to its boundary datum satisfies the Euler-Lagrange equation

$$\int_\Omega A_i^\alpha(Du) D_\alpha \varphi^i dx = 0, \quad \forall \varphi \in W_0^{1,2}(\Omega, \mathbb{R}^m). \tag{8.6}$$

Indeed, using a Taylor expansion and (8.5), we have for $\varphi \in W_0^{1,2}(\Omega, \mathbb{R}^m)$

$$\begin{aligned} 0 &\leq \int_\Omega \frac{F(Du + tD\varphi) - F(Du)}{t} dx \\ &= \int_\Omega \big(A_i^\alpha(Du) D_\alpha \varphi^i + O(t) M |D\varphi|^2\big) dx. \end{aligned}$$

Taking the limit as $t \to 0$, and also replacing φ with $-\varphi$, we obtain (8.6).

The associated quasilinear elliptic system

We may differentiate (8.6) using the *difference quotient* method, to get

Proposition 8.6 *Let $u \in W^{1,2}(\Omega, \mathbb{R}^m)$ be a weak solution to the the elliptic system (8.6) where A_i^α satisfy the growth condition (8.5). Then $u \in W^{2,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ and, for $1 \le s \le n$, $D_s u$ satisfies the elliptic system*

$$\int_\Omega D_{p^j_\beta} A_i^\alpha(Du) D_\beta(D_s u^j) D_\alpha \varphi^i dx = 0, \quad \forall \varphi \in W_0^{1,2}(\Omega, \mathbb{R}^m). \tag{8.7}$$

Proof. Let $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{R}^n$, and fix an integer s, $1 \le s \le n$. Given a test function $\varphi \in C_c^\infty(\Omega, \mathbb{R}^m)$, for every $h > 0$ small enough, also $\varphi(x - he_s)$ is a test function, thus

$$\int_\Omega \Big[A_i^\alpha(Du(x + he_s)) - A_i^\alpha(Du(x))\Big] D_\alpha \varphi^i(x) dx = 0.$$

For almost every $x \in \Omega$ we have

$$\begin{aligned}
&A_i^\alpha(Du(x + he_s)) - A_i^\alpha(Du(x)) \\
&= \int_0^1 \frac{d}{dt} A_i^\alpha(tDu(x + he_s) + (1 - t)Du(x))dt \\
&= \int_0^1 D_{p^j_\beta} A_i^\alpha(tDu(x + he_s) + (1 - t)Du(x)) D_\beta\big[u^j(x + he_s) - u^j(x)\big]dt.
\end{aligned}$$

Setting

$$\widetilde{A}^{\alpha\beta}_{ij(h)}(x) := \int_0^1 D_{p^j_\beta} A_i^\alpha(tDu(x + he_s) + (1 - t)Du(x))dt$$

we have

$$\int_\Omega \widetilde{A}^{\alpha\beta}_{ij(h)}(x) D_\beta \frac{u^j(x + he_s) - u^j(x)}{h} D_\alpha \varphi^i dx = 0, \tag{8.8}$$

where $\widetilde{A}^{\alpha\beta}_{ij(h)}(x)$ satisfy

$$\begin{aligned}
|\widetilde{A}^{\alpha\beta}_{ij(h)}(x)| &\le M \\
\widetilde{A}^{\alpha\beta}_{ij(h)}(x)\xi^i_\alpha \xi^j_\beta &\ge \lambda|\xi|^2.
\end{aligned} \tag{8.9}$$

Insert now the test function

$$\varphi(x) := \frac{u(x + he_s) - u(x)}{h} \eta^2$$

into (8.8), where $\eta \in C_c^\infty(B_R(x_0))$, $B_{3R}(x_0) \subset \Omega$, $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{\frac{R}{2}}(x_0)$, and $|D\eta| \le \frac{c}{R}$. From (8.9), Hölder's inequality and Proposition 4.8 (i), we get

$$\begin{aligned}\int_\Omega \Big|\frac{Du(x+he_s)-Du(x)}{h}\Big|^2 \eta^2 dx &\le c_1 \int_\Omega \Big|\frac{u(x+he_s)-u(x)}{h}\Big|^2 |D\eta|^2 dx\\ &\le \frac{c_2}{R^2}\int_\Omega |Du|^2 dx.\end{aligned}$$

Thus

$$\int_{B_{\frac{R}{2}}(x_0)} \Big|\frac{Du(x+he_s)-Du(x)}{h}\Big|^2 dx \le c_3$$

where c_3 is independent of h. By Proposition 4.8 (ii) and a covering argument, $Du \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^m)$. Passing to the limit as $h \to 0$ in (8.8) we obtain (8.7). □

The regularity of critical points of class C^1

A bootstrap procedure based on Theorem 5.17 and Schauder estimates, shows that if the first derivatives of u are continuous, then u is of class C^∞:

Theorem 8.7 *If $u \in C^1(\Omega, \mathbb{R}^m)$ is a solution of the elliptic system* (8.6) *then $u \in C^\infty(\Omega, \mathbb{R}^m)$.*

Proof. If $u \in C^1(\Omega, \mathbb{R}^m)$ then, by Proposition 8.6, Du solves

$$D_\alpha\big(A_{ij}^{\alpha\beta} D_\beta(Du)\big) = 0,$$

and the coefficients $A_{ij}^{\alpha\beta}(x) := D_{p^j_\beta} A_i^\alpha(Du(x))$ are continuous and elliptic. By Theorem 5.17 and its corollary, Du is Hölder continuous. Thus the coefficients $A_{ij}^{\alpha\beta}$ are Hölder continuous and, by Theorem 5.19, $Du \in C^{1,\sigma}(\Omega)$ for some σ, giving $A_{ij}^{\alpha\beta} \in C^{1,\sigma}(\Omega)$. Finally Theorem 5.20 yields $u \in C^\infty(\Omega, \mathbb{R}^m)$. □

The C^1-regularity of critical points

By Theorem 8.7 a critical point of the variational integral (8.4) is smooth as soon as its first derivatives are continuous. Thus we need, or better it suffices to show that the solutions $D_s u$ to the elliptic system (8.7) are continuous. This is false in general (we shall see counterexamples) but it is true in the scalar case as proved by Ennio De Giorgi in 1957 [24], and independently by John F. Nash [83].

Before stating this theorem we observe that system (8.7), in the scalar case ($m = 1$), reduces to the elliptic equation in the unknown $D_s u$

$$\int_\Omega F_{p_\alpha p_\beta}(Du) D_\beta(D_s u) D_\alpha \varphi dx = 0, \quad \forall \varphi \in W_0^{1,2}(\Omega),$$

thus $v := D_s u$ solves an equation of the type

$$D_\alpha(A^{\alpha\beta} D_\beta v) = 0,$$

where $A^{\alpha\beta} \in L^\infty$ and $A^{\alpha\beta}\xi_\alpha\xi_\beta \geq |\xi|^2$, and in this case we would like to show that v is Hölder continuous. This is exactly the claim of De Giorgi's theorem, often referred to also as De Giorgi-Nash-Moser theorem.

8.3 De Giorgi's theorem: essentially the original proof

De Giorgi's class

Definition 8.8 *Define the De Giorgi class $DG(\Omega)$ as the set of functions $u \in W^{1,2}(\Omega)$ for which there is a constant c such that for all $x_0 \in \Omega$, $0 < \rho < R < \operatorname{dist}(x_0, \partial\Omega)$*

$$\int_{A(k,\rho)} |Du|^2 dx \leq \frac{c}{(R-\rho)^2} \int_{A(k,R)} |u-k|^2 dx, \quad \forall k \in \mathbb{R}, \tag{8.10}$$

where $A(k,r) := \{x \in B_r(x_0) : u(x) > k\}$, $r > 0$.

To simplify the notation we usually don't indicate the point x_0 involved in the definition of $A(k,R)$.

This definition is motivated by the fact that, as we shall see, a solution to an elliptic equation belongs to the De Giorgi class.

Exercise 8.9 Prove that if $u, -u \in DG(\Omega)$, then u satisfies the following Caccioppoli inequality:

$$\int_{B_\rho(x_0)} |Du|^2 dx \leq \frac{c}{(R-\rho)^2} \int_{B_R(x_0)} |u-\lambda|^2 dx,$$

for every $\lambda \in \mathbb{R}$ and $B_\rho(x_0) \Subset B_R(x_0) \Subset \Omega$.
[Hint: Use Proposition 3.23.]

Exercise 8.10 Prove that if $u \in DG(\Omega)$ and $\lambda \in \mathbb{R}$, then also $u + \lambda \in DG(\Omega)$.

A subsolution of the elliptic equation

$$\begin{aligned} &D_\alpha(A^{\alpha\beta} D_\beta u) = 0, \\ &\lambda|\xi|^2 \leq A^{\alpha\beta}\xi_\alpha\xi_\beta \leq \Lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^n \end{aligned} \tag{8.11}$$

is a function $u \in W^{1,2}(\Omega)$ such that

$$\int_\Omega A^{\alpha\beta} D_\beta u D_\alpha \varphi dx \leq 0, \quad \forall \varphi \in W_0^{1,2}(\Omega), \;\; \varphi \geq 0. \tag{8.12}$$

Lemma 8.11 *Let $u \in W^{1,2}(\Omega)$ be a subsolution of* (8.11)*.*

1. *If $f \in C^2(\mathbb{R})$ is a non-negative, convex, monotone increasing function with $f' \in L^\infty(\mathbb{R})$, then also $f \circ u$ is a subsolution of* (8.11)*.*
2. *For any $k \in \mathbb{R}$, $(u-k)^+$ is a subsolution of* (8.11)

Proof. By density it is enough to prove that $f \circ u$ satisfies (8.12) for every non-negative $\varphi \in C_c^\infty(\Omega)$. For $\varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$ define the non-negative test function

$$\zeta(x) := f'(u(x))\varphi(x) \in W_0^{1,2}(\Omega).$$

We have by (8.11)

$$\begin{aligned} A^{\alpha\beta} D_\beta u D_\alpha \zeta &= A^{\alpha\beta} D_\beta (f \circ u) D_\alpha \varphi + f''(u) A^{\alpha\beta} D_\beta u D_\alpha u \varphi \\ &\geq A^{\alpha\beta} D_\beta (f \circ u) D_\alpha \varphi. \end{aligned}$$

Integrating yields

$$0 \geq \int_\Omega A^{\alpha\beta} D_\beta u D_\alpha \zeta dx \geq \int_\Omega A^{\alpha\beta} D_\beta (f \circ u) D_\alpha \varphi dx, \tag{8.13}$$

which proves the first claim.

To prove the second claim, notice first that $(u-k)^+ \in W^{1,2}(\Omega)$ by Corollary 3.25. Up to a translation we can assume $k = 0$. Set

$$f_\varepsilon(t) := \begin{cases} \sqrt{t^2 + \varepsilon^2} - \varepsilon & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

and $f(t) := \max\{t, 0\}$. Then (8.13) holds with f_ε instead of f and by Proposition 3.22 and dominated convergence

$$\begin{aligned} \int_\Omega A^{\alpha\beta} D_\beta u^+ D_\alpha \varphi dx &= \int_\Omega A^{\alpha\beta} D_\beta (f \circ u) D_\alpha \varphi dx \\ &= \int_{\{x \in \Omega : u(x) > 0\}} A^{\alpha\beta} f'(u) D_\beta u D_\alpha \varphi dx \\ &= \lim_{\varepsilon \to 0} \int_{\{x \in \Omega : u(x) > 0\}} A^{\alpha\beta} f_\varepsilon'(u) D_\beta u D_\alpha \varphi dx \\ &\leq 0. \end{aligned}$$

$\square$

Corollary 8.12 *For any subsolution (resp. supersolution) u to* (8.11), *we have $u \in DG(\Omega)$ (resp. $-u \in DG(\Omega)$).*

Proof. By Lemma 8.11, $(u-k)^+$ is a subsolution, hence it satisfies Caccioppoli's inequality (Theorem 4.4 with $f_i, F_i^\alpha = 0$; notice that the proof works for subsolutions, not only for solutions):

$$\int_{B_\rho(x_0)} |D(u-k)^+|^2 dx \le \frac{c}{(R-\rho)^2} \int_{B_R(x_0)\setminus B_\rho(x_0)} ((u-k)^+)^2 dx, \quad (8.14)$$

which implies (8.10) (and is actually a slightly stronger) thanks to Corollary 3.25.

To conclude simply observe that if u is a supersolution, then $-u$ is a subsolution. □

The theorem and its proof

Theorem 8.13 (De Giorgi) *If $u, -u \in DG(\Omega)$, then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0,1)$. Moreover letting $\omega(R)$ denotes the oscillation of u in a $B_R(x_0) \Subset \Omega$ and $0 < \rho < R$, then we have*

$$\omega(\rho) \le c\Big(\frac{\rho}{R}\Big)^\alpha \omega(R), \quad (8.15)$$

and

$$\sup_{B_{\frac{R}{2}}(x_0)} |u| \le c\Big(\fint_{B_R(x_0)} |u|^2 dx\Big)^{\frac{1}{2}}. \quad (8.16)$$

for some constant $c > 0$ independent of x_0, ρ and R.

In particular if $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak solution to the elliptic equation with L^∞ coefficients

$$D_\alpha(A^{\alpha\beta} D_\beta u) = 0,$$

$$\lambda|\xi|^2 \le A^{\alpha\beta}\xi_\alpha\xi_\beta \le \Lambda|\xi|^2,$$

then, by Corollary 8.12, u and $-u$ belong to $DG(\Omega)$ so that u is Hölder continuous. A consequence of (8.15) and (8.16) we have, when $u, -u \in DG(\Omega)$

$$\int_{B_\rho(x_0)} |u - u_{x_0,\rho}|^2 dx \le c\Big(\frac{\rho}{R}\Big)^{n+2\alpha} \int_{B_R(x_0)} |u - u_{x_0,R}|^2 dx \quad (8.17)$$

for $B_\rho(x_0) \Subset B_R(x_0) \Subset \Omega$. Indeed, assuming $R \ge 2\rho$ (otherwise (8.17) is obvious)

$$
\begin{aligned}
\int_{B_\rho(x_0)} |u - u_{x_0,\rho}|^2 dx &\le c_1 \rho^n (\omega(\rho))^2 \\
&\le c_2 \rho^n \Big(\frac{\rho}{R}\Big)^{2\alpha} (\omega(R/2))^2 \\
&\le c_3 \rho^n \Big(\frac{\rho}{R}\Big)^{2\alpha} \sup_{B_{R/2}(x_0)} |u - u_{x_0,R}|^2 \\
&\le c_4 \Big(\frac{\rho}{R}\Big)^{n+2\alpha} \int_{B_R(x_0)} |u - u_{x_0,R}|^2 dx,
\end{aligned}
$$

where in the last inequality we applied (8.16) to $u - u_{x_0,R}$ (compare Exercise 8.10).

Together with the Caccioppoli and Poincaré inequalities (see Exercise 8.9 and Proposition 3.12), from (8.17) we also infer

$$
\begin{aligned}
\int_{B_\rho(x_0)} |Du|^2 dx &\le \frac{c}{\rho^2} \int_{B_{2\rho}(x_0)} |u - u_{x_0,2\rho}| dx \\
&\le \frac{c'}{\rho^2} \Big(\frac{\rho}{R}\Big)^{n+2\alpha} \int_{B_R(x_0)} |u - u_{x_0,R}| dx \\
&\le c'' \Big(\frac{\rho}{R}\Big)^{n-2+2\alpha} \int_{B_R(x_0)} |Du|^2 dx.
\end{aligned}
\tag{8.18}
$$

In the following we will assume $n \ge 3$. The case $n = 2$ essentially follows by the hole-filling technique of Widman (Section 4.4), by slightly modifying the definition of $DG(\Omega)$ (the integral on the right-hand side of (8.10) should be taken, for instance, over $A(k,R) \setminus A(k,\rho)$, as in (8.14)) and noticing that it implies (4.23).

Lemma 8.14 *If* $u \in DG(\Omega)$, $0 < \rho < R \le \operatorname{dist}(x_0, \partial\Omega)$, *then*

$$
\int_{A(k,\rho)} |u - k|^2 dx \le \frac{C}{(R-\rho)^2} \int_{A(k,R)} |u - k|^2 dx \cdot |A(k,R)|^{\frac{2}{n}}, \quad \forall k \in \mathbb{R}, \tag{8.19}
$$

for some constant $C > 0$ *independent of* x_0, ρ, R, k.

Proof. By (8.10) we have, for $(u-k)^+ := \max(0, u-k)$,

$$
\int_{B_{\frac{R+\rho}{2}}(x_0)} |D(u-k)^+|^2 dx \le \frac{4c_1}{(R-\rho)^2} \int_{B_R(x_0)} ((u-k)^+)^2 dx.
$$

Choose a cut-off function $\eta \in C_c^\infty(B_{\frac{R+\rho}{2}}(x_0))$ such that

$$
\eta \equiv 1 \text{ on } B_\rho(x_0), \quad 0 \le \eta \le 1 \quad \text{and} \quad |D\eta| \le \frac{4}{R-\rho}.
$$

Then expanding $|D(\eta(u-k)^+)|^2$ we infer

$$\begin{aligned}\int_{B_{\frac{R+\rho}{2}}(x_0)} |D(\eta(u-k)^+)|^2 dx &\leq \int_{B_{\frac{R+\rho}{2}}(x_0)} \eta^2 |D((u-k)^+)|^2 dx \\ &\quad + \frac{c_2}{(R-\rho)^2}\int_{B_{\frac{R+\rho}{2}}} ((u-k)^+)^2 dx \\ &\leq \frac{c_3}{(R-\rho)^2}\int_{B_R(x_0)} ((u-k)^+)^2 dx.\end{aligned}$$

Using Sobolev's inequality with vanishing boundary value, Theorem 7.29, we then bound

$$\begin{aligned}\left(\int_{B_\rho(x_0)} |(u-k)^+|^{2^*} dx\right)^{\frac{2}{2^*}} &\leq \left(\int_{B_{\frac{R+\rho}{2}}(x_0)} |\eta(u-k)^+|^{2^*} dx\right)^{\frac{2}{2^*}} \\ &\leq c_4 \int_{B_{\frac{R+\rho}{2}}(x_0)} |D(\eta(u-k)^+)|^2 dx \\ &\leq \frac{c_5}{(R-\rho)^2}\int_{B_R(x_0)} |(u-k)^+|^2 dx,\end{aligned}$$

where $c_4 = c_4(n)$ does not depend on ρ, as a simple scaling argument implies. By Hölder's inequality

$$\begin{aligned}\int_{B_\rho(x_0)} |(u-k)^+|^2 dx &= \int_{A(k,\rho)} |u-k|^2 dx \\ &\leq \left(\int_{A(k,\rho)} |u-k|^{2^*} dx\right)^{\frac{2}{2^*}} |A(k,\rho)|^{1-\frac{2}{2^*}},\end{aligned}$$

which yields (8.19), since $1-\frac{2}{2^*} = \frac{2}{n}$. □

Proposition 8.15 *If $u \in DG(\Omega)$ there exists a constant $c > 0$ such that, for $0 < R \leq \operatorname{dist}(x_0, \partial\Omega)$, we have*

$$\sup_{B_{\frac{R}{2}}(x_0)} u \leq k + c\left(\frac{1}{R^n}\int_{A(k,R)} |u-k|^2 dx\right)^{\frac{1}{2}} \left(\frac{|A(k,R)|}{R^n}\right)^{\frac{\theta-1}{2}}, \quad \forall k \in \mathbb{R}, \tag{8.20}$$

for some $\theta(n) > 1$ to be defined.

Proof. Step 1. Since, for $h > k$ and $0 < \rho \leq R$ we have $A(h,\rho) \subset A(k,R)$ and, for $x \in A(h,\rho)$, $(h-k) < (u(x)-k)$, we see that

$$|h-k|^2 |A(h,\rho)| = \int_{A(h,\rho)} |h-k|^2 dx \leq \int_{A(k,R)} |u-k|^2 dx. \tag{8.21}$$

Define

$$u(h,\rho) := \int_{A(h,\rho)} |u-h|^2 dx \le \int_{A(k,\rho)} |u-k|^2 dx =: u(k,\rho),$$

so that (8.19) and (8.21) become

$$\begin{aligned} u(h,\rho) &\le \frac{C}{(R-\rho)^2} u(k,R)|A(k,R)|^{\frac{2}{n}}, \\ |A(h,\rho)| &\le \frac{1}{(h-k)^2} u(k,R). \end{aligned}$$

It follows that, for $\xi > 0$ we have

$$u(h,\rho)^\xi |A(h,\rho)| \le \frac{C^\xi}{(R-\rho)^{2\xi}} \frac{1}{(h-k)^2} u(k,R)^{\xi+1} |A(k,R)|^{\frac{2\xi}{n}}. \tag{8.22}$$

Set

$$\xi = \frac{n}{2}\theta, \quad \text{where} \quad \theta = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2}{n}} > 1$$

is the positive solution of $\theta^2 - \theta - \frac{2}{n} = 0$. Then

$$\xi + 1 = \theta\xi, \quad \text{and} \quad \frac{2\xi}{n} = \theta.$$

If we define

$$\Phi(h,\rho) := u(h,\rho)^\xi |A(h,\rho)|,$$

then (8.22) becomes

$$\Phi(h,\rho) \le \frac{C^\xi}{(R-\rho)^{2\xi}} \frac{1}{(h-k)^2} \Phi(k,R)^\theta, \quad \text{for } 0 < \rho < R, \ h > k. \tag{8.23}$$

Step 2. We claim that, for any $k \in \mathbb{R}$, we have

$$\Phi(k+d, R/2) = 0, \tag{8.24}$$

for

$$d := \left(R^{-2\xi} 2^{(2\xi+2)\frac{\theta}{\theta-1}} C^\xi \Phi(k,R)^{\theta-1} \right)^{\frac{1}{2}}.$$

To see this, we set for $\ell \in \mathbb{N}$

$$\begin{aligned} k_\ell &= k + d - \frac{d}{2^\ell} \\ \rho_\ell &= \frac{R}{2} + \frac{R}{2^\ell}. \end{aligned}$$

Then (8.23) gives

$$\begin{aligned}\Phi(k_{\ell+1},\rho_{\ell+1}) &\leq \frac{C^\xi}{(\rho_{\ell+1}-\rho_\ell)^{2\xi}(k_{\ell+1}-k_\ell)^2}\Phi(k_\ell,\rho_\ell)^\theta\\ &= \Phi(k_\ell,\rho_\ell)\left(\Phi(k_\ell,\rho_\ell)^{\theta-1}C^\xi\frac{2^{(\ell+1)(2\xi+2)}}{R^{2\xi}d^2}\right),\end{aligned}$$

which setting

$$\psi_\ell := 2^{\mu\ell}\Phi(k_\ell,\rho_\ell), \quad \mu := \frac{2\xi+2}{\theta-1}$$

becomes

$$\psi_{\ell+1} \leq \psi_\ell\left(\psi_\ell^{\theta-1}2^{(2\xi+2)\frac{\theta}{\theta-1}}C^\xi R^{-2\xi}d^{-2}\right).$$

But with our choice of d we have

$$\psi_0^{\theta-1}2^{(2\xi+2)\frac{\theta}{\theta-1}}C^\xi R^{-2\xi}d^{-2} = 1,$$

hence by induction we verify $\psi_\ell \leq \psi_0$ for every $\ell \in \mathbb{N}$, i.e.

$$\Phi(k_\ell,\rho_\ell) \leq \frac{\Phi(k_0,R)}{2^{\mu\ell}}.$$

Letting $\ell \to \infty$ we get (8.24).

Step 3. Now (8.24) implies that either $u(k+d,R/2) = 0$, or $|A(k+d,R/2)| = 0$. In both cases

$$\sup_{B_{\frac{R}{2}}(x_0)} u \leq k+d.$$

Since d is (up to choosing the proper constant c) the second addend on the right side of (8.20), the proof is complete. □

Corollary 8.16 *If $u, -u \in DG(\Omega)$, then there is a constant $c > 0$ such that*

$$\sup_{B_{\frac{R}{2}}(x_0)} |u| \leq c\Big(\fint_{B_R(x_0)} |u|^2dx\Big)^{\frac{1}{2}}, \tag{8.25}$$

for every $x_0 \in \Omega$, $0 < R \leq \mathrm{dist}(x_0,\partial\Omega)$.

Proof. Being $u \in DG(\Omega)$ (8.20) holds; choosing $k = 0$ and observing that $\frac{|A(k,R)|}{R^n} \leq c_1$, we obtain

$$\sup_{B_{\frac{R}{2}}(x_0)} u^+ \leq c\Big(\fint_{B_R(x_0)} |u^+|^2dx\Big)^{\frac{1}{2}}. \tag{8.26}$$

Similarly from $-u \in DG(\Omega)$ we have

$$\sup_{B_{\frac{R}{2}}(x_0)} u^- \leq c\Big(\fint_{B_R(x_0)} |u^-|^2 dx \Big)^{\frac{1}{2}}. \tag{8.27}$$

Now $\sup|u| = \max\{\sup u^+, \sup u^-\}$ and we conclude. $\square$

Proposition 8.17 *Let $u \in DG(\Omega)$ and, for $0 < \rho \leq R \leq \operatorname{dist}(x_0, \partial\Omega)$, set*

$$M(\rho) := \sup_{B_\rho(x_0)} u, \quad m(\rho) := \inf_{B_\rho(x_0)} u, \quad k_0 := \frac{M(2R) - m(2R)}{2}. \tag{8.28}$$

Assume that

$$|A(k_0, R)| \leq \frac{1}{2}|B_R(x_0)|.$$

Then there is a monotone increasing function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ (independent of x_0) with $\lim_{t\to 0^+} \zeta(t) = 0$, such that

$$|A(h, R)| \leq \zeta(M(2R) - h), \quad h < M(2R).$$

In particular

$$\lim_{h\to M(2R)} |A(h, R)| = 0.$$

Proof. For $h > k > k_0$ define $v(x) := \min\{u, h\} - \min\{u, k\}$. Then

$$\begin{aligned} \big|\{x \in B_R(x_0) : v(x) = 0\}\big| &= |B_R(x_0)\backslash A(k, R)| \\ &\geq |B_R(x_0)\backslash A(k_0, R)| \\ &\geq \frac{|B_R(x_0)|}{2} \end{aligned}$$

This and the Poincaré inequality (3.4) imply

$$\frac{c}{R}\int_{B_R(x_0)} |v|dx \leq \int_{B_R(x_0)} |Dv|dx \tag{8.29}$$

thus

$$\|v\|_{W^{1,1}(B_R(x_0))} \leq c_1 \int_{B_R(x_0)} |Dv|dx$$

and the Sobolev embedding theorem gives

$$\Big(\int_{B_R(x_0)} |v|^{1^*} dx\Big)^{\frac{1}{1^*}} \leq c_2 \int_{B_R(x_0)} |Dv|dx, \quad 1^* = \frac{n}{n-1}$$

which implies

$$
\begin{aligned}
(h-k)^{1^*}|A(h,R)| &\le \int_{A(h,R)} (\min\{u,h\}-k)^{1^*}\,dx \\
&\le \int_{B_R(x_0)} |v|^{1^*}\,dx \\
&\le c_3 \left(\int_{B_R(x_0)} |Dv|dx\right)^{\frac{n}{n-1}} \\
&\le c_3 \left(\int_{A(k,R)\setminus A(h,R)} |Du|dx\right)^{\frac{n}{n-1}}.
\end{aligned}
\tag{8.30}
$$

Taking the $\frac{2n-2}{n}$-th power in (8.30) and using Hölder's inequality we infer

$$
\begin{aligned}
(h-k)^2|A(h,R)|^{\frac{2n-2}{n}} &\le c_4|A(k,R)\setminus A(h,R)| \int_{A(k,R)\setminus A(h,R)} |Du|^2 dx \\
&\le c_4|A(k,R)\setminus A(h,R)| \int_{A(k,R)} |Du|^2 dx.
\end{aligned}
\tag{8.31}
$$

Since $u \in DG(\Omega)$ we have, by (8.10),

$$
\int_{A(k,R)} |Du|^2 dx \le \frac{c_5}{R^2} \int_{A(k,2R)} (u-k)^2 dx \le c_6 R^{n-2}(M(2R)-k)^2,
$$

which, together with (8.31), yields

$$
(h-k)^2|A(h,R)|^{\frac{2n-2}{n}} \le c_7 R^{n-2}(M(2R)-k)^2|A(k,R)\setminus A(h,R)|. \tag{8.32}
$$

Now set

$$
k_i := M(2R) - \frac{M(2R)-k_0}{2^i}, \quad i \in \mathbb{N},
$$

so that

$$
k_i - k_{i-1} = \frac{M(2R)-k_0}{2^i}, \quad M(2R) - k_{i-1} = \frac{M(2R)-k_0}{2^{i-1}}.
$$

Set $h = k_i$, $k = k_{i-1}$ in (8.32) to obtain

$$
|A(k_i,R)|^{\frac{2n-2}{n}} \le 4c_8 R^{n-2}\big(|A(k_{i-1},R)| - |A(k_i,R)|\big).
$$

Summing for $1 \le i \le N$ and using $|A(k_{i-1},R)| \ge |A(k_i,R)|$ we obtain

$$
\begin{aligned}
N|A(k_N,R)|^{\frac{2n-2}{n}} &\le 4c_8 R^{n-2}\big(|A(k_0,R)| - |A(k_N,R)|\big) \\
&\le 4c_8 R^{n-2}|A(k_0,R)|.
\end{aligned}
\tag{8.33}
$$

As $N \to \infty$ we have $k_N \to M(2R)$ and $|A(k_N, R)| \to 0$. We see at once from (8.33) that the rate of convergence doesn't depend on x_0, so that the function ζ exists. □

Proof of De Giorgi's theorem. Fix $x_0 \in \Omega$. With the same notation as in (8.28) we assume

$$|A(k_0, R)| = |\{x \in B_R(x_0) : u(x) \geq k_0\}| \leq \frac{1}{2}$$

(otherwise we work with $-u$). By (8.20) with

$$k = k_\nu := M(2R) - \frac{M(2R) - m(2R)}{2^{\nu+1}},$$

we have

$$M(R) \leq k_\nu + c(M(2R) - k_\nu)\left(\frac{|A(k_\nu, 2R)|}{R^n}\right)^{\frac{\theta-1}{2}}.$$

Thanks to Proposition 8.17 we may choose ν large enough and independent of x_0, such that

$$c\left(\frac{|A(k_\nu, 2R)|}{R^n}\right)^{\frac{\theta-1}{2}} < \frac{1}{2}.$$

Then

$$\begin{aligned} M(R) &\leq M(2R) - \frac{M(2R) - m(2R)}{2^{\nu+1}} + \frac{1}{2}(M(2R) - k_\nu) \\ &= M(2R) - \frac{M(2R) - m(2R)}{2^{\nu+2}}. \end{aligned}$$

Subtract $m(R)$ and obtain

$$\begin{aligned} M(R) - m(R) &\leq M(2R) - m(R) - \frac{M(2R) - m(2R)}{2^{\nu+2}} \\ &\leq M(2R) - m(2R) - \frac{M(2R) - m(2R)}{2^{\nu+2}} \\ &= (M(2R) - m(2R))\left(1 - \frac{1}{2^{\nu+2}}\right), \end{aligned}$$

that is

$$\omega(R) \leq \sigma\omega(2R), \tag{8.34}$$

$\sigma < 1$ independent of x_0, $\omega(\rho) := M(\rho) - m(\rho)$ being the oscillation on the ball $B_\rho(x_0)$. Iterating it follows that

$$\omega(\rho) \leq c_1 \left(\frac{\rho}{R}\right)^\alpha \omega(R).$$

with $\alpha = -\frac{\log \sigma}{\log 2} \in (0,1)$. To conclude that u is locally Hölder continuous, define

$$\omega(\rho, x) := \sup_{B_\rho(x)} u - \inf_{B_\rho(x)} u$$

and for $x, y \in B_R(x_0)$, $3R < \operatorname{dist}(x_0, \partial\Omega)$, we have

$$\begin{aligned}
|u(x) - u(y)| &\le \omega(|x-y|, x) \\
&\le c_1 \left(\frac{|x-y|}{2R}\right)^\alpha \omega(2R, x) \\
&\le c_2 \left(\frac{|x-y|}{R}\right)^\alpha \omega(3R, x_0).
\end{aligned}$$

□

A remark

In the right-hand side of (8.25) one can take the L^p norm of u instead of its L^2 norm, for any $p > 0$. For $p > 2$ this follows of course from Jensen's inequality, while from $p \in (0,2)$ it can be proven using the following lemma.

Lemma 8.18 *Let $\phi : [0,T] \to \mathbb{R}$ be a non-negative bounded function. Suppose that for $0 \le \rho < R \le T$ we have*

$$\phi(\rho) \le A(R-\rho)^{-\alpha} + \varepsilon\phi(R)$$

for some $A, \alpha > 0$, $0 \le \varepsilon < 1$. Then there exists a constant $c = c(\alpha, \varepsilon)$ such that for $0 \le \rho < R \le T$ we have

$$\phi(\rho) \le cA(R-\rho)^{-\alpha}.$$

Proof. For some $0 < \tau < 1$, define

$$\begin{cases} t_0 := \rho \\ t_{i+1} := t_i + (1-\tau)\tau^i(R-\rho), \quad i \ge 0. \end{cases}$$

Notice that $t_i < R$ since

$$\sum_{i=1}^\infty \tau^i = \frac{\tau}{1-\tau},$$

and prove inductively that

$$\phi(t_0) \le \varepsilon^k \phi(t_k) + A(1-\tau)^{-\alpha}(R-\rho)^{-\alpha} \sum_{i=0}^{k-1} \varepsilon^i \tau^{-i\alpha}.$$

Choose τ in such a way that $\frac{\varepsilon}{\tau^\alpha} < 1$ and letting $k \to \infty$ we get

$$\phi(\rho) \le c(\alpha, \varepsilon) \frac{A}{(R-\rho)^\alpha}.$$

□

Proposition 8.19 *For every $p > 0$ and every u such that $u, -u \in DG(\Omega)$, there is a constant $c > 0$ such that for*

$$x_0 \in \Omega, \quad 0 < \rho < R < \operatorname{dist}(x_0, \partial\Omega), \quad p > 0$$

one has

$$\sup_{B_\rho(x_0)} |u| \le \frac{c}{(R-\rho)^{\frac{n}{p}}} \Big(\int_{B_R(x_0)} |u|^p dx \Big)^{\frac{1}{p}}, \tag{8.35}$$

or, equivalently, there exists $C > 0$ such that

$$\sup_{B_{\frac{R}{2}}(x_0)} |u| \le C \Big(\fint_{B_R(x_0)} |u|^p dx \Big)^{\frac{1}{p}}, \tag{8.36}$$

where the constants are independent of x_0.

Proof. Step 1. First we prove that for any $0 < \tau < 1$ we have

$$\sup_{B_{\tau R}(x_0)} |u| \le \frac{c_1}{(1-\tau)^{\frac{n}{2}}} \Big(\fint_{B_R(x_0)} |u|^2 dx \Big)^{\frac{1}{2}},$$

or, equivalently, for $0 < \rho < R$

$$\sup_{B_\rho(x_0)} |u| \le \frac{c_1'}{(R-\rho)^{\frac{n}{2}}} \Big(\int_{B_R(x_0)} |u|^2 dx \Big)^{\frac{1}{2}}. \tag{8.37}$$

For any $\varepsilon > 0$ there exists $x_1 \in B_{\tau R}(x_0)$ such that

$$u(x_1)^2 > \sup_{B_{\tau R}(x_0)} |u|^2 - \varepsilon.$$

Then

$$\begin{aligned}
\sup_{B_{\tau R}(x_0)} |u|^2 &\le \varepsilon + \sup_{B_{(1-\tau)\frac{R}{4}}(x_1)} |u|^2 dx \\
&\underbrace{\le}_{(8.25)} \varepsilon + c \fint_{B_{(1-\tau)\frac{R}{2}}(x_1)} |u|^2 dx \\
&\le \varepsilon + \frac{2^n c}{(1-\tau)^n} \frac{1}{|B_R|^n} \int_{B_{(1-\tau)\frac{R}{2}}(x_1)} |u|^2 dx \\
&\le \varepsilon + \frac{c_1}{(1-\tau)^n} \fint_{B_R(x_0)} |u|^2 dx.
\end{aligned}$$

By the arbitrariness of ε, we conclude. This takes care of the case $p=2$ and, by Jensen's inequality, also of the case $p>2$.

Step 2. Now assume $p\in(0,2)$. From (8.37) we get

$$\begin{aligned}\sup_{B_\rho(x_0)}|u| &\le \frac{c}{(R-\rho)^{\frac{n}{2}}}\Big(\int_{B_R(x_0)}|u|^2dx\Big)^{\frac12}\\ &\le \frac{c}{(R-\rho)^{\frac{n}{2}}}\Big(\int_{B_R(x_0)}|u|^p dx\Big)^{\frac12}\sup_{B_R(x_0)}|u|^{\frac{2-p}{2}}\\ &\le \varepsilon\sup_{B_R(x_0)}|u|+\frac{c(\varepsilon,p)}{(R-\rho)^{\frac{n}{p}}}\Big(\int_{B_R(x_0)}|u|^p dx\Big)^{\frac1p},\end{aligned}\tag{8.38}$$

using

$$ab\le \varepsilon a^{\frac{2}{p}}+c(\varepsilon,p)b^{\frac{2}{2-p}}.$$

Applying Lemma 8.18 to (8.38) with

$$\phi(\rho):=\sup_{B_\rho(x_0)}|u|,\quad T=R$$

we obtain (8.35). $\square$

8.4 Moser's technique and Harnack's inequality

We present here Moser's proof [79] [80] of Harnack's inequality that, as a corollary, yields De Giorgi's theorem.

The iteration technique

Proposition 8.20 *Let $u\in W^{1,2}(\Omega)$ be a subsolution of*

$$\begin{aligned}&D_\alpha(A^{\alpha\beta}D_\beta u)=0,\\ &\lambda|\xi|^2\le A^{\alpha\beta}\xi_\alpha\xi_\beta\le\Lambda|\xi|^2,\quad \forall\xi\in\mathbb{R}^n.\end{aligned}\tag{8.39}$$

Then for every $p>0$ there exists a constant $k_1=k_1(p,n,\lambda,\Lambda)$ such that

$$\sup_{B_{\frac{R}{2}}(x_0)}u\le k_1\Big(\fint_{B_R(x_0)}|u|^p dx\Big)^{\frac1p},\quad \forall x_0\in\Omega,\ 0<R<\operatorname{dist}(x_0,\partial\Omega).\tag{8.40}$$

If u is a positive *supersolution, then for every $q<0$ there exists a constant $k_2=k_2(q,n,\lambda,\Lambda)$ such that*

$$\inf_{B_{\frac{R}{2}}(x_0)}u\ge k_2\Big(\fint_{B_R(x_0)}|u|^q dx\Big)^{\frac1q},\quad \forall x_0\in\Omega,\ 0<R<\operatorname{dist}(x_0,\partial\Omega).\tag{8.41}$$

Proof. We divide the proof in several steps
Step 1. Let u be a subsolution. It is enough to consider the case $u \geq 0$, since otherwise we can work with u^+, which is again a subsolution thanks to Lemma 8.11, and get

$$\sup_{B_{\frac{R}{2}}(x_0)} u \leq \sup_{B_{\frac{R}{2}}(x_0)} u^+ \leq k_1 \Big(\fint_{B_R(x_0)} |u^+|^p dx \Big)^{\frac{1}{p}} \leq k_1 \Big(\fint_{B_R(x_0)} |u|^p dx \Big)^{\frac{1}{p}}.$$

For any $0 < \rho < R$, take $\eta \in C_c^\infty(B_R(x_0))$, with $\eta \equiv 1$ on $B_\rho(x_0)$ and $|D\eta| \leq \frac{2}{R-\rho}$. Choose $\xi \geq 1$, and consider the test function $u^\xi \eta^2 \geq 0$:

$$\begin{aligned} 0 &\geq \int_\Omega A^{\alpha\beta} D_\beta u D_\alpha (u^\xi \eta^2) dx \\ &= \xi \int_{B_R(x_0)} A^{\alpha\beta} D_\beta u D_\alpha u u^{\xi-1} \eta^2 dx + 2 \int_{B_R(x_0)} A^{\alpha\beta} D_\beta u D_\alpha \eta u^\xi \eta dx. \end{aligned}$$

By ellipticity and boundedness of $A^{\alpha\beta}$ this becomes

$$\int_{B_R(x_0)} |Du|^2 u^{\xi-1} \eta^2 dx \leq \frac{c(\lambda, \Lambda)}{\xi} \int_{B_R(x_0)} |Du| u^{\frac{\xi-1}{2}} \eta u^{\frac{\xi+1}{2}} |D\eta| dx,$$

and using $2ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon}$ with $a = |Du| u^{\frac{\xi-1}{2}} \eta$, we get

$$\int_{B_R(x_0)} |Du|^2 u^{\xi-1} \eta^2 dx \leq \frac{c_1}{\xi^2} \int_{B_R(x_0)} u^{\xi+1} |D\eta|^2 dx.$$

Since $\left| D(u^{\frac{\xi+1}{2}}) \right|^2 = \left(\frac{\xi+1}{2} \right)^2 u^{\xi-1} |Du|^2$, we infer

$$\int_{B_R(x_0)} \left| D(u^{\frac{\xi+1}{2}}) \right|^2 \eta^2 dx \leq c_2 \left(\frac{\xi+1}{\xi} \right)^2 \int_{B_R(x_0)} u^{\xi+1} |D\eta|^2 dx.$$

Now use

$$\left| D\left(\eta u^{\frac{\xi+1}{2}} \right) \right|^2 \leq 2 \left| D(u^{\frac{\xi+1}{2}}) \right|^2 \eta^2 + 2 u^{\xi+1} |D\eta|^2$$

to get

$$\begin{aligned} \int_{B_R(x_0)} \left| D\Big(\eta u^{\frac{\xi+1}{2}} \Big) \right|^2 dx &\leq c_3 \int_{B_R(x_0)} u^{\xi+1} |D\eta|^2 dx \\ &\leq \frac{4c_3}{(R-\rho)^2} \int_{B_R(x_0)} u^{\xi+1} dx, \end{aligned}$$

where c_3 is independent of $\xi \geq 1$. By the Sobolev inequality (Theorem 7.29) we then have

$$\begin{aligned}\left(\int_{B_\rho(x_0)} u^{\frac{\xi+1}{2}2^*}dx\right)^{\frac{2}{2^*}} &\leq \left(\int_{B_R(x_0)} \left(u^{\frac{\xi+1}{2}}\eta\right)^{2^*} dx\right)^{\frac{2}{2^*}} \\ &\leq c_4 \int_{B_R(x_0)} \left|D\left(\eta u^{\frac{\xi+1}{2}}\right)\right|^2 dx \\ &\leq \frac{c_5}{(R-\rho)^2}\int_{B_R(x_0)} u^{\xi+1}dx.\end{aligned} \tag{8.42}$$

Finally, setting $\mu := \frac{2^*}{2} = \frac{n}{n-2}$, $p := \xi+1$, (8.42) becomes

$$\left(\int_{B_\rho(x_0)} u^{\mu p}dx\right)^{\frac{1}{\mu}} \leq \frac{c_5}{(R-\rho)^2}\int_{B_R(x_0)} u^p dx, \quad \forall p \geq 2, \tag{8.43}$$

Step 2. Thanks to Lemma 8.11, u^μ is still a subsolution of (8.39), so we can iterate the bound (8.43). Define, for any $p \geq 2$,

$$\sigma_i := p\mu^i, \quad R_i := \rho + \frac{R-\rho}{2^i}, \quad (R_i - R_{i+1})^2 = \frac{(R-\rho)^2}{2^{2i+2}}.$$

By (8.43) we have

$$\begin{aligned}\left(\int_{B_{R_{i+1}}(x_0)} u^{\sigma_{i+1}}dx\right)^{\frac{1}{\mu^{i+1}}} &= \left(\int_{B_{R_{i+1}}(x_0)} u^{\mu\sigma_i}dx\right)^{\frac{1}{\mu^i}\frac{1}{\mu}} \\ &\leq \left(\frac{c_5 2^{2i+2}}{(R-\rho)^2}\right)^{\frac{1}{\mu^i}}\left(\int_{B_{R_i}(x_0)} u^{\sigma_i}dx\right)^{\frac{1}{\mu^i}} \\ &\leq \prod_{k=0}^{i}\left(\frac{c_5 2^{2k+2}}{(R-\rho)^2}\right)^{\frac{1}{\mu^k}}\int_{B_R(x_0)} u^p dx.\end{aligned}$$

Since

$$\begin{aligned}\log\left(\prod_{k=0}^{i}\left(c_5 2^{2k+2}\right)^{\frac{1}{\mu^k}}\right) &\leq \sum_{k=0}^{\infty}\frac{1}{\mu^k}((2k+2)\log(2)+\log c_5) < \infty, \\ \prod_{k=0}^{i}\left(\frac{1}{(R-\rho)^2}\right)^{\frac{1}{\mu^k}} &= \left(\frac{1}{(R-\rho)^2}\right)^{\sum_{k=0}^{i}\frac{1}{\mu^k}} \\ \sum_{k=0}^{\infty}\frac{1}{\mu^k} &= \frac{n}{2},\end{aligned}$$

we have

$$\lim_{i\to\infty}\prod_{k=0}^{i}\left(\frac{c_5 2^{2k+2}}{(R-\rho)^2}\right)^{\frac{1}{\mu^k}} \leq c_6\left(\frac{1}{R-\rho}\right)^n,$$

where c_6 depends only on p, λ and Λ. Therefore

$$\Big(\int_{B_\rho(x_0)} u^{\sigma_i}\,dx\Big)^{\frac{1}{\sigma_i}} \le \Big(\int_{B_{R_i}(x_0)} u^{\sigma_i}\,dx\Big)^{\frac{1}{\sigma_i}}$$
$$\le c_6(R-\rho)^{-\frac{n}{p}}\Big(\int_{B_R(x_0)} u^p\,dx\Big)^{\frac{1}{p}}.$$

For $i \to +\infty$ the left hand side converges to $\sup_{B_\rho(x_0)} u$ and this completes the proof of (8.40) in the case $p \ge 2$.

Step 3. We now prove (8.40) for $0 < p < 2$. By step 2 we have

$$\sup_{B_\rho(x_0)} u \le c_6(R-\rho)^{-\frac{n}{2}}\Big(\int_{B_R(x_0)} u^2\,dx\Big)^{\frac{1}{2}},$$

which implies

$$\sup_{B_\rho(x_0)} u \le c_7(R-\rho)^{-\frac{n}{2}}\big(\sup_{B_R(x_0)} u\big)^{1-\frac{p}{2}}\Big(\int_{B_R(x_0)} u^p\,dx\Big)^{\frac{1}{2}}.$$

Next we use Young's inequality $ab \le \varepsilon a^q + c(\varepsilon,q)b^{q'}$, with $\frac{1}{q} := 1-\frac{p}{2}$, $q' := \frac{2}{p}$,

$$a := \left(\sup_{B_R(x_0)} u\right)^{1-\frac{p}{2}}, \qquad b := \left(c_7(R-\rho)^{-n}\int_{B_R(x_0)} u^p\,dx\right)^{\frac{1}{2}},$$

inferring

$$\sup_{B_\rho(x_0)} u \le \frac{1}{2}\sup_{B_R(x_0)} u + c_8(R-\rho)^{-\frac{n}{p}}\Big(\int_{B_R(x_0)} u^p\,dx\Big)^{\frac{1}{p}}.$$

Setting $\phi(\rho) = \sup_{B_\rho(x_0)} u$ the conclusion follows at once from Lemma 8.18.

Step 4. Suppose u is a positive supersolution. Proceding exactly as for subsolutions, but taking first $\xi < -1$ and then $p = \xi + 1 < 0$ one easily deduces (8.41). □

The Harnack inequality

Proposition 8.20 in conjunction with John-Nirenberg's theorem yields

Theorem 8.21 (Harnack's inequality) *Let $u \in W^{1,2}(\Omega)$ be a positive solution of the elliptic equation (8.39). Then there exists a constant*

$$c = c(n,\lambda,\Lambda) \in (0,1)$$

such that

$$\inf_{B_{\frac{R}{2}}(x_0)} u \geq c \sup_{B_{\frac{R}{2}}(x_0)} u, \tag{8.44}$$

for any $x_0 \in \Omega$, $0 < R \leq \operatorname{dist}(x_0, \partial\Omega)$.

Proof. Consider a cut-off function η compactly supported in $B_\rho(x_0)$ with $\eta \equiv 1$ in $B_{\frac{\rho}{2}}(x_0)$ and $|D\eta| \leq \frac{4}{\rho}$. From

$$\int_\Omega A^{\alpha\beta} D_\alpha u D_\beta \left(\frac{\eta^2}{u}\right) dx = 0,$$

we infer

$$\begin{aligned}\int_\Omega A^{\alpha\beta} D_\alpha u D_\beta u \frac{\eta^2}{u^2} dx &= 2\int_\Omega A^{\alpha\beta} D_\alpha u D_\beta \eta \frac{\eta}{u} dx\\ &\leq c\Lambda \int_\Omega \frac{|Du|}{u} |D\eta| \eta dx.\end{aligned}$$

Using ellipticity and boundedness of $A^{\alpha\beta}$ and $2ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon}$, we obtain

$$\int_\Omega \frac{|Du|^2}{u^2} \eta^2 dx \leq c_1 \int_\Omega |D\eta|^2 dx \leq c_2 \rho^{n-2}. \tag{8.45}$$

Observe that $|D\log u|^2 = \frac{|Du|^2}{u^2}$; by the Poincaré inequality and the properties of η we then have

$$\int_{B_\rho(x_0)} |\log u - (\log u)_{x_0,\rho}|^2 dx \leq c_3 \rho^2 \int_{B_\rho(x_0)} |D\log u|^2 dx \leq c_4 \rho^n,$$

and since x_0 is arbitrary we have $\log u \in \mathcal{L}^{2,n}(\Omega_0) \cong BMO(\Omega_0)$ for $\Omega_0 \Subset \Omega$. By Theorem 6.25 part 4, this implies the existence of $\gamma = \gamma(n) > 0$ and $c_5 = c_5(n)$ such that the function $v := e^{\gamma \log u}$ satisfies

$$\fint_{B_R(x_0)} v dx \fint_{B_R(x_0)} \frac{1}{v} dx \leq c_5.$$

We now choose $p = \gamma$, $q = -\gamma$ in Proposition 8.20 to conclude

$$\begin{aligned}\inf_{B_{\frac{R}{2}}(x_0)} u &\geq k_2 \Big(\fint_{B_R(x_0)} u^{-\gamma} dx\Big)^{\frac{-1}{\gamma}}\\ &\geq k_2 c_5^{-\frac{1}{\gamma}} \Big(\fint_{B_R(x_0)} u^\gamma dx\Big)^{\frac{1}{\gamma}}\\ &\geq \frac{k_2}{k_1} c_5^{-\frac{1}{\gamma}} \sup_{B_{\frac{R}{2}}(x_0)} u.\end{aligned}$$

□

A first consequence of Harnack's inequality is the theorem of De Giorgi.

Corollary 8.22 *A solution $u \in W^{1,2}(\Omega)$ of equation* (8.39) *is locally Hölder continuous.*

Proof. Define $M(\rho)$ and $m(\rho)$ as in (8.28), with $B_{2R}(x_0) \subset \Omega$. We have that $M(2R) - u \geq 0$ in $B_{2R}(x_0)$ is a solution of (8.11), hence by virtue of the Harnack's inequality, we get

$$M(2R) - m(R) \leq \frac{1}{c}\big(M(2R) - M(R)\big).$$

Similarly, with $u - m(2R) \geq 0$ we obtain

$$M(R) - m(2R) \leq \frac{1}{c}\big(m(R) - m(2R)\big).$$

Summing we obtain

$$\omega(2R) + \omega(R) \leq \frac{1}{c}\big(\omega(2R) - \omega(R)\big)$$

for some $c \in (0,1)$, hence

$$\omega(R) \leq \frac{1-c}{1+c}\omega(2R).$$

Since $\frac{1-c}{1+c} < 1$ and c doesn't depend on x_0 and R we conclude as after equation (8.34). □

A second consequence is

Theorem 8.23 (Strong maximum principle) *Let u be a non-negative solution of equation* (8.39). *Then either $u > 0$ or $u \equiv 0$.*

Proof. It suffices to apply the Harnack inequality to $u + \varepsilon$ for some $\varepsilon > 0$ and let $\varepsilon \to 0$. □

8.5 Still another proof of De Giorgi's theorem

We report here about another proof of De Giorgi's theorem due to P. Tilli [106], at least assuming that u is bounded.

Theorem 8.24 (Oscillation lemma) *Assume that u is a bounded solution to* (8.11) *in the ball $B_4(0)$. If*

$$\big|\{x \in B_1(0) \mid u \leq 0\}\big| \geq \frac{1}{2}|B_1(0)|, \tag{8.46}$$

then

$$\sup_{B_1(0)} u^+ \leq c_0\big|\{x \in B_2(0) \mid u > 0\}\big|^{\frac{1}{2n}} \sup_{B_4(0)} u^+,$$

for a dimensional constant c_0.

Proof. As in the proof of Caccioppoli's inequality, testing with the function $(u-k)^+\eta$ and then letting η tend to the characteristic function of $B_r(0)$, we find

$$\int_{B_r(0)} |D(u-k)^+|^2 dx \le c\int_{\partial B_r(0)} |D(u-k)^+|(u-k)^+ d\mathcal{H}^{n-1}. \quad (8.47)$$

Now set for $\rho \in [0,1]$

$$g(\rho) := \int_{B_{2-\rho}(0)} |D(u-k\rho)^+|(u-k\rho)^+ dx.$$

The function $g(\rho)$ is absolutely continuous and differentiation yields for a.e. $\rho \in (0,1)$

$$-g'(\rho) = a(\rho) + kb(\rho)$$

where

$$a(\rho) := \int_{\partial B_{2-\rho}(0)} |D(u-k\rho)^+|(u-k\rho)^+ d\mathcal{H}^{n-1}$$

and

$$b(\rho) := \int_{B_{2-\rho}(0)} |D(u-k\rho)^+| dx.$$

Setting $M_4 := \sup_{B_4(0)} u^+$ and using (8.47), we also get

$$\begin{aligned} g^2(\rho) &\le \int_{B_{2-\rho}(0)} |D(u-k\rho)^+|^2 dx \int_{B_{2-\rho}(0)} ((u-k\rho)^+)^2 dx \\ &\le M_4^{\frac{n-2}{n-1}} \int_{B_{2-\rho}(0)} |D(u-k\rho)^+|^2 dx \int_{B_{2-\rho}(0)} ((u-k\rho)^+)^{\frac{n}{n-1}} dx \\ &\le cM_4^{\frac{n-2}{n-1}} a(\rho) \int_{B_{2-\rho}(0)} ((u-k\rho)^+)^{\frac{n}{n-1}} dx, \end{aligned}$$

having used (8.47) in the last inequality. Since $(u-k\rho)^+$ for $k \ge 0$ vanishes on a large portion of $B_{2-\rho}(0)$ because of (8.46), we also have by the Sobolev embedding and the Poincaré inequality (Proposition (3.15))

$$\int_{B_{2-\rho}(0)} |(u-k\rho)^+|^{\frac{n}{n-1}} dx \le c\Big(\int_{B_{2-\rho}(0)} |D(u-k\rho)^+| dx\Big)^{\frac{n}{n-1}},$$

concluding

$$g^2(\rho) \le CM_4^{\frac{n-2}{n-1}} a(\rho) b(\rho)^{\frac{n}{n-1}}.$$

Using

$$ab \le \Big(\frac{a}{\varepsilon}\Big)^p + (\varepsilon b)^q \quad \text{with} \quad p = \frac{2n-1}{n-1}, \quad q = \frac{2n-1}{n},$$

we find for every $\varepsilon > 0$

$$\begin{aligned}
\frac{g(\rho)^2}{CM_4^{\frac{n-2}{n-1}}} &\leq \left(\frac{a(\rho)^p}{\varepsilon^p} + \varepsilon^q b(\rho)^{q\frac{n}{n-1}}\right) \\
&= \left(\frac{a(\rho)^p}{\varepsilon^p} + \varepsilon^q b(\rho)^p\right) \\
&\leq \left(\frac{a(\rho)}{\varepsilon} + \varepsilon^{\frac{q}{p}} b(\rho)\right)^p \\
&= \left(\frac{a(\rho)}{\varepsilon} + \frac{\varepsilon^{\frac{n-1}{n}}}{k} kb(\rho)\right)^{\frac{2n-1}{n-1}}.
\end{aligned}$$

Finally, if we choose $\varepsilon = k^{\frac{n}{2n-1}}$, we find

$$g(\rho)^2 \frac{k^{\frac{n}{n-1}}}{CM_4^{\frac{n-2}{n-1}}} \leq (-g'(\rho))^{\frac{2n-1}{n-1}} \qquad \text{a.e. } \rho \in (0,1). \tag{8.48}$$

Now we claim that, if we choose

$$k = c_0 M_4^{\frac{n-2}{n}} g(0)^{\frac{1}{n}} = c_0 M_4^{\frac{n-2}{n}} \left(\int_{B_2(0)} |Du^+| u^+ dx\right)^{\frac{1}{n}},$$

where c_0 is large enough, then $g(1) = 0$. Indeed, if $g(1) > 0$, then $g(\rho) > 0$ for a.e. ρ and (8.48) gives

$$\frac{k^{\frac{n}{2n-1}}}{\tilde{C} M_4^{\frac{n-2}{2n-1}}} \leq -\frac{d}{d\rho}\left(g(\rho)^{\frac{1}{2n-1}}\right),$$

i.e.

$$0 < g(1)^{\frac{1}{2n-1}} \leq g(0)^{\frac{1}{2n-1}} - \frac{k^{\frac{n}{2n-1}}}{\tilde{C} M_4^{\frac{n-2}{2n-1}}} = g(0)^{\frac{1}{2n-1}} \left(1 - \frac{c_0^{\frac{n}{2n-1}}}{\tilde{C}}\right),$$

a contradiction if we choose c_0 large enough. Thus $g(1) = 0$, i.e.

$$\begin{aligned}
\sup_{B_1(0)} u^+ \leq k = c_0 M_4^{\frac{n-2}{n}} &\left(\int_{B_2(0)} |Du^+| u^+ dx\right)^{\frac{1}{n}} \\
&\leq c_0 M_4^{\frac{n-1}{n}} \left(\int_{B_2(0)} |Du^+| dx\right)^{\frac{1}{n}} \\
&\leq c_0 M_4^{\frac{n-1}{n}} \left|\{x \in B_2(0) \mid u(x) > 0\}\right|^{\frac{1}{2n}} \left(\int_{B_2(0)} |Du^+|^2 dx\right)^{\frac{1}{2n}} \\
&\leq c_0' M_4^{\frac{n-1}{n}} \left|\{x \in B_2(0) \mid u(x) > 0\}\right|^{\frac{1}{2n}} \left(\int_{B_4(0)} |u^+|^2 dx\right)^{\frac{1}{2n}} \\
&\leq c_0' \left|\{x \in B_2(0) \mid u(x) > 0\}\right|^{\frac{1}{2n}} \sup_{B_4(0)} u^+,
\end{aligned}$$

and the proof is complete. □

Theorem 8.25 *A bounded solution of* (8.11) *is Hölder continuous.*

Proof. Let u be a solution. By translation and scaling, we can assume

$$\sup_{B_4(0)} u = 1, \qquad \inf_{B_4(0)} u = -1$$

and, possibly considering $-u$, that (8.46) holds. The oscillation lemma, applied to $u - 1 + \varepsilon$, then yields

$$\sup_{B_1(0)} u \le 1 - \varepsilon + \varepsilon c_0 \big|\big\{x \in B_2(0) \;\big|\; u > 1 - \varepsilon\big\}\big|^{\frac{1}{2n}}. \tag{8.49}$$

On the other hand,

$$\begin{aligned}\big|\big\{x \in B_2(0) : u(x) > 1 - \varepsilon\big\}\big| \log \frac{1}{\varepsilon} &\le \int_{\{x \in B_2(0): u(x) > 1-\varepsilon\}} - \log(1-u)dx \\ &\le \int_{B_2(0)} \max\{-\log(1-u), 0\}dx\end{aligned}$$

and since $\max\{-\log(1-u), 0\} = 0$ on a large subest of $B_2(0)$, by the Sobolev, Poincaré and Jensen inequalities (see Proposition 3.15) we have

$$\int_{B_2(0)} \max\{-\log(1-u), 0\}dx \le \left(\int_{B_2(0)} \frac{|Du|^2}{(1-u)^2} dx \right)^{\frac{1}{2}},$$

and the integral on the right-hand side can be bounded as in (8.45), since $(1-u) \ge 0$.

Together with (8.49), and choosing ε sufficiently small, we conclude

$$\sup_{B_1(0)} u \le 1 - \theta, \qquad \theta > 0,$$

or, since $\inf_{B_4(0)} u = -1$,

$$\operatorname*{osc}_{B_1(0)} u \le (1-\theta) - (-1) = \frac{2-\theta}{2} \operatorname*{osc}_{B_4(0)} u.$$

By scaling and iterating this inequality, we then easily conclude. □

8.6 The weak Harnack inequality

We have seen that solutions to an elliptic equation with bounded coefficients satisfy the Harnack inequality (8.44). Actually a theorem of Di Benedetto and Trudinger [28] shows that any function u such that $u, -u \in DG(\Omega)$ satisfies the Harnack inequality. For simplicity we state it on cubes.

Theorem 8.26 *If $-u \in DG(Q_4)$ and $u > 0$, then there exist constants $p > 0$ and $c > 0$ such that*

$$\inf_{Q_1} u \geq c\Big(\fint_{Q_1} u^p dx\Big)^{\frac{1}{p}}. \tag{8.50}$$

Inequality (8.50) is called *weak Harnack inequality.* Before giving its proof we observe that Harnack's inequality (Theorem 8.21) is a straightforward consequence of (8.50) with (8.36). Then, with the same proof of Corollary 8.22 we obtain another proof of De Giorgi's theorem.

The proof of Theorem 8.26 uses several propositions. The first is essentially De Giorgi's oscillation lemma.

Theorem 8.27 (De Giorgi) *Suppose that $u > 0$ in the cube Q_4, $\tau > 0$, $\delta \in (0, 1)$ and $-u \in DG(Q_4)$. If*

$$|\{x \in Q_2 : u(x) < \tau\}| \leq \delta |Q_2|,$$

then

$$\inf_{Q_1} u \geq c(\delta)\tau,$$

where $c(\delta) \in (0, 1)$ is non-increasing with respect to δ.

Proof. Step 1. We first prove the Proposition when δ is sufficiently small. With $-\tau$ and $-u$ in place of k and u respectively, (8.20) and a covering argument yield

$$\sup_{Q_1}(-u) \leq -\tau + c\Big(\frac{1}{|Q_2|}\int_{\{x\in Q_2 : u(x)<\tau\}} (-u+\tau)^2 dx\Big)^{\frac{1}{2}} \times \left(\frac{|\{x \in Q_2 : u(x) < \tau\}|}{|Q_2|}\right)^{\frac{\theta-1}{2}},$$

hence

$$\begin{aligned}\inf_{Q_1} u &\geq \tau - c\Big(\frac{1}{|Q_2|}\int_{\{x\in Q_2 : u(x)<\tau\}} (\tau - u)^2 dx\Big)^{\frac{1}{2}} \\ &\quad \times \left(\frac{|\{x \in Q_2 : u(x) < \tau\}|}{|Q_2|}\right)^{\frac{\theta-1}{2}} \\ &\geq \tau - c\tau\left(\frac{|\{x \in Q_2 : u(x) < \tau\}|}{|Q_2|}\right)^{\frac{\theta}{2}} \\ &\geq \tau\big(1 - c\delta^{\frac{\theta}{2}}\big).\end{aligned}$$

Then

$$\inf_{Q_1} u \geq \frac{1}{2}\tau$$

for $\delta = \delta_0$ sufficiently small.

Step 2. Let now $\delta \in (0,1)$ be arbitrary. From (8.32) applied to $-u$ and (8.26) applied to $-u-k$ for some $k \le 0$ we obtain

$$\begin{aligned}(h-k)^2|\{x \in & Q_2 : -u(x) > h\}|^{\frac{2n-2}{n}} \\ &\le c\int_{Q_4}((-u-k)^+)^2 dx |\{x \in Q_2 : h \ge -u(x) > k\}| \\ &\le c|Q_4|((-k)^+)^2|\{x \in Q_2 : h \ge -u(x) > k\}|,\end{aligned} \tag{8.51}$$

where c depends on δ. In fact $\delta \in (0,1/2]$ in Proposition 8.17, but (8.32) holds for $\delta \in (1/2,1)$ as well with constant $c = (1-\delta)^{-2}c$, since it depends on (8.29), which for δ close to 1 can be replaced by

$$\frac{c}{R}\int_{B_R(x_0)}|v|dx \le \frac{1}{1-\delta}\int_{B_R(x_0)}|Dv|dx,$$

which follows from Proposition 3.15. We now apply (8.51) with

$$k = -2^{-s}\tau, \quad h = -2^{-s-1}\tau \quad \text{for some } s \in \mathbb{N},$$

getting

$$\begin{aligned}\left|\left\{x \in Q_2 : u(x) < \frac{\tau}{2^{s+1}}\right\}\right|^{\frac{2n-2}{n}} \le c_1\Bigg(&\left|\left\{x \in Q_2 : u(x) < \frac{\tau}{2^{s}}\right\}\right| \\ &- \left|\left\{x \in Q_2 : u(x) < \frac{\tau}{2^{s+1}}\right\}\right|\Bigg).\end{aligned}$$

Summing for $0 \le s \le \nu - 1$ we obtain

$$\nu\left|\left\{x \in Q_2 : u(x) < \frac{\tau}{2^{\nu}}\right\}\right|^{\frac{2n-2}{n}} \le c_1\left|\left\{x \in Q_2 : u(x) < \tau\right\}\right| \le c_1 2^n,$$

with c_1 depending on δ, hence, choosing ν large enough (depending on δ_0 and δ) we obtain

$$\left|\left\{x \in Q_2 : u(x) < \frac{\tau}{2^{\nu}}\right\}\right| \le \delta_0|Q_2|,$$

so that by the previous step

$$\inf_{Q_1} u \ge \frac{\tau}{2^{\nu+1}},$$

hence the theorem is proven with $c(\delta) = 2^{-\nu-1}$. □

We need to slightly extend the previous result.

Proposition 8.28 *Assume that* $u > 0$ *in* Q_4, $\tau > 0$, $\delta \in (0,1)$ *and* $-u \in DG(Q_4)$. *If* $|\{x \in Q_1 : u(x) \geq \tau\}| \geq \delta |Q_1|$, *then*

$$\inf_{Q_1} u \geq c(\delta)\tau,$$

where $c(\delta) \in (0,1)$ *is non-decreasing with respect to* δ.

Proof. Indeed

$$|\{x \in Q_2 : u(x) \geq \tau\}| \geq |\{x \in Q_1 : u(x) \geq \tau\}| \geq \delta|Q_1| \geq \frac{\delta}{2^n}|Q_2|,$$

hence

$$|\{x \in Q_2 : u(x) < \tau\}| \leq \left(1 - \frac{\delta}{2^n}\right)|Q_2|$$

and we can apply Theorem 8.27. □

Proposition 8.29 *Suppose that* $u > 0$ *in the cube* Q_1, $\tau > 0$, *and* $-u \in DG(Q_4)$. *If*

$$|\{x \in Q_1 : u(x) \geq \tau\}| \geq 2^{-s}|Q_1|$$

for some positive integer s, *then*

$$\inf_{Q_1} u \geq c^s \tau,$$

$c = c(\overline{\delta}) \in (0,1)$ *being as in Proposition 8.28, with* $\overline{\delta} := 2^{-n-1}$.

Proof. For $s = 1$ the claim is true by Proposition 8.28. Let us assume the claim true for some s and prove it for $s+1$. By hypothesis, if we set

$$E_0 := \{x \in Q_1 : u \geq \tau\},$$

we have $|E_0| \geq 2^{-s-1}|Q_1|$.

If $|E_0| \geq 2^{-s}|Q_1|$, then by the inductive hypothesis

$$\inf_{Q_1} u \geq c^s \tau \geq c^{s+1}\tau$$

and we are done. Otherwise

$$2^{-s-1}|Q_1| \leq |E_0| < 2^{-s}|Q_1|.$$

Set $f := \chi_{E_0}$, and apply the Calderón-Zygmund argument to f in Q_1 with parameter $\frac{1}{2}$ to find a sequence of dyadic cubes $\{Q_j\}_{j \in J}$ such that

(i) $\dfrac{1}{2} < \displaystyle⨍_{Q_j} f\,dx \leq \frac{1}{2}2^n$, i.e., $|E_0 \cap Q_j| > \frac{1}{2}|Q_j|$;

(ii) $f \leq \frac{1}{2}$ a.e. in $Q \setminus \bigcup_j Q_j$, i.e. $E_0 \subset \bigcup_j Q_j$ for $j \in J$ up to a set of measure 0;

(iii) if Q_j is one of the 2^n subcubes of P_i arising during the Calderón-Zygmund process, then

$$\fint_{P_i} f \leq \frac{1}{2}, \quad \text{i.e.} \quad |E_0 \cap P_i| \leq \frac{1}{2}|P_i|.$$

From (ii) and (iii) we infer

$$|E_0| = |E_0 \cap (\cup_i P_i)| = \sum_i |E_0 \cap P_i| \leq \frac{1}{2} \sum_i |P_i|.$$

On the other hand, if P is one of the $P_i's$ and Q_j is one of its subcubes, we have

$$|E_0 \cap P| \geq |E_0 \cap Q_j| \geq \frac{1}{2}|Q_j| \geq \frac{1}{2 \cdot 2^n}|P|;$$

therefore we can apply Proposition 8.28 to conclude $\inf_P u \geq c\tau$, i.e.

$$P \subset E_1 := \{x \in Q_1 \mid u > c\tau\}.$$

Consequently

$$2^{-s-1}|Q_1| \leq |E_0| \leq \frac{1}{2} \sum_i |P_i| \leq \frac{1}{2}|E_1|,$$

i.e.,

$$|E_1| \geq 2^{-s}|Q_1|.$$

Then, by inductive hypothesis,

$$\inf_{Q_1} u \geq c(c^s)\tau = c^{s+1}\tau.$$

$\square$

Proof of Theorem 8.26. Given any τ such that

$$0 < \tau < t_1 := \sup_{Q_1} u$$

and choose an integer s such that

$$\lambda_\tau := |\{x \in Q_1 : u(x) \geq \tau\}| \geq 2^{-s}|Q_1|,$$

i.e.

$$s \geq \log\left(\frac{\lambda_\tau}{|Q_1|}\right) \cdot \left(\log \frac{1}{2}\right)^{-1}.$$

Then, according to Proposition 8.29, we have

$$\begin{aligned}\inf_{Q_1} u &\geq c^s \tau \\ &\geq c^{\log\left(\frac{\lambda_\tau}{|Q_1|}\right)\cdot(\log\frac{1}{2})^{-1}} \tau \\ &= \exp\left(\frac{\log c}{\log\frac{1}{2}}\log\left(\frac{\lambda_\tau}{|Q_1|}\right)\right)\tau \\ &= \left(\frac{\lambda_\tau}{|Q_1|}\right)^\gamma \tau, \qquad \gamma := \frac{\log c}{\log\frac{1}{2}},\end{aligned}$$

i.e.

$$\frac{\lambda_\tau}{|Q_1|} \leq \tau^{-\frac{1}{\gamma}} \inf_{Q_1} u^{\frac{1}{\gamma}}.$$

Taking into account (6.1), we conclude for

$$t_0 := \inf_{Q_1} u, \quad p < \frac{1}{\gamma}$$

and using (6.1)

$$\begin{aligned}\fint_{Q_1} u^p dx &= p\int_{t_0}^{t_1} \tau^{p-1}\frac{\lambda_\tau}{|Q_1|}d\tau + \frac{\lambda_{t_0}}{|Q_1|}t_0^p \\ &\leq c_1 \inf_{Q_1} u^{\frac{1}{\gamma}} \int_{t_0}^{\infty} \tau^{p-1-\frac{1}{\gamma}} d\tau + \left(c_2 \inf_{Q_1} u\right)^p \\ &= c_3 \inf_{Q_1} u^p,\end{aligned}$$

as was to be shown. □

8.7 Differentiability of minimizers of non-differentiable variational integrals

We conclude the chapter proving two results of Giaquinta-Giusti [41] [42], [37]. Consider a variational integral

$$\mathcal{F}(u,\Omega) := \int_\Omega F(x,u,Du)dx, \tag{8.52}$$

that is not necessarily differentiable in $W^{1,2}$.

Theorem 8.30 *Let $\mathcal{F}$ be as in* (8.52) *and assume that*

$$|p|^2 \leq F(x,u,p) \leq \Lambda|p|^2, \tag{8.53}$$

and let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ *be a* quasi-minimum *of* $\mathcal{F}$ *meaning that there exists* $Q \geq 1$ *such that*

$$\mathcal{F}(u, \operatorname{spt}(u-v)) \leq Q\mathcal{F}(v, \operatorname{spt}(u-v))$$

for all $v \in W^{1,2}_{\mathrm{loc}}(\Omega)$ *with* $\operatorname{spt}(u-v) \Subset \Omega$. *Then both* u *and* $-u$ *belong to* $DG(\Omega)$, *hence* u *is locally Hölder continuous by De Giorgi's theorem.*

Proof. Let $x_0 \in \Omega$ and $B_R(x_0) \Subset \Omega$. For $k > 0$, $0 < s < R$ set as before

$$A(k,s) := \{x \in B_s(x_0) : u(x) > k\}.$$

For any t with $0 < t < s$, let $\eta \in C^\infty_c(\Omega)$, $\operatorname{spt}\eta \subset B_s(x_0)$, $\eta \equiv 1$ on $B_t(x_0)$, $|D\eta| \leq \frac{2}{s-t}$ and define

$$w := (u-k)^+ = \max\{u-k, 0\}, \qquad v := u - \eta w.$$

Observing that u differs from v only on $A(k,s)$, using the minimality of u and (8.53), we find

$$\begin{aligned}\int_{A(k,s)} |Du|^2 dx &\leq \int_{A(k,s)} F(x,u,Du)dx \\ &\leq Q\int_{A(k,s)} F(x,v,Dv)dx \leq Q\Lambda \int_{A(k,s)} |Dv|^2 dx \\ &\leq c_1\Big\{\int_{A(k,s)} (1-\eta)^2|Du|^2dx + \int_{A(k,s)} w^2|D\eta|^2dx\Big\}.\end{aligned}$$

Observing that $\eta \equiv 1$ on $B_t(x_0)$ and $|D\eta| \leq \frac{2}{s-t}$, we get

$$\int_{A(k,t)} |Du|^2dx \leq c_1\Big\{\int_{A(k,s)\setminus A(k,t)} |Du|^2dx + \frac{4}{(s-t)^2}\int_{A(k,R)} (u-k)^2dx\Big\}.$$

Summing $c_1 \int_{A_{k,t}} |Du|^2dx$ to both sides, we obtain

$$\int_{A(k,t)} |Du|^2dx \leq \frac{c_1}{1+c_1}\int_{A(k,s)} |Du|^2dx + \frac{4c_1}{(s-t)^2}\int_{A(k,R)} (u-k)^2dx,$$

and Lemma 8.18 yields at once

$$\int_{A(k,\rho)} |Du|^2dx \leq \frac{c_2}{(R-\rho)^2}\int_{A(k,R)} (u-k)^2dx, \tag{8.54}$$

for $0 < \rho < R < \operatorname{dist}(x_0, \partial\Omega)$. Therefore u belongs to the De Giorgi class $DG(\Omega)$. The same reasoning applied to $-u$ implies that also $-u \in DG(\Omega)$ hence, by virtue of De Giorgi's theorem, we conclude that u is Hölder continuous. □

Remark 8.31 The notion of quasi-minima was introduced in [43]. Its interest consists in the fact that, under very soft assumptions, minimizers, solutions of elliptic systems, solutions of problems with obstacles, etc. are all quasi-minima. The interested reader is referred to [39] [52].

Theorem 8.32 *Let u be a minimizer of the variational integral $\mathcal{F}$. Assume that the growth condition* (8.53) *holds and moreover*

(i) *for every $(x,u) \in \Omega \times \mathbb{R}^m$, $F(x,u,p)$ is twice differentiable in p, and for some $\lambda, M > 0$*

$$\begin{aligned} |F_{pp}(x,u,p)| &\leq M \\ F_{p_\alpha p_\beta}(x,u,p)\xi_\alpha \xi_\beta &\geq \lambda |\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^n. \end{aligned} \tag{8.55}$$

(ii) *The function $(1+|p|^2)^{-1}F(x,u,p)$ is Hölder continuous in (x,u) uniformly in p, i.e. there exist contants $A > 0$, $\sigma \in (0,1)$ such that*

$$|F(x,u,p) - F(y,v,p)| \leq A(|x-y|^2 + |u-v|^2)^{\sigma/2}|p|^2. \tag{8.56}$$

Then $Du \in C^{1,\sigma}_{\mathrm{loc}}(\Omega)$.

Proof. Step 1. Take any $x_0 \in \Omega$, $B_R(x_0) \Subset \Omega$, and let v be a minimizer of

$$\mathcal{F}^0(v, B_R(x_0)) := \int_{B_R(x_0)} F(x_0, u_{x_0,R}, Dv)dx, \quad u_{x_0,R} := \fint_{B_R(x_0)} u dx,$$

among the functions in $W^{1,2}(B_R(x_0))$ taking the value u on $\partial B_R(x_0)$. Such a minimizer exists because, thanks to (8.53), a minimizing sequence v_j is bounded in $W^{1,2}(B_R(x_0))$, hence it has a weakly converging subsequence. On the other hand (8.55) implies convexity in p, thus weak semicontinuity. By Proposition 8.6, Dv satisfies an elliptic equation with bounded coefficients hence, by De Giorgi's theorem, $Dv \in C^{0,\delta}_{\mathrm{loc}}(B_R(x_0))$ for some $\delta \in (0,1)$ and we have by (8.17) the estimate

$$\int_{B_\rho(x_0)} |Dv - (Dv)_{x_0,\rho}|^2 dx \leq c_1 \Big(\frac{\rho}{R}\Big)^{n+2\delta} \int_{B_R(x_0)} |Dv - (Dv)_{x_0,R}|^2 dx.$$

Therefore as in (5.22)

$$\begin{aligned} \int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 dx \leq & c_2 \Big\{ \Big(\frac{\rho}{R}\Big)^{n+2\delta} \int_{B_R(x_0)} |Du - (Du)_{x_0,R}|^2 dx \\ & + \int_{B_R(x_0)} |D(u-v)|^2 dx \Big\}. \end{aligned}$$

Then, noticing that $D_s v$ satisfies an elliptic equation with bounded coefficients for every s, namely

$$-D_\alpha(F_{p^\alpha p^\beta} D_\beta(D_s v)) = 0,$$

we can bound with De Giorgi's Theorem, (8.16) in particular,

$$\fint_{B_\rho(x_0)} |Dv|^2 dx \le \sup_{B_\rho(x_0)} |v|^2 \le \sup_{B_{R/2}(x_0)} |v|^2 \le c \fint_{B_R(x_0)} |v|^2 dx$$

(for $\rho \le R/2$, otherwise the inequality is elementary), hence as in (5.22)

$$\int_{B_\rho(x_0)} |Du|^2 dx \le c_2 \Big\{ \Big(\frac{\rho}{R}\Big)^n \int_{B_R(x_0)} |Du|^2 dx + \int_{B_R(x_0)} |D(u-v)|^2 dx \Big\}. \tag{8.57}$$

Step 2. We now claim that

$$\int_{B_R(x_0)} |D(u-v)|^2 dx \le \frac{2}{\lambda} \big[\mathcal{F}^0(u, B_R(x_0)) - \mathcal{F}^0(v, B_R(x_0)) \big]. \tag{8.58}$$

To see that, set $F^0(p) := F(x_0, u_{x_0,R}, p)$, $w := u - v$. Taking into account (8.55) we find

$$\begin{aligned} F^0(Du) - F^0(Dv) &= F^0_{p_\alpha}(Dv) D_\alpha w \\ &\quad + \int_0^1 (1-t) F_{p_\alpha p_\beta}(tDu + (1-t)Dv) D_\alpha w D_\beta w dt \\ &\ge F^0_{p_\alpha}(Dv) D_\alpha w + \frac{\lambda}{2} |Dw|^2. \end{aligned}$$

Integrating over $B_R(x_0)$ and observing that v satisfies that Euler-Lagrange equation

$$\int_{B_R(x_0)} F^0_{p_\alpha}(Dv) D_\alpha \varphi dx = 0 \qquad \text{for all } \varphi \in W^{1,2}_0(\Omega),$$

(8.58) follows at once.

Step 3. We have

$$\begin{aligned} \mathcal{F}^0(u, B_R(x_0)) - \mathcal{F}^0(v, B_R(x_0)) \\ &= \int_{B_R(x_0)} \big[F(x_0, u_{x_0,R}, Du) - F(x, u, Du) \big] dx \\ &\quad + \int_{B_R(x_0)} \big[F(x, v, Dv) - F(x_0, u_{x_0,R}, Dv) \big] dx \\ &\quad + \mathcal{F}(u, B_R(x_0)) - \mathcal{F}(v, B_R(x_0)) \\ &\le \int_{B_R(x_0)} A(|x - x_0|^2 + |u - u_{x_0,R}|)^{\sigma/2} |Du|^2 dx \\ &\quad + \int_{B_R(x_0)} A(|x - x_0|^2 + |v - u_{x_0,R}|)^{\sigma/2} |Dv|^2 dx. \end{aligned}$$

Using (8.58), the minimizing property of u, and taking into account (8.56) and the Hölder continuity of u (Theorem 8.30) and v (De Giorgi's theorem), we infer

$$\int_{B_R(x_0)} |D(u-v)|^2 dx \le c_3 R^{\gamma\sigma} \int_{B_R(x_0)} |Du|^2 dx, \quad \text{for some } \gamma \in (0,1). \tag{8.59}$$

Combining with (8.57) we get

$$\int_{B_\rho(x_0)} |Du|^2 dx \le c_4 \Big[\Big(\frac{\rho}{R}\Big)^n + \omega_1(R) \Big] \int_{B_R(x_0)} |Du|^2 dx,$$

where $\omega(R) \le cR^{\gamma\sigma}$. This implies, by means of Lemma 5.13, that for every $\varepsilon > 0$ there is a constant c_5 such that

$$\int_{B_\rho(x_0)} |Du|^2 dx \le c_5 \Big(\frac{\rho}{R}\Big)^{n-\varepsilon} \int_{B_R(x_0)} |Du|^2 dx, \quad 0 < \rho < R. \tag{8.60}$$

Notice that this implies $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for every $\alpha \in (0,1)$, by Theorem 5.7.

Step 4. To conclude, first observe that (8.60) together with (8.59) implies

$$\int_{B_R(x_0)} |D(u-v)|^2 dx \le c_6 R^{n+2\alpha\sigma-\varepsilon}.$$

Taking $\varepsilon = \alpha\sigma$ we get in conclusion

$$\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 dx \le c_7 \Big(\frac{\rho}{R}\Big)^{n+2\delta} \int_{B_R(x_0)} |Du - (Du)_{x_0,R}|^2 dx + c_7 R^{n+\alpha\sigma}.$$

By Lemma 5.13

$$\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 dx \le c_8 \rho^{n+2r}, \qquad r = \min\Big(\frac{\alpha\sigma}{2}, \frac{\delta}{2}\Big),$$

for all x_0 in an open set, hence $Du \in \mathcal{L}^{2,n+2r}_{\mathrm{loc}}(\Omega) \cong C^{0,r}_{\mathrm{loc}}(\Omega)$. $\square$

Chapter 9
Partial regularity in the vector-valued case

No genaralizations nor counterexamples to the theorem of De Giorgi were found during the years 1957-67 for the vector case $m > 1$. With the exception of the special case $n = 2$, general regularity results for elliptic systems were not available, and in fact, as shown from 1968 on, not valid.

9.1 Counterexamples to everywhere regularity

Here we present only three classical counterexamples. For a more comprehensive discussion we refer to [37] and to the more recent works [105] and [65].

9.1.1 De Giorgi's counterexample

The following example is due to De Giorgi [27]. In $B_1(0) \subset \mathbb{R}^n$, $n \geq 3$ consider on $W^{1,2}(B_1(0), \mathbb{R}^n)$ the regular variational integral

$$\begin{aligned}\mathcal{F}(u) &= \int_{B_1(0)} F(x, Du)dx \\ &= \frac{1}{2}\int_{B_1(0)} \left\{|Du|^2 + \Big[\sum_{i,\alpha=1}^{n} \Big((n-2)\delta_{i\alpha} + n\frac{x_i x_\alpha}{|x|^2}\Big) D_\alpha u^i\Big]^2\right\}dx.\end{aligned} \tag{9.1}$$

Its Euler-Lagrange equation is

$$\int_{B_1(0)} A_{ij}^{\alpha\beta} D_\beta u^j D_\alpha \varphi^i dx = 0, \quad \forall \varphi \in W_0^{1,2}(B_1(0), \mathbb{R}^n), \tag{9.2}$$

with

$$A_{ij}^{\alpha\beta}(x) = \delta_{\alpha\beta}\delta_{ij} + \Big[(n-2)\delta_{\alpha i} + n\frac{x_i x_\alpha}{|x|^2}\Big]\Big[(n-2)\delta_{\beta j} + n\frac{x_j x_\beta}{|x|^2}\Big].$$

Though these coefficients are bounded and satisfy the Legendre condition, the vector valued map

$$u(x) := \frac{x}{|x|^\gamma}, \qquad \gamma := \frac{n}{2}\left[1 - \left((2n-2)^2+1\right)^{-\frac{1}{2}}\right],$$

which belongs to $W^{1,2}(B_1(0),\mathbb{R}^n)$ but is not bounded, is an extremal of $\mathcal{F}$, hence it satisfies the elliptic system with bounded coefficients (9.2).

9.1.2 Giusti and Miranda's counterexample

A slight modification of De Giorgi's counterexample is the following (compare [54]).

$$\begin{aligned}\mathcal{F}(u) &= \int_{B_1(0)} F(u,Du)dx \\ &:= \int_{B_1(0)} |Du|^2 + \left\{\sum_{i,j=1}^n \left(\delta_{ij} + \frac{4}{n-2}\frac{u^iu^j}{1+|u|^2}\right)D_iu^j\right\}^2,\end{aligned}$$

where $B_1(0) \subset \mathbb{R}^n$, $n \geq 3$. For n large, the unique minimizer of $\mathcal{F}$ in $W^{1,2}(B_1(0),\mathbb{R}^n)$ is

$$u(x) = \frac{x}{|x|}$$

and it satisfies an elliptic system with bounded coefficients $A^{\alpha\beta}_{ij}(u)$, where the dependence on u is real analytic.

9.1.3 The minimal cone of Lawson and Osserman

The area functional for the graph of a vector valued function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is

$$\mathcal{A}(u) = \int_\Omega F(Du)dx = \int_\Omega \sqrt{\det\left(I + Du^*Du\right)}dx.$$

Its critical points (whose graphs are called *minimal*) satisfy the elliptic system -called *minimal surface system*-

$$\begin{cases}\sum_{i=1}^n D_\alpha\left(\sqrt{g}g^{\alpha\beta}\right) = 0 & \beta = 1,\dots,n \\ \sum_{i,j=1}^n D_\alpha\left(\sqrt{g}g^{\alpha\beta}D_\beta u^i\right) = 0 & i = 1,\dots,m.\end{cases} \tag{9.3}$$

where $g_{\alpha\beta} := 1 + \sum_{i=1}^m D_\alpha u^i D_\beta u^i$, $(g^{\alpha\beta}) := (g_{\alpha\beta})^{-1}$.

Let $\eta : S^3 \subset \mathbb{R}^4 \to S^2 \subset \mathbb{R}^3$ be the Hopf's map defined by

$$\eta(z_1, z_2) = (|z_1|^2 - |z_2|^2, 2z_1\overline{z}_2) \in \mathbb{R} \times \mathbb{C} \cong \mathbb{R}^3,$$

where $(z_1, z_2) \in \mathbb{C}^2 \cong \mathbb{R}^4$. Then, as proven by Lawson and Osserman [68], the Lipschitz but not C^1 map defined by

$$u : \mathbb{R}^4 \to \mathbb{R}^3$$

$$u(x) = \frac{\sqrt{5}}{2}|x|\eta\left(\frac{x}{|x|}\right), \quad x \neq 0 \tag{9.4}$$

and $u(0) = 0$ satisfies the minimal surface system (9.3).

9.2 Partial regularity

The counterexamples given in last section show that everywhere regularity results for critical points or minimizers of regular variational integrals are in general not possible. Here we shall see some *partial regularity* results, i.e. we prove that minimizers of variational integrals or solutions to nonlinear elliptic systems are regular except in a closed set of small Hausdorff dimension.

9.2.1 Partial regularity of minimizers

Consider the functional

$$\mathcal{F}(u) := \int_\Omega F(Du)dx, \quad u \in W^{1,2}(\Omega, \mathbb{R}^m),$$

where $F : \mathbb{R}^{m\times n} \to \mathbb{R}$ is smooth, satisfies the growth condition (8.5) and

$$\frac{1}{\sigma}|p|^2 \leq F(p) \leq \sigma|p|^2$$

for some $\sigma > 0$.

Given a minimizer $u \in W^{1,2}(\Omega, \mathbb{R}^m)$ of $\mathcal{F}$, define the *singular set* of u

$$\Sigma(u) := \left\{x \in \Omega : \liminf_{\rho\to 0} \fint_{B_\rho(x)} |Du - (Du)_{x,\rho}|^2 dx > 0\right\}.$$

Notice that if Du is continuous at x, then $x \notin \Sigma(u)$. In fact the converse is true, as the next theorem shows.

Theorem 9.1 *The set $\Sigma(u)$ is closed, $u \in C^{1,\sigma}(\Omega\backslash\Sigma(u))$ for any $\sigma \in (0,1)$, and*

$$\mathcal{H}^{n-2}(\Sigma(u)) = 0,$$

where $\mathcal{H}^{n-2}$ is the $(n-2)$-dimensional Hausdorff measure.

Proof. Observe that for every $p, \overline{p} \in \mathbb{R}^{m\times n}$,

$$\begin{aligned} F(p) =& F(\overline{p}) + F_{p^i_\alpha}(\overline{p})(p^i_\alpha - \overline{p}^i_\alpha) \\ &+ \int_0^1 (1-t) F_{p^i_\alpha p^j_\beta}(tp + (1-t)\overline{p})(p^i_\alpha - \overline{p}^i_\alpha)(p^j_\beta - \overline{p}^j_\beta)dt. \end{aligned} \tag{9.5}$$

Fix $\overline{p} := (Du)_{x_0,R}$ for some $x_0 \in \Omega$, $0 < R < \frac{\operatorname{dist}(x_0,\partial\Omega)}{2}$ and define

$$\mathcal{G}(v) = \int_{B_R(x_0)} G(Dv)dx,$$

where G is the approximation of F given by

$$G(p) := F(\overline{p}) + F_{p^i_\alpha}(\overline{p})(p^i_\alpha - \overline{p}^i_\alpha) + \frac{1}{2} F_{p^i_\alpha p^j_\beta}(\overline{p})(p^i_\alpha - \overline{p}^i_\alpha)(p^j_\beta - \overline{p}^j_\beta). \tag{9.6}$$

Since $\mathcal{G}$ is the sum of a quadratic coercive form, a linear functional and a constant, by Theorem 3.39, there exists a unique minimizer v for $\mathcal{G}$ in the class

$$\left\{\zeta \in W^{1,2}(B_R(x_0), \mathbb{R}^m) : \zeta - u \in W^{1,2}_0(B_R(x_0), \mathbb{R}^m)\right\}$$

and it satisfies the elliptic system

$$-D_\alpha \left(F_{p^i_\alpha p^j_\beta}(\overline{p})(Dv^j_\beta - \overline{p}^j_\beta)\right) = 0, \quad \text{in } B_R(x_0),$$

thus the energy estimate (5.14). By Proposition 8.6, $Du \in W^{1,2}_{\mathrm{loc}}(\Omega)$, hence the Sobolev embedding theorem yields $Du \in L^{2^*}_{\mathrm{loc}}(\Omega)$. Now thanks to the L^p-estimates, Theorem 7.1 (actually applied to the function $v^i(x) - \overline{p}^i_\alpha x^\alpha$), for every $q \in [2, 2^*]$, there exists a constant $c = c(q, \lambda, \Lambda)$ such that

$$\int_{B_R(x_0)} |Dv - \overline{p}|^q dx \leq c \int_{B_R(x_0)} |Du - \overline{p}|^q dx.$$

As in equation (5.22), we obtain

$$\begin{aligned} \int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 dx \leq c\Big(\frac{\rho}{R}\Big)^{n+2} \int_{B_R(x_0)} |Du - (Du)_{x_0,R}|^2 dx \\ + c\int_{B_R(x_0)} |Du - Dv|^2 dx. \end{aligned} \tag{9.7}$$

As usual we want to estimate $\int_{B_R(x_0)} |Du - Dv|^2$. By a Taylor expansion

$$
\begin{aligned}
&\int_{B_R(x_0)} \big[F(Dv) - F(Du)\big]dx = \int_{B_R(x_0)} \Big[F_{p^i_\alpha}(Du)(D_\alpha v^i - D_\alpha u^i)\\
&+ \int_0^1 (1-t)F_{p^i_\alpha p^j_\beta}(tDu + (1-t)Dv)(D_\alpha v^i - D_\alpha u^i)(D_\beta v^j - D_\beta u^j)dt\Big]dx\\
&= \int_{B_R(x_0)} \Big[\int_0^1 (1-t)F_{p^i_\alpha p^j_\beta}(tDu + (1-t)Dv)\\
&\qquad\qquad \times (D_\alpha v^i - D_\alpha u^i)(D_\beta v^j - D_\beta u^j)dt\Big]dx\\
&\geq \frac{\lambda}{2}\int_{B_R(x_0)} |Du - Dv|^2 dx,
\end{aligned}
\tag{9.8}
$$

where the second identity comes from the Euler-Lagrange equation of $\mathcal{F}$ (u is a minimizer and $u - v$ is a test function) and the last inequality is due to the ellipticity of $F_{p^i_\alpha p^j_\beta}$, i.e. (8.5). Observing that

$$
|F(p) - G(p)| \leq \omega(|p - \overline{p}|^2)|p - \overline{p}|^2,
$$

being ω the modulus of continuity of $F_{p^i_\alpha p^j_\beta}$,[1] we get

$$
\begin{aligned}
\int_{B_R(x_0)} |Du - Dv|^2 dx \leq& \frac{2}{\lambda}\int_{B_R(x_0)} \big[F(Dv) - F(Du)\big]dx\\
\leq& \frac{2}{\lambda}\int_{B_R(x_0)} \Big\{\big[F(Dv) - G(Dv)\big] + \underbrace{\big[G(Dv) - G(Du)\big]}_{\leq 0}\\
&\qquad + \big[G(Du) - F(Du)\big]\Big\}dx\\
\leq& \frac{2}{\lambda}\int_{B_R(x_0)} \Big\{\omega(|Dv - \overline{p}|^2)|Dv - \overline{p}|^2 + \omega(|Du - \overline{p}|^2)|Du - \overline{p}|^2 dx\Big\}\\
\leq& c_1\Big(\int_{B_R(x_0)} |Dv - \overline{p}|^{2^*} dx\Big)^{\frac{2}{2^*}}\Big(\int_{B_R(x_0)} \omega\big(|Dv - \overline{p}|^2\big)dx\Big)^{\frac{2}{n}}\\
&+ c_1\Big(\int_{B_R(x_0)} |Du - \overline{p}|^{2^*} dx\Big)^{\frac{2}{2^*}}\Big(\int_{B_R(x_0)} \omega\big(|Du - \overline{p}|^2\big)dx\Big)^{\frac{2}{n}},
\end{aligned}
$$

where we used the boundedness of ω, so that $\omega^{\frac{n}{2}} = \omega \cdot \omega^{\frac{n}{2}-1} \leq c_2\omega$. Using the Poincaré-Sobolev inequality (Proposition 3.27) and the Caccioppoli

[1] ω is concave, bounded and $|F_{p^i_\alpha p^j_\beta}(p_1) - F_{p^i_\alpha p^j_\beta}(p_2)| \leq \omega(|p_1 - p_2|^2)$ for every $p_1, p_2 \in \mathbb{R}^{n\times m}$.

inequality (Theorem 4.4) we obtain

$$\begin{aligned}\Big(\int_{B_R(x_0)}|Dv-\overline{p}|^{2^*}dx\Big)^{\frac{2}{2^*}} &\le c_2\Big(\int_{B_R(x_0)}|Du-\overline{p}|^{2^*}dx\Big)^{\frac{2}{2^*}}\\ &\le c_3\int_{B_R(x_0)}|D^2u|^2dx \qquad (9.9)\\ &\le \frac{c_4}{R^2}\int_{B_{2R}(x_0)}|Du-\overline{p}|^2dx.\end{aligned}$$

Putting together (9.8) and (9.9) gives

$$\begin{aligned}\int_{B_R(x_0)}|Du-Dv|^2dx &\le \frac{c_5}{R^2}\Big(\int_{B_{2R}(x_0)}|Du-\overline{p}|^2dx\Big)\\ &\quad\times\Big(\int_{B_{2R}(x_0)}\Big[\omega(|Du-\overline{p}|^2)+\omega(|Dv-\overline{p}|^2)\Big]dx\Big)^{\frac{2}{n}}\\ &\underbrace{\le}_{\text{Jensen}} c_6\Big(\int_{B_{2R}(x_0)}|Du-\overline{p}|^2dx\Big)\\ &\quad\times\omega\Big(\fint_{B_{2R}(x_0)}|Du-\overline{p}|^2dx\Big)^{\frac{2}{n}}\end{aligned}$$

Inserting this into (9.7) and recalling that $\overline{p}=(Du)_{x_0,R}$ yield

$$\begin{aligned}\int_{B_\rho(x_0)}|Du-(Du)_{x_0,\rho}|^2 &\le c_7\Big[\Big(\frac{\rho}{R}\Big)^{n+2}+\omega\Big(\fint_{B_{2R}(x_0)}|Du-(Du)_{x_0,2R}|^2dx\Big)\Big]\\ &\quad\times\int_{B_{2R}(x_0)}|Du-(Du)_{x_0,2R}|^2dx\end{aligned}$$

By Lemma 5.13 we infer that, if $\omega\Big(\fint_{B_{2R}(x_0)}|Du-(Du)_{x_0,2R}|^2dx\Big)$ is small enough, then for any $\beta<n+2$

$$\int_{B_\rho(x_0)}|Du-(Du)_{x_0,\rho}|^2dx\le c_8\rho^\beta. \qquad (9.10)$$

As the function

$$x_0\mapsto\int_{B_\rho(x_0)}|Du-(Du)_{x_0,\rho}|^2dx$$

is continuous, (9.10) holds for ρ small in a neigborhood U of any point $x_0\in\Omega\backslash\Sigma(u)$, thus $Du\in\mathcal{L}^{2,\beta}(U)$ and is Hölder continuous by Campanato's lemma. The estimate on $\dim^{\mathcal{H}}(\Sigma(u))$ is a direct consequence of Proposition 9.21 below. □

9.2.2 Partial regularity of solutions to quasilinear elliptic systems

The proof above is taken from [45]. It shows that a key point is the higher integrability of the gradient of the solution. The same idea yields partial regularity of solutions of systems in variation, Theorem 9.2 below, due to Giusti-Miranda [53] and Morrey [78]. Recall that, thanks to Proposition 8.6, given a minimizer u of a variational integral of the form (8.4) satifying the growth condition (8.5), the derivatives of u satisfy the elliptic system (8.7). The proof below is taken from [40] [46].

Theorem 9.2 *Let $u \in W^{1,2}(\Omega, \mathbb{R}^m)$ be a solution of*

$$D_\alpha \big(A_{ij}^{\alpha\beta}(x, u(x)) D_\beta u^i(x)\big) = 0, \tag{9.11}$$

with coefficients $A_{ij}^{\alpha\beta} : \Omega \times M^{m\times n} \to \mathbb{R}$ uniformly continuous, bounded and satisfying the Legendre condition: there is a $\lambda > 0$ such that

$$A_{ij}^{\alpha\beta}(x,u)\xi_\alpha^i \xi_\beta^j \geq \lambda|\xi|^2 \quad \textit{for every } (x,u) \in \Omega \times \mathbb{R}^m.$$

Then, defined the singular set $\Sigma(u)$ as

$$\Sigma(u) := \Big\{x \in \Omega : \liminf_{\rho\to 0} \fint_{B_\rho(x)} |u - u_{x,\rho}|^2 dx > 0\Big\},$$

we have that $u \in C^{0,\sigma}(\Omega \backslash \Sigma(u))$ for every $\sigma \in (0,1)$ and

$$\dim^{\mathcal{H}} \Sigma(u) < n-2.$$

Remark 9.3 The singular set may be characterized in terms of Du by

$$\Sigma(u) := \Big\{x \in \Omega : \liminf_{\rho\to 0} \frac{1}{\rho^{n-2}} \int_{B_\rho(x)} |Du|^2 dx > 0\Big\}.$$

Indeed, if

$$\liminf_{\rho\to 0} \frac{1}{\rho^{n-2}} \int_{B_\rho(x)} |Du|^2 dx = 0, \tag{9.12}$$

then

$$\liminf_{\rho\to 0} \fint_{B_\rho(x)} |u - u_{x,\rho}|^2 dx = 0 \tag{9.13}$$

by Poincaré's inequality. The converse is consequence of Caccioppoli's inequality.

The following lemma will be crucial in the proof of the theorem.

Lemma 9.4 *In the same hypothesis of Theorem 9.2 there exists $p > 2$ and $c = c(n, m, \lambda, \Lambda)$ such that $Du \in L^p_{\mathrm{loc}}(\Omega)$ and*

$$\Big(\fint_{B_R(x_0)} |Du|^p dx \Big)^{\frac{1}{p}} \le c \Big(\fint_{B_{2R}(x_0)} |Du|^2 dx \Big)^{\frac{1}{2}}.$$

Proof. By Caccioppoli and Sobolev-Poincaré's inequalities (Theorem 4.4 and Proposition 3.27) we have

$$\begin{aligned}\int_{B_R(x_0)} |Du|^2 dx &\le \frac{c_1}{R^2} \int_{B_{2R}(x_0)} |u - u_{x_0,2R}|^2 dx \\ &\le \frac{c_2}{R^2} \Big(\int_{B_{2R}(x_0)} |Du|^{2_*} dx \Big)^{\frac{2}{2_*}},\end{aligned}$$

where $2_* := \frac{2n}{n+2}$, so that $(2_*)^* = 2$. Dividing by R^n we obtain

$$\Big(\fint_{B_R(x_0)} |Du|^2 dx \Big)^{\frac{1}{2}} \le \Big(\fint_{B_{2R}(x_0)} |Du|^{2_*} dx \Big)^{\frac{1}{2_*}},$$

so that we may apply Theorem 6.38 to $f := |Du|^{2_*}$ with $q = 2$. □

Proof of Theorem 9.2. Fix $x_0 \in \Omega$ and $R > 0$ such that $B_{2R}(x_0) \subset \Omega$. Freezing the coefficients we get

$$\begin{aligned}&\int_{B_R(x_0)} A^{\alpha\beta}_{ij}(x_0, u_{x_0,R}) D_\beta u^i D_\alpha \varphi^j dx \\ &\qquad = \int_{B_R(x_0)} \big[A^{\alpha\beta}_{ij}(x_0, u_{x_0,R}) - A^{\alpha\beta}_{ij}(x, u) \big] D_\beta u^i D_\alpha \varphi^j dx,\end{aligned} \tag{9.14}$$

for all $\varphi \in W^{1,2}_0(B_R(x_0), \mathbb{R}^m)$. Write $u = v + (u - v)$, where v is the solution of

$$\begin{cases} \displaystyle\int_{B_R(x_0)} A^{\alpha\beta}_{ij}(x_0, u_{x_0,R}) D_\beta v^i D_\alpha \varphi^j = 0, & \forall \varphi \in W^{1,2}_0(B_R(x_0), \mathbb{R}^m) \\ v - u \in W^{1,2}_0(B_R(x_0), \mathbb{R}^m). \end{cases}$$

Since Dv satisfies (5.13) by Proposition 5.8, similar to (5.22), we obtain

$$\int_{B_\rho(x_0)} |Du|^2 dx \le c \Big(\frac{\rho}{R} \Big)^n \int_{B_R(x_0)} |Du|^2 dx + c \int_{B_R(x_0)} |D(u - v)|^2 dx.$$

Inserting $\varphi = u - v$ in (9.14), using ellipticity of A and $ab \le \frac{a^2}{\varepsilon} + \varepsilon b^2$ yields

$$\begin{aligned}\int_{B_R(x_0)} |D(u - v)|^2 dx &\le \int_{B_R(x_0)} |A(x_0, u_{x_0,R}) - A(x, u)|^2 |Du|^2 dx \\ &\le \int_{B_R(x_0)} \omega(|x - x_0|^2 + |u - u_{x_0,R}|^2)^2 |Du|^2 dx,\end{aligned}$$

where ω is the modulus of continuity of A, bounded, concave, and satisfying $\lim_{t\to 0^+}\omega(t)=0$. You may observe that we have followed the proof of Theorem 5.17, but now we cannot say that ω is small if R is, since $|u-u_{x_0,R}|$ may be large on $B_R(x_0)$. Using the higher integrability of the gradient and the boundedness of ω (so that $\omega^q \le c_1(q)\omega$ if $q>1$), we obtain with $p>2$ given by Lemma 9.4 and Hölder's inequality

$$
\begin{aligned}
&\int_{B_R(x_0)} \omega(|x-x_0|^2+|u-u_R|^2)^2|Du|^2dx\\
&\le c_2\Big(\int_{B_R(x_0)}|Du|^p dx\Big)^{\frac{2}{p}}\Big(\int_{B_R(x_0)}\omega(|x-x_0|^2+|u-u_R|^2)^{2\frac{p}{p-2}}dx\Big)^{\frac{p-2}{p}}\\
&\le c_3\int_{B_R(x_0)}|Du|^2dx\Big(\fint_{B_R(x_0)}\omega(|x-x_0|^2+|u-u_R|^2)dx\Big)^{\frac{p-2}{p}}.
\end{aligned}
$$

Since ω is concave, we can use Jensen's inequality and get

$$
\fint_{B_R(x_0)}\omega(|x-x_0|^2+|u-u_R|^2)dx \le \omega\Big(R^2+\fint_{B_R(x_0)}|u-u_R|^2dx\Big).
$$

In conclusion

$$
\begin{aligned}
\int_{B_\rho(x_0)}|Du|^2dx \le c\Big[\Big(\frac{\rho}{R}\Big)^n+\omega\Big(R^2+\fint_{B_R(x_0)}|u-u_{x_0,R}|^2dx\Big)^{\frac{p-2}{p}}\Big]&\\
\times\int_{B_R(x_0)}|Du|^2dx.&
\end{aligned}
\tag{9.15}
$$

If $x_0\in\Omega\backslash\Sigma(u)$, we may take $R>0$ small enough in order to have

$$
\omega\Big(R^2+\fint_{B_R(x_0)}|u-u_{x_0,R}|^2dx\Big)^{\frac{p-2}{p}}<\varepsilon_0
$$

and apply Lemma 5.13 to obtain that, for every $\rho<R$, we have

$$
\int_{B_\rho(x_0)}|Du|^2dx\le c\rho^{n-\varepsilon}, \tag{9.16}
$$

where ε can be taken arbitrarily small, if R is chosen correspondingly small (depending on ε). For $x\in B_{\frac{R}{2}}(x_0)$, observe that

$$
\begin{aligned}
\fint_{B_{\frac{R}{2}}(x)}|u-u_{x,\frac{R}{2}}|^2dx &\le \fint_{B_{\frac{R}{2}}(x)}|u-u_{x_0,R}|^2dx\\
&\le 2^n\fint_{B_R(x_0)}|u-u_{x_0,R}|^2dx,
\end{aligned}
$$

thus what we have done applies to every x in $B_{\frac{R}{2}}(x_0)$, with the consequence that $Du \in L^{2,n-\varepsilon}(B_{\frac{R}{2}}(x_0))$ for every $\varepsilon > 0$ (R depending on ε). Thanks to Morrey's Theorem 5.7, we have that u is Hölder continuous in $B_{\frac{R}{2}}(x_0)$, hence locally in $\Omega \setminus \Sigma(u)$. Now the estimate on the dimension of $\Sigma(u)$ follows immediately from the characterization of the singular set in Remark 9.3, and by Lemma 9.4 and Proposition 9.21. □

Remark 9.5 The conclusion of the theorem above can be made more precise if the coefficients do not depend on x: $A = A(u)$. In this case inequality (9.15) becomes

$$\int_{B_\rho(x_0)} |Du|^2 dx \le c\Big[\Big(\frac{\rho}{R}\Big)^n + \omega\Big(\fint_{B_R(x_0)} |u - u_{x_0,R}|^2 dx\Big)^{2\frac{p-2}{p}}\Big] \times \int_{B_R(x_0)} |Du|^2 dx.$$

By Lemma 5.13 and the discussion at the end of the proof, we infer that there exists $\varepsilon = \varepsilon(n, m, \lambda, \sup|A|, \omega)$ such that if

$$\fint_{B_R(x_0)} |u - u_{x_0,R}|^2 dx \le \varepsilon,$$

then $u \in C^{0,\alpha}(B_{\frac{R}{2}}(x_0))$ and we have the following estimate

$$\|u\|_{C^{0,\alpha}(B_{\frac{R}{2}}(x_0))} \le c_1 \|Du\|_{L^{2,n-\varepsilon}(B_{\frac{R}{2}}(x_0))} \le c(n, m, \varepsilon, A)\|Du\|_{L^2(B_R(x_0))}.$$

A similar conclusion holds in the general case $A = A(x, u)$ for R small enough.

9.2.3 Partial regularity of solutions to quasilinear elliptic systems with quadratic right-hand side

Of interest are also systems of quasilinear equations with right-hand side that grows naturally, i.e., systems of the type

$$-D_\beta\big(A_{ij}^{\alpha\beta}(x, u(x))D_\alpha u^i\big) = f_j\big(x, u(x), Du(x)\big), \tag{9.17}$$

where $A_{ij}^{\alpha\beta}$ and f_j are smooth functions so that A is very strongly elliptic, i.e.

$$A_{ij}^{\alpha\beta}(x, u)\xi_\alpha^i \xi_\beta^j \ge \lambda|\xi|^2, \qquad \forall \xi \in \mathbb{R}^{n\times m}, \tag{9.18}$$

and f satisfies the so-called *natural growth condition*

$$|f(x, u, p)| \le a(M)|p|^2, \quad \forall x, u, p \text{ with } |u| \le M, \tag{9.19}$$

where a is a nondecreasing function, or even *diagonal systems* as

$$-\Delta u^i(x) = f^i(x, u(x), Du(x)). \tag{9.20}$$

As first noticed by S. Hildebrandt and K. O. Widman, weak solutions of (9.20) need not be regular everywhere. Indeed $u(x) := \frac{x}{|x|}$ is a weak solution in $\mathbb{R}^3$ of the system

$$-\Delta u = u|Du|^2.$$

But the argument in the proof of Theorem 9.2 easily extends to prove

Theorem 9.6 *Let $u \in W^{1,2} \cap L^\infty(\Omega, \mathbb{R}^m)$ be a weak solution of the system* (9.17) *where* (9.18), (9.19) *and $|u| \leq M$ hold. Assume*

$$2a(M)M \leq \lambda. \tag{9.21}$$

Then u is Hölder continuous in an open set $\Omega_0 \subset \Omega$ and the closed singular set has Hausdorff dimension strictly less than $n-2$.

Proof. In fact (9.21) allows us to prove Caccioppoli's inequality and, consequently, higher integrability of the gradient of u. The rest of the proof proceeds exactly as previously. □

The next theorem provides also the possibility of bootstrapping regularity by means of Schauder results.

Theorem 9.7 *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ be a weak solution to system*

$$-D_\beta\big(A^{\alpha\beta}_{ij}(x) D_\alpha u^i\big) = f_j\big(x, u(x), Du(x)\big). \tag{9.22}$$

Suppose that $A^{\alpha\beta}_{ij} \in C^{0,\mu}(\Omega)$ for some $\mu \in (0,1)$ and satisfy the Legendre condition, f is smooth and satisfies (9.19). *If $u \in C^{0,\mu}(\Omega_0)$ for some open set $\Omega_0 \subset \Omega$, then $Du \in C^{0,\mu}(\Omega)$.*

Proof. For the sake of simplicity, we present the details of the proof in the simpler case of diagonal systems, leaving the rest for the reader. So let $u \in W^{1,2} \cap C^{0,\mu}(\Omega_0, \mathbb{R}^m)$ be a weak solution of (9.20). For all $x_0 \in \Omega_0$, $\rho < R < \operatorname{dist}(x_0, \partial\Omega_0)$, letting $H \in W^{1,2}(B_R(x_0), \mathbb{R}^m)$ be the solution to

$$\Delta H = 0, \quad H - u \in W^{1,2}_0(B_R(x_0), \mathbb{R}^m),$$

we have

$$\int_{B_\rho(x_0)} |Du|^2 dx \leq c\Big(\frac{\rho}{R}\Big)^n \int_{B_R(x_0)} |Du|^2 dx + c\int_{B_R(x_0)} |D(u-H)|^2 dx \tag{9.23}$$

$$\begin{aligned}\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 dx \leq{}& c\Big(\frac{\rho}{R}\Big)^{n+2} \int_{B_R(x_0)} |Du - (Du)_{x_0,R}|^2 dx \\ &+ c\int_{B_R(x_0)} |D(u-H)|^2 dx,\end{aligned} \tag{9.24}$$

From (9.22) we infer

$$\begin{aligned}\int_{B_R(x_0)} |D(u-H)|^2 dx &= \int_{B_R(x_0)} (u-H)\cdot f(x,u,Du)dx \\ &\le c\int_{B_R(x_0)} |u-H||Du|^2 dx \qquad (9.25)\\ &\le cR^\mu \int_{B_R(x_0)} |Du|^2 dx,\end{aligned}$$

since $u \in C^{0,\mu}(\Omega_0)$ implies $H \in C^{0,\mu}(\Omega_0)$. Therefore we conclude from (9.23) and Lemma 5.13 that for ρ sufficiently small

$$\int_{B_\rho(x_0)} |Du|^2 dx \le c_\sigma \rho^{n-2+2\sigma}, \quad \text{for all } \sigma,\ 0<\sigma<1,$$

and, as in (9.25), for some $\varepsilon > 0$

$$\int_{B_R(x_0)} |D(u-H)|^2 dx \le cR^{n+\varepsilon}.$$

The estimate (9.24) then yields, as in Schauder theory, that Du is Hölder continuous with some small positive exponent, which is enough to get Hölder continuity of Du with all exponents. □

For later use, we also consider the case of continuous solution.

Theorem 9.8 *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ be a weak solution to system*

$$-\Delta u^i = f_j\big(x, u(x), Du(x)\big), \qquad (9.26)$$

where f is smooth and satisfies (9.19). *If $u \in C^0(\Omega)$, we get $u \in C^{0,\mu}_{\mathrm{loc}}(\Omega)$.*

Proof. Following the proof of Theorem 9.7, we obtain, instead of (9.25)

$$\int_{B_R(x_0)} |D(u-H)|^2 dx \le c\omega(R) \int_{B_R(x_0)} |Du|^2 dx, \qquad (9.27)$$

where ω is the modulus of continuity of u. Then, again with Lemma 5.13, we get

$$\int_{B_\rho(x_0)} |Du|^2 dx \le c_\varepsilon \rho^{n-\varepsilon}$$

for any given $\varepsilon > 0$ and ρ sufficiently small. Taking into account the inequality of Poincaré, we then get

$$\int_{B_\rho(x_0)} |u - u_{x_0,\rho}|^2 dx \le c_\varepsilon \rho^{n+2-\varepsilon},$$

hence Hölder continuity of u follows from Campanato's theorem (Theorem 5.5). $\square$

Remark 9.9 Actually, bounded solutions of diagonal systems for which (9.21) (and, in fact, a weaker condition) holds are regular everywhere, as shown by S. Hildebrandt and K. O. Widman, see [61], while bounded solutions of scalar equations with right-hand side of natural growth are everywhere regular without assuming any smallness condition like (9.21), see [67]. We shall not deal with these topics, the interested reader is referred, besides the works already mentioned, to [37] for an account.

Here we would like to present an alternative proof of Theorem 9.6. This proof, taken from [29], has its origin in Simon's proof of the regularity theorem of Allard, see Chapter 11, compare [97] and [11], and avoids the use of the higher integrability result. To illustrate the ideas, we confine ourselves to the case of diagonal systems (9.20); the reader can easily supply the missing details to treat the general case.

Let u be a bounded weak solution of (9.20). Fix a ball $B_R(x_0) \Subset \Omega$ and let H be a harmonic function in $B_R(x_0)$ with

$$\int_{B_R(x_0)} |DH|^2 dx \le \int_{B_R(x_0)} |Du|^2 dx.$$

As we have seen several times, we then have for $\rho < R$

$$\begin{aligned}\int_{B_\rho(x_0)} |u - u_{x_0,\rho}|^2 dx &\le 2\int_{B_\rho(x_0)} |u-H|^2 dx + 2\int_{B_\rho(x_0)} |H - H_{x_0,\rho}|^2 dx \\ &\le 2\int_{B_R(x_0)} |u-H|^2 dx + c\Big(\frac{\rho}{R}\Big)^{n+2} \int_{B_R(x_0)} |H - H_{x_0,R}|^2 dx \\ &\le c\int_{B_R(x_0)} |u-H|^2 dx + c\Big(\frac{\rho}{R}\Big)^{n+2} \int_{B_R(x_0)} |u - u_{x_0,R}|^2 dx,\end{aligned} \tag{9.28}$$

and the point is to estimate the last term. This is accomplished by means of the following two propositions.

Proposition 9.10 *Given any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $g \in W^{1,2}(B_R(x_0), \mathbb{R}^m)$ satisfying*

$$⨍_{B_R(x_0)} |Dg|^2 dx \le 1,$$

$$\left| ⨍_{B_R(x_0)} DgD\varphi dx \right| \le \delta \sup_{B_R(x_0)} |D\varphi|, \qquad \forall \varphi \in C^1_c(B_R(x_0), \mathbb{R}^m),$$

then there exists a harmonic function H *with* $\fint_{B_R(x_0)} |DH|^2 dx \leq 1$ *satisfying*

$$\frac{1}{R^{n+2}} \int_{B_R(x_0)} |H - g|^2 dx \leq \varepsilon.$$

Proof. We can assume $x_0 = 0$, $R = 1$, as the result will follow by rescaling. Were the conclusion false, we could find $\varepsilon > 0$, $g_k \in W^{1,2}(B_1(0), \mathbb{R}^m)$ such that

$$\begin{cases} \fint_{B_1(0)} |g_k - H|^2 dx \geq \varepsilon & \forall H \text{ harmonic, } \fint_{B_1(0)} |DH|^2 \leq 1 \\ \fint_{B_1(0)} |Dg_k|^2 dx \leq 1 & \\ \left| \int_{B_1(0)} Dg_k D\varphi dx \right| \leq \frac{1}{k} \sup_{B_1(0)} |D\varphi| & \forall \varphi \in C^1_c(B_R(x_0), \mathbb{R}^m). \end{cases} \tag{9.29}$$

Without loss of generality we can assume $\fint_{B_1(0)} g_k dx = 0$; therefore, by Poincaré's inequality, the g_k are equibounded in $W^{1,2}$, and up to a subsequence

$$g_k \rightharpoonup g \text{ weakly in } W^{1,2}, \qquad g_k \to g \text{ in } L^2, \qquad \fint_{B_1(0)} |Dg|^2 dx \leq 1.$$

Consequently

$$\int_{B_1(0)} DgD\varphi dx = 0 \qquad \forall \varphi \in C^1_c(B_R(x_0), \mathbb{R}^m),$$

and that contradicts (9.29) with H replaced by g and k large enough. $\square$

Proposition 9.11 *Given any* $\varepsilon > 0$*, there exists* $C > 0$ *such that for any* $B_R(x_0)$ *and any* $g \in W^{1,2}(B_R(x_0), \mathbb{R}^m)$ *we have*

$$\begin{aligned} \inf_{H \in \mathcal{A}_g} \left(\fint_{B_R(x_0)} |H - g|^2 dx \right)^{\frac{1}{2}} \leq & C \sup_{\substack{\varphi \in C^\infty_c(B_R(x_0), \mathbb{R}^m) \\ \|D\varphi\|_\infty \leq \frac{1}{R}}} \frac{1}{R^{n-2}} \int_{B_R(x_0)} DgD\varphi dx \\ & + \varepsilon \left(\frac{1}{R^{n-2}} \int_{B_R(x_0)} |Dg|^2 dx \right)^{\frac{1}{2}}, \end{aligned}$$

where

$$\mathcal{A}_g := \left\{ H \text{ harmonic on } B_R(x_0) \,\middle|\, \fint_{B_R(x_0)} |DH|^2 dx \leq \fint_{B_R(x_0)} |Dg|^2 dx \right\}$$

Proof. By a rescaling argument it suffices to consider the case $x_0 = 0$, $R = 1$. Set $B := B_1(0)$, and let δ be the constant in Proposition 9.10. First assume that

$$\sup\left\{\int_B DgD\varphi dx \;\middle|\; \varphi \in C_c^\infty(B_1(0), \mathbb{R}^m),\ \|D\varphi\|_\infty < 1\right\} \le \delta\|Dg\|_{L^2}. \tag{9.30}$$

Then

$$\begin{aligned}
&\inf\left\{\left(\int_B |H-g|^2 dx\right)^{\frac{1}{2}} \;\middle|\; H \text{ harmonic in } B\right\}\\
&= \|Dg\|_{L^2} \inf\left\{\left(\int_B \left|\frac{H}{\|Dg\|_{L^2}} - \frac{g}{\|Dg\|_{L^2}}\right|^2 dx\right)^{\frac{1}{2}} \;\middle|\; H \text{ harmonic in } B\right\}\\
&\le \varepsilon\|Dg\|_{L^2}
\end{aligned}$$

by Proposition 9.10. If (9.30) does not hold, we have by the Poincaré inequality

$$\begin{aligned}
&\inf\left\{\left(\int_B |H-g|^2 dx\right)^{\frac{1}{2}} \;\middle|\; H \text{ harmonic in } B\right\}\\
&\le \left(\int_B \left|g(x) - \fint_B g(y)dy\right|^2 dx\right)^{\frac{1}{2}} \le c\|Dg\|_{L^2}\\
&\le \frac{c}{\delta}\sup\left\{\int_B DgD\varphi dx \;\middle|\; \varphi \in C_c^\infty(B, \mathbb{R}^m),\ \|D\varphi\|_\infty \le 1\right\}.
\end{aligned}$$

This completes the proof. □

Returning to (9.28), we now estimate the term $\fint_{B_R(x_0)} |u-H|^2 dx$ using Proposition 9.11 by

$$\begin{aligned}
&\left|\frac{1}{R^{n-2}}\int_{B_R(x_0)} DuD\varphi dx\right|^2 + \frac{\varepsilon^2}{R^{n-2}}\int_{B_R(x_0)} |Du|^2 dx\\
&\qquad\le \left(\frac{1}{R^{n-2}}\int_{B_R(x_0)} |Du|^2 dx + \varepsilon^2\right)\frac{1}{R^{n-2}}\int_{B_R(x_0)} |Du|^2 dx.
\end{aligned}$$

Next we observe that the smallness condition (9.21) allows us to prove Caccioppoli's inequality

$$\int_{B_R(x_0)} |Du|^2 dx \le \frac{c}{R^2}\int_{B_{2R}(x_0)} |u - u_{x_0,2R}|^2 dx,$$

to conclude that

$$\begin{aligned}
\int_{B_\rho(x_0)} |u-u_{x_0,\rho}|^2 dx \le c\left[\left(\frac{\rho}{R}\right)^{n+2} + \fint_{B_R(x_0)} |u-u_{x_0,R}|^2 dx + \varepsilon\right]\\
\times \int_{B_R(x_0)} |u - u_{x_0,R}|^2 dx.
\end{aligned}$$

Now if

$$\fint_{B_R(x_0)} |u - u_{x_0,R}|^2 dx \le \varepsilon$$

is true at a point x_0, it remains true in a neighborhood of x_0, if we allow for a larger c. Then the Hölder continuity of u follows by Lemma 5.13 and Campanato's lemma, as we have seen several times.

Remark 9.12 The proof outlined above is indirect. It is to be mentioned that indirect methods, a blow-up technique that originates in the works of De Giorgi [25] and in [5] were used in the original works of Giusti-Miranda and Morrey. Indirect methods were also used for quasiconvex functionals by [94] and [30], for a direct approach see [38].

9.2.4 Partial regularity of minimizers of non-differentiable quadratic functionals

The study of the regularity of non-differentiable functionals differs from the study of smooth functionals in the lack of the Euler-Lagrange equation, and consequently, of Caccioppoli's inequality. On the other hand, for quadratic functionals, i.e. functionals of the form

$$\mathcal{F}(u) := \int_\Omega A_{ij}^{\alpha\beta}(x,u) D_\alpha u^i D_\beta u^j \, dx,$$

Caccioppoli's inequality is still available, as we see in the following

Proposition 9.13 (Caccioppoli inequality) *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ be a local minimizer of the functional*

$$\mathcal{F}(u) := \int_\Omega A_{ij}^{\alpha\beta}(x,u) D_\alpha u^i D_\beta u^j \, dx,$$

with $A_{ij}^{\alpha\beta}$ bounded and elliptic: $\lambda|\xi|^2 \le A_{ij}^{\alpha\beta}\xi_\alpha^i \xi_\beta^j \le \Lambda|\xi|^2$. Then there exists a constant $c = c(\lambda, \Lambda)$ such that

$$\int_{B_{\frac{R}{2}}(x_0)} |Du|^2 dx \le \frac{c}{R^2} \int_{B_R(x_0)\setminus B_{\frac{R}{2}}(x_0)} |u - u_{x_0,R}|^2 dx \tag{9.31}$$

for all $x_0 \in \Omega$, $0 < R < \mathrm{dist}(x_0, \partial\Omega)$.

Proof. Take $x_0 \in \Omega$, $0 < t < s < R$, and choose a cut-off function $\eta \in C_c^\infty(\Omega)$ with

1. $\mathrm{spt}\,\eta \subset B_s(x_0)$ and $\eta \equiv 1$ in $B_t(x_0)$;
2. $|D\eta| \le \frac{2}{s-t}$.

Consider the test function

$$v := u - \eta(u - u_{x_0,R})$$

and use the minimality of u, together with the ellipticity and boundedness of $A_{ij}^{\alpha\beta}$ to get

$$\begin{aligned}\int_{B_s(x_0)} |Du|^2 dx &\le \frac{1}{\lambda}\int_{B_s(x_0)} A_{ij}^{\alpha\beta}(x,u) D_\alpha u^i D_\beta u^j\, dx \\ &\le \frac{1}{\lambda}\int_{B_s(x_0)} A_{ij}^{\alpha\beta}(x,v) D_\alpha v^i D_\beta v^j\, dx \\ &\le \frac{\Lambda}{\lambda}\int_{B_s(x_0)} |D[u - \eta(u - u_{x_0,R})]|^2 dx \\ &\le c_1\Big\{\int_{B_s(x_0)} (1-\eta)^2 |Du|^2 dx \\ &\qquad + \int_{B_s(x_0)} |u - u_{x_0,R}|^2 |D\eta|^2 dx\Big\}.\end{aligned}$$

Therefore

$$\begin{aligned}\int_{B_t(x_0)} |Du|^2 dx \le c_1 &\int_{B_s(x_0)\setminus B_t(x_0)} |Du|^2 dx \\ &+ \frac{4c_1}{(s-t)^2}\int_{B_s(x_0)\setminus B_t(x_0)} |u - u_{x_0,R}|^2 dx.\end{aligned}$$

Adding c_1 times the left-hand side to both sides, we get

$$\begin{aligned}\int_{B_t(x_0)} |Du|^2 dx \le \frac{c_1}{c_1+1} &\int_{B_s(x_0)} |Du|^2 dx \\ &+ \frac{c_2}{(s-t)^2}\int_{B_s(x_0)\setminus B_t(x_0)} |u - u_{x_0,R}|^2 dx.\end{aligned}$$

Setting $\phi(s) := \int_{B_s(x_0)} |Du|^2 dx$, Lemma 8.18 implies that there exists a constant c depending on $c_1 = c_1(\lambda, \Lambda)$ such that

$$\int_{B_\rho(x_0)} |Du|^2 dx \le \frac{c}{(R-\rho)^2}\int_{B_R(x_0)} |u - u_{x_0,R}|^2 dx,$$

and the result follows taking $\rho = \frac{R}{2}$. $\square$

With the same proof of Lemma 9.4 we get

Lemma 9.14 *In the hypothesis of Proposition 9.13, there exist $p > 2$ and $c = c(n, m, \lambda, \Lambda)$ such that $Du \in L^p_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ and*

$$\Big(\fint_{B_R(x_0)} |Du|^p dx\Big)^{\frac{1}{p}} \le c\Big(\fint_{B_{2R}(x_0)} |Du|^2 dx\Big)^{\frac{1}{2}},$$

whenever $B_{2R}(x_0) \Subset \Omega$.

The following result is due to Giaqunta and Giusti [41]

Theorem 9.15 *Consider a local minimizer u of the variational integral*

$$\mathcal{F}(u) := \int_\Omega A_{ij}^{\alpha\beta}(x,u) D_\alpha u^i D_\beta u^j \, dx,$$

where the coefficients $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$ are uniformily continuous in (x,u) and satisfy the Legendre condition

$$A_{ij}^{\alpha\beta}(x,u)\xi_\alpha^i \xi_\beta^j \geq \lambda|\xi|^2, \quad \forall x \in \Omega, u \in \mathbb{R}^m, \xi \in \mathbb{R}^{n\times m}.$$

Then for any $\sigma \in (0,1)$ there exist $\varepsilon_0 = \varepsilon_0(n,m,\lambda,\omega,\sigma)$ (ω being the modulus of continuity of $A_{ij}^{\alpha\beta}$) such that $u \in C^{0,\sigma}_{\mathrm{loc}}(\Omega \backslash \Sigma(u))$, where

$$\Sigma(u) := \Big\{ x \in \Omega : \liminf_{R\to 0} \frac{1}{R^{n-2}} \int_{B_R(x_0)} |Du|^2 dx > \varepsilon_0 \Big\}.$$

Moreover $\dim^{\mathcal{H}}(\Sigma(u)) < n-2$.

Proof. Step 1. Fix $x_0 \in \Omega$ and R with $0 < 2R < \mathrm{dist}(x_0, \partial\Omega)$, and consider the functional

$$\mathcal{F}_0(v) := \int_{B_R(x_0)} A_{ij}^{\alpha\beta}(x_0, u_{x_0,R}) D_\alpha v^i D_\beta v^j \, dx.$$

Thanks to the ellipticity of the coefficients, the functional is coercive thus, following the proof of Theorem 3.29, it admits a unique minimizer v in the class

$$\Big\{ \zeta \in W^{1,2}(B_R(x_0)) : \zeta - u \in W_0^{1,2}(B_R(x_0)) \Big\}.$$

By Lemma 9.14, $Du \in L^p(B_R(x_0))$ for some $p > 2$. Since the coefficients of $\mathcal{F}_0$ are constant, the corresponding Euler-Lagrange equation is elliptic with constant coefficients:

$$\int_{B_R(x_0)} A_{ij}^{\alpha\beta}(x_0, u_{x_0,R}) D_\alpha v^i D_\beta \varphi^j \, dx = 0, \quad \forall \varphi \in W_0^{1,2}(\Omega, \mathbb{R}^m). \tag{9.32}$$

Step 2. By L^p-theory, Theorem 7.1, we have

$$\int_{B_R(x_0)} |Dv|^p dx \leq c_1 \int_{B_R(x_0)} |Du|^p dx, \quad c_1 = c_1(p,\lambda,\Lambda), \tag{9.33}$$

and by Proposition 5.8, for every $0 < \rho < R$

$$\int_{B_\rho(x_0)} |Dv|^2 dx \leq c_2 \Big(\frac{\rho}{R}\Big)^n \int_{B_R(x_0)} |Dv|^2 dx.$$

If we set $w := u - v$, the last equation easily implies

$$\int_{B_\rho(x_0)} |Du|^2 dx \le c_3 \Big(\frac{\rho}{R}\Big)^n \int_{B_R(x_0)} |Du|^2 dx + c_3 \int_{B_R(x_0)} |Dw|^2 dx. \quad (9.34)$$

Step 3. To estimate the term $\int_{B_R(x_0)} |Dw|^2 dx$ we first observe that

$$\lambda \int_{B_R(x_0)} |Dw|^2 dx \le \int_{B_R(x_0)} A_{ij}^{\alpha\beta}(x_0, u_{x_0,R}) D_\alpha w^i D_\beta w^j dx,$$

and by (9.32) with $\varphi = w \in W_0^{1,2}(\Omega, \mathbb{R}^m)$

$$\begin{aligned}
&\int_{B_R(x_0)} A_{ij}^{\alpha\beta}(x_0, u_{x_0,R}) D_\alpha w^i D_\beta w^j dx \\
&= \int_{B_R(x_0)} A_{ij}^{\alpha\beta}(x_0, u_{x_0,R}) D_\alpha u^i D_\beta w^j dx \\
&= \int_{B_R(x_0)} \big[A_{ij}^{\alpha\beta}(x_0, u_{x_0,R}) - A_{ij}^{\alpha\beta}(x, u)\big] D_\alpha (u^i + v^i) D_\beta w^j dx \\
&\quad + \int_{B_R(x_0)} \big[A_{ij}^{\alpha\beta}(x, v) - A_{ij}^{\alpha\beta}(x, u)\big] D_\alpha v^i D_\beta v^j dx \\
&\quad + \int_{B_R(x_0)} A_{ij}^{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^j dx - \int_{B_R(x_0)} A_{ij}^{\alpha\beta}(x, v) D_\alpha v^i D_\beta v^j dx.
\end{aligned}$$

The sum of the last two terms is non-positive because u is a minimizer. For the other two terms, after using $ab \le \varepsilon a^2 + \frac{b^2}{\varepsilon}$, we get

$$\begin{aligned}
&\int_{B_R(x_0)} |Dw|^2 dx \\
&\le c_4 \int_{B_R(x_0)} \big[|Du|^2 + |Dv|^2\big] \Big(\omega(R^2 + |u - u_{x_0,R}|) + \omega(|u - v|^2)\Big) dx,
\end{aligned}$$

where ω is the modulus of continuity of $A_{ij}^{\alpha\beta}$.[2] Set

$$\omega_1 = \omega(R^2 + |u - u_{x_0,R}|), \quad \omega_2 = \omega(|u - v|^2).$$

Using the boundedness of ω and the higher integrability of Du we get

$$\begin{aligned}
&\int_{B_R(x_0)} |Du|^2 (\omega_1 + \omega_2) dx \\
&\qquad \le c_5 \Big(\int_{B_R(x_0)} |Du|^p dx\Big)^{\frac{2}{p}} \Big(\int_{B_R(x_0)} (\omega_1 + \omega_2) dx\Big)^{1 - \frac{2}{p}} \\
&\qquad \le c_6 \int_{B_{2R}(x_0)} |Du|^2 dx \Big(\fint_{B_R(x_0)} (\omega_1 + \omega_2) dx\Big)^{1 - \frac{2}{p}},
\end{aligned}$$

[2] $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is a concave increasing function satisfying $\lim_{r \to 0^+} \omega(r) = 0$ and
$$|A_{ij}^{\alpha\beta}(x, u) - A_{ij}^{\alpha\beta}(y, v)| \le \omega(|x - y|^2 + |u - v|^2).$$

and by (9.33) together with the preceding equation,

$$\int_{B_R(x_0)} |Dv|^2(\omega_1+\omega_2)dx \le c_6 \int_{B_R(x_0)} |Du|^2 dx \Big(\fint_{B_{2R}(x_0)} (\omega_1+\omega_2)dx\Big)^{1-\frac{2}{p}}.$$

Now by the concavity of ω, we get

$$\begin{aligned}
\fint_{B_R(x_0)} \omega_2 dx &\le \omega\Big(R^2 + \fint_{B_R(x_0)} |u-v|^2 dx\Big) \\
&\underbrace{\le}_{\text{Poincaré}} \omega\Big(R^2 + c_7 \frac{1}{R^{n-2}} \int_{B_R(x_0)} |Dw|^2 dx\Big) \\
&\underbrace{\le}_{L^2 \text{ theory}} \omega\Big(R^2 + c_8 \frac{1}{R^{n-2}} \int_{B_R(x_0)} |Du|^2 dx\Big),
\end{aligned}$$

where in the last inequality we used that

$$\begin{aligned}
\int_{B_R(x_0)} |Dw|^2 dx &\le 2\int_{B_R(x_0)} |Du|^2 dx + 2\int_{B_R(x_0)} |Dv|^2 dx \\
&\le (2+c)\int_{B_R(x_0)} |Du|^2 dx,
\end{aligned}$$

and the last inequality follows from L^2-theory, since $u = v$ on $\partial B_R(x_0)$ and v satisfies the elliptic system with constant coefficients (9.32). Similarly

$$\fint_{B_R(x_0)} \omega_1 dx \le \omega\Big(R^2 + c_8 \frac{1}{R^{n-2}} \int_{B_R(x_0)} |Du|^2 dx\Big).$$

Step 4. Now estimate (9.34) may be rewritten as

$$\begin{aligned}
&\int_{B_\rho(x_0)} |Du|^2 dx \\
&\le c_9 \Big[\Big(\frac{\rho}{R}\Big)^n + \omega\Big(R^2 + c_8 \frac{1}{R^{n-2}} \int_{B_R(x_0)} |Du|^2 dx\Big)^{1-\frac{2}{p}}\Big] \int_{B_R(x_0)} |Du|^2 dx,
\end{aligned}$$

valid for every $x_0 \in \Omega$, $0 < \rho < R \le \frac{\operatorname{dist}(x_0,\partial\Omega)}{2}$ and with constants c_8 and c_9 depending on $n, m, \lambda, \Lambda, p$. Since $\lim_{r\to 0^+} \omega(r) = 0$, Lemma 5.13 implies that given $\sigma \in (0,1)$ there are R_0 and ε_0 depending on n, m, λ, Λ and σ such that whenever $R \le R_0$ and

$$\frac{1}{R^{n-2}} \int_{B_R(x_0)} |Du|^2 dx \le \varepsilon_0,$$

(and this last condition can be met when $x_0 \in \Omega \setminus \Sigma(u)$) then

$$\int_{B_\rho(x_0)} |Du|^2 dx \le c\Big(\frac{\rho}{R}\Big)^{n-2+2\sigma}, \quad 0 < \rho < R_0. \tag{9.35}$$

Since for a fixed R, $\frac{1}{R^{n-2}}\int_{B_R(x_0)}|Du|^2dx$ varies continuously with respect to x_0, (9.35) holds in a neighborhood V of x_0, yielding $Du \in L^{2,n-2+2\sigma}(V)$; by Morrey's Theorem 5.7, $u \in C^{0,\sigma}(V)$.

The estimate on $\dim^{\mathcal{H}}(\Sigma(u))$ follows from the characterization of $\Sigma(u)$, together with Lemma 9.14 and Proposition 9.21. □

In fact one can show, see [37], [42], [45]:

Theorem 9.16 *Let $u \in W^{1,r}(\Omega)$ (for some fixed $r \geq 2$) be a minimizer of the functional*

$$\int_\Omega F(x,u,Du)dx, \tag{9.36}$$

where

(i) $\lambda|p|^r \leq F(x,u,p) \leq \Lambda|p|^r$, *for some* $\lambda,\Lambda > 0$ *and every* $(x,u,p) \in \Omega\times\mathbb{R}^m\times\mathbb{R}^{n\times m}$,

(ii) F *is twice differentiable in* p *and for some* $L, A > 0$

$$\begin{aligned}&|F_{pp}(x,u,p)| \leq L(1+|p|)^{r-2}\\ &F_{p^i_\alpha p^j_\beta}(x,u,p)\xi^i_\alpha\xi^j_\beta \geq (1+|p|)^{r-2}|\xi|^2, \qquad \forall\xi\in\mathbb{R}^{n\times m}.\end{aligned}$$

(iii) *The function* $(1+|p|^2)^{1-r}F(x,u,p)$ *is continuous in* x,u *uniformly with respect to* p, *and*

$$|F(x,u,p)-F(y,v,p)| \leq L(1+|p|^r)\omega(|x-y|^r+|u-v|^r),$$

where $\omega(t) \leq A\min\{t^\sigma,1\}$, *for some* $\sigma > 0$.

Then there exists an open set $\Omega_0 \subset \Omega$ such that u has Hölder continuous first derivatives in Ω_0. Moreover $\Omega\backslash\Omega_0 = \Sigma_1\cup\Sigma_2$, where

$$\begin{aligned}\Sigma_1 &:= \left\{x_0\in\Omega \;\middle|\; \sup_{r>0}|(Du)_{x_0,r}| = +\infty\right\}\\ \Sigma_2 &:= \left\{x_0\in\Omega \;\middle|\; \liminf_{r\to 0}\fint_{B_r(x_0)}|Du-(Du)_{x_0,r}|^2dx > 0\right\},\end{aligned}$$

hence $\mathrm{meas}(\Omega\backslash\Omega_0) = 0$.

Also considering the previous results, it is natural to ask whether or not $\dim^{\mathcal{H}}(\Sigma_1\cup\Sigma_2) < n$. A positive answer was given by Kristensen and Mingione in [66]; indeed (similar to the case $r = 2$) there exists a higher integrability exponent $q > r$, depending only on n, m, λ, Λ, but otherwise independent of the minimizer and of the functional considered, such that

$$Du \in L^q_{\mathrm{loc}}(\Omega)\,.$$

Then we have

Theorem 9.17 *Let u be a minimizer of the functional* (9.36) *under the assumptions of Theorem 9.16. Then*

$$\dim^{\mathcal{H}}(\Omega \setminus \Omega_0) \leq n - \min\{\sigma, q - r\}.$$

Moreover, in the low dimensional case case $n \leq r+2$ the previous estimate improves in

$$\dim^{\mathcal{H}}(\Omega \setminus \Omega_0) \leq n - \sigma.$$

Finally, $\Omega_0 = \Omega$ holds in the two dimensional case $n = 2$.

In the case of solutions to nonlinear elliptic systems similar estimates were obtained by Mingione in [73, 74]. Finally, when the functional has a splitting type, special structure, of the type considered in [42], it was shown in [66] that the dimension estimates improve in every dimension.

Theorem 9.18 *Let u be a minimizer of the functional*

$$\int_\Omega f(x, Du) + g(x, u)\, dx$$

under the assumptions of Theorem 9.16 satisfied by the integrand $F = f + g$; then

$$\dim^{\mathcal{H}}(\Omega \setminus \Omega_0) \leq n - \sigma.$$

A further refinement of the previous result eventually leads to consider measurable dependence of $x \mapsto g(x, \cdot)$; further cases are also considered in [66].

9.2.5 The Hausdorff dimension of the singular set

We briefly recall the definitions of Hausdorff measure and dimension.

Definition 9.19 *For $k > 0$* integer, *define ω_k to be the volume of the unit ball in $\mathbb{R}^k$, given by*

$$\omega_k = \frac{2\pi^{\frac{k}{2}}}{k\Gamma(\frac{k}{2})}, \tag{9.37}$$

where Γ is the Euler function

$$\Gamma(t) := \int_0^{+\infty} x^{t-1} e^{-x} dx, \quad t \geq 0. \tag{9.38}$$

Since Γ is defined for every positive number we shall use (9.37) *to define ω_k for any real number $k > 0$.*

Given a set $A \subset \mathbb{R}^n$ and $k, \delta > 0$, define

$$\mathcal{H}^k_\delta(A) := \inf\left\{\sum_{j=0}^\infty \omega_k \rho_j^k : A \subset \bigcup_{j=0}^\infty B_{\rho_j}(x_j),\ \rho_j \leq \delta,\ x_j \in \mathbb{R}^n\right\}.$$

Definition 9.20 *The k-dimensional Hausdorff measure $\mathcal{H}^k(A)$ of a set $A \subset \mathbb{R}^n$ is defined as*

$$\mathcal{H}^k(A) := \sup_{\delta>0} \mathcal{H}^k_\delta(A).$$

The Hausdorff dimension of A is defined as

$$\dim^{\mathcal{H}}(A) := \inf\Big\{k \geq 0 : \mathcal{H}^k(A) = 0\Big\}.$$

We also recall that for every $k > \dim^{\mathcal{H}}(A)$, we have $\mathcal{H}^k(A) = 0$, and for every $k < \dim^{\mathcal{H}}(A)$, $\mathcal{H}^k(A) = +\infty$.

Proposition 9.21 *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \in L^1_{\mathrm{loc}}(\Omega)$, $0 \leq \alpha < n$. Define*

$$\Sigma_\alpha := \Big\{x \in \Omega : \limsup_{\rho\to 0} \frac{1}{\rho^\alpha} \int_{B_\rho(x)} |f|dx > 0\Big\}.$$

Then $\mathcal{H}^\alpha(\Sigma_\alpha) = 0$. In particular $\dim^{\mathcal{H}}(\Sigma_\alpha) \leq \alpha$.

Proof. For $s = 1, 2, \dots$ define

$$F_s := \Big\{x \in \Omega : \limsup_{\rho\to 0} \frac{1}{\rho^\alpha} \int_{B_\rho(x)} |f|dx > \frac{1}{s}\Big\}.$$

Then $\Sigma_\alpha = \bigcup_{s=1}^{+\infty} F_s$ and it suffices to show that $\mathcal{H}^\alpha_\delta(F_s) = 0$ for every s, since it can be easily seen that

$$\mathcal{H}^\alpha_\delta(\Sigma_\alpha) = \lim_{s\to\infty} \mathcal{H}^\alpha_\delta(F_s).$$

By definition of F_s, for every $\delta > 0$ and $x \in F_s$ there exists $r = r(x,\delta) \leq \delta$ such that

$$\frac{1}{r^\alpha} \int_{B_r(x)} |f|dx > \frac{1}{s}.$$

Then F_s is covered by $\{B_{r(x,\delta)}(x) : x \in F_s\}$ and, by Besicovitch's covering lemma, there exists a disjoint countable subfamily $\{B_{r_i}(x_i)\}$ such that $\{B_{5r_i}(x_i)\}$ covers F_s, $r_i = r(x_i,\delta)$. Now

$$\sum_{i=1}^{+\infty} r_i^\alpha \leq \sum_{i=1}^{+\infty} s \int_{B_{r_i}(x_i)} |f|dx = s \int_{\cup_{i=1}^\infty B_{r_i}(x_i)} |f|dx, \tag{9.39}$$

because the balls are disjoint. Therefore

$$\mathcal{L}^n\Big(\bigcup_{i=1}^{+\infty} B_{r_i}(x_i)\Big) = \omega_n \sum_{i=1}^{+\infty} r_i^n \leq \omega_n \delta^{n-\alpha} \sum_{i=1}^{+\infty} r_i^\alpha \leq \omega_n \delta^{n-\alpha} s \int_\Omega |f|dx.$$

Hence, as $\delta \to 0$, the last integral in (9.39) vanishes (absolute continuity of Lebesgue's integral). Therefore

$$\mathcal{H}^{\alpha}_{\delta}(F_s) \leq \sum_{i=1}^{n} \omega_{\alpha}(5r_i)^{\alpha} \leq 5^{\alpha} s \int_{\cup_{i=1}^{+\infty} B_{r_i}(x_i)} |f| dx \to 0, \quad \text{as } \delta \to 0.$$

$\square$

Chapter 10
Harmonic maps

A harmonic map between two Riemannian manifolds (M, g) and (N, γ) of dimension n and m respectively is, roughly speaking, a critical point for the Dirichlet integral

$$\mathcal{E}(u) := \int_M |\nabla u|^2 d\,\mathrm{vol}_M,$$

where, for $x \in M$ and charts φ and ψ at x and $u(x)$ respectively, and $\overline{u} := \psi \circ u \circ \varphi^{-1}$,

$$|\nabla u|^2(x) := \gamma_{ij} g^{\alpha\beta} D_\alpha \overline{u}^i_{\varphi(x)} D_\beta \overline{u}^j_{\varphi(x)},$$

with $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$. If $M = \Omega \subset \mathbb{R}^n$ and $N = \mathbb{R}^n$, then harmonic maps are simply maps whose components are harmonic functions. In general the curvature of N introduces an important *nonlinearity* in the problem.

In this chapter we shall present the results of Giaquinta-Giusti and Schoen-Uhlenbeck about the regularity of local minimizers of the Dirichlet integral.

10.1 Basic material

Thanks to a theorem of John Nash we can assume that the target manifold (N, γ) is isometrically embedded into $\mathbb{R}^p$ for some p. For the sake of simplicity, we shall also assume that the manifold M is an open set Ω of $\mathbb{R}^n$ with the standard Euclidean metric.[1] Then the Dirichlet energy becomes

$$\mathcal{E}(u) := \int_\Omega |Du|^2 dx, \tag{10.1}$$

[1] The general case of the results we shall present may be obtained with minor changes, see [98].

where

$$|Du|^2 := \sum_{i=1}^{p}\sum_{\alpha=1}^{n} |D_\alpha u^i|^2.$$

Definition 10.1 *Given Ω and N as above, we define*

$$W^{1,2}_{\mathrm{loc}}(\Omega, N) := \big\{u \in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^p) : u(x) \in N, \text{ for a.e. } x \in \Omega\big\}.$$

Definition 10.2 (Local minimizers) *A map $u \in W^{1,2}_{\mathrm{loc}}(\Omega, N)$ is a local minimizer of the Dirichlet energy* (10.1) *if for every ball $B_\rho(x_0) \Subset \Omega$ and every $v \in W^{1,2}(B_\rho(x_0), N)$ with $v = u$ on $\partial B_\rho(x_0)$, we have*

$$\int_{B_\rho(x_0)} |Du|^2 dx \le \int_{B_\rho(x_0)} |Dv|^2 dx. \tag{10.2}$$

10.1.1 The variational equations

Consider a local minimizer u. For a ball $B_\rho(x_0) \Subset \Omega$ and some $\delta > 0$ suppose that there exists a family of maps $\{u_s\}_{s\in(-\delta,\delta)} \subset W^{1,2}(B_\rho(x_0), N)$ such that

1. $u_0 \equiv u$;
2. $u_s \equiv u$ on $\partial B_\rho(x_0)$ for every $s \in (-\delta, \delta)$.

Then by (10.2) we have

$$\frac{d}{ds}\left(\int_{B_\rho(x_0)} |Du_s|^2 dx\right)\bigg|_{s=0} = 0, \tag{10.3}$$

whenever the derivative exists.

There are two particularly useful ways of choosing families $\{u_s\}$ as above: we shall refer to them as *inner* and *outer variations*.

Outer variations

For any $\zeta \in C^\infty_c(B_\rho(x_0), \mathbb{R}^p)$, set

$$u_s := \Pi \circ (u + s\zeta),$$

where Π is the nearest point projection onto N. Clearly for s small enough the image of $u + s\zeta$ lies in a tubolar neighborhood of N, so that u_s is well defined. By the Taylor expansion, we find

$$D_\alpha u_s = D_\alpha u + s(d\Pi_u(D_\alpha \zeta) + d^2\Pi_u(\zeta, D_\alpha u)) + O(s^2).$$

As one can verify (see e.g. [98, Sec. 2.2]), together with (10.3), this yields

$$\sum_{\alpha=1}^{n} \int_{B_\rho(x_0)} \Big[D_\alpha u \cdot D_\alpha \zeta - \zeta \cdot A_u(D_\alpha u, D_\alpha u) \Big] dx = 0, \tag{10.4}$$

where A_u is the second fundamental form of N at $u(x)$, compare Section 11.1.3. We can also write (10.4) in the form

$$\Delta u + \sum_{\alpha=1}^{n} A_u(D_\alpha u, D_\alpha u) = 0. \tag{10.5}$$

Inner variations

For any $\zeta \in C_c^\infty(B_\rho(x_0), \mathbb{R}^n)$, define

$$u_s(x) := u(x + s\zeta(x)),$$

well defined for s small enough. Then (10.3) implies

$$\int_{B_\rho(x_0)} \Big[\frac{1}{2} |Du|^2 \operatorname{div} \zeta - D_\alpha u^i D_\beta u^i D_\alpha \zeta^\beta \Big] dx = 0. \tag{10.6}$$

Proof. Set $\mathcal{Q}_t(x) := x + t\zeta(x)$. For $|t|$ small enough $\mathcal{Q}_t$ is a diffeomorphism of $B_\rho(x_0)$ onto itself. Set $\mathcal{U}_t(x) := u(\mathcal{Q}_t^{-1}(x))$. Then

$$\mathcal{U}_t \in W^{1,2}(B_\rho(x_0), N), \quad \mathcal{U}_0 \equiv u$$

and $\mathcal{U}_t$ agrees with u in a neighborhood on $\partial B_\rho(x_0)$. From (10.3) we have

$$\frac{d}{dt} \left(\int_{B_\rho(x_0)} |D\mathcal{U}_t|^2 dx \right) \Big|_{t=0} = 0.$$

Together with

$$\int_{B_\rho(x_0)} |D\mathcal{U}_t|^2 dx = \int_{B_\rho(x_0)} |Du(x) D\mathcal{Q}_t^{-1}(\mathcal{Q}_t(x))|^2 \det D\mathcal{Q}_t(x) dx,$$

$$Du(x) \cdot D\mathcal{Q}_t^{-1}(\mathcal{Q}_t(x)) = Du(x)(I - tD\zeta(x) + O(t^2)),$$

$$\det D\mathcal{Q}_t(x) = 1 + t \operatorname{div} \zeta(x) + O(t^2),$$

as $t \to 0$, gives (10.6). □

Definition 10.3 *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, N)$. Then u is said to be a*

1. weakly harmonic map *if it satisfies* (10.4) *for every ball $B_\rho(x_0) \Subset \Omega$ and every $\zeta \in C_c^\infty(B_\rho(x_0, \mathbb{R}^p))$;*

2. stationary harmonic map *if it is weakly harmonic and satisfies* (10.6) *for every $B_\rho(x_0) \Subset \Omega$, $\zeta \in C_c^\infty(B_\rho(x_0), \mathbb{R}^n)$.*

Remark 10.4 If $u \in C^2(\Omega, N)$, integration by parts yields that (10.4) implies (10.6). However this is false in general for $u \in W^{1,2}_{\mathrm{loc}}(\Omega, N)$.

10.1.2 The monotonicity formula

Proposition 10.5 (Monotonicity formula) *Let u be an inner extremal of the Dirichlet integral, that is* (10.6) *holds for every ball $B_\rho(x_0) \Subset \Omega$, and every $\zeta \in C_c^\infty(B_\rho(x_0, \mathbb{R}^n))$. Then, setting*

$$r := \frac{x - x_0}{|x - x_0|}, \qquad R := |x - x_0|,$$

we have for any $x_0 \in \Omega$ and almost every $\rho \in (0, \operatorname{dist}(x_0, \partial\Omega))$,

$$\frac{d}{d\rho}\Big(\frac{1}{\rho^{n-2}}\int_{B_\rho(x_0)} |Du|^2 dx\Big) = 2\frac{d}{d\rho}\Big(\int_{B_\rho(x_0)} \frac{1}{R^{n-2}}\Big|\frac{\partial u}{\partial r}\Big|^2 dx\Big), \tag{10.7}$$

and for every $x_0 \in \Omega$ and $0 < \sigma < \rho < \operatorname{dist}(x_0, \partial\Omega)$

$$\begin{aligned}\frac{1}{\rho^{n-2}}\int_{B_\rho(x_0)} |Du|^2 dx - \frac{1}{\sigma^{n-2}}\int_{B_\sigma(x_0)} |Du|^2 dx&\\ = 2\int_{B_\rho(x_0)\setminus B_\sigma(x_0)} \frac{1}{R^{n-2}}\Big|\frac{\partial u}{\partial r}\Big|^2 dx.&\end{aligned} \tag{10.8}$$

Proof. Since (10.8) easily follows by integrating (10.7), we only need to prove (10.7). Fix a smooth radial cut-off function $\eta = \eta(|x - x_0|)$ with $\operatorname{spt}\eta \subset B_\rho(x_0)$, $0 \le \eta \le 1$, $\eta(0) = 1$. Inserting $\zeta(x) = (x - x_0)\eta(|x - x_0|)$ in (10.6), we find

$$\begin{aligned}&\frac{1}{2}\int_{B_\rho(x_0)} |Du|^2(n\eta(|x - x_0|) + R\eta'(|x - x_0|))dx\\ &=\int_{B_\rho(x_0)} D_\alpha u^i D_\beta u^i \Big(\delta^{\alpha\beta}\eta(|x - x_0|)+\eta'(|x - x_0|)\frac{(x^\alpha - x_0^\alpha)(x^\beta - x_0^\beta)}{|x - x_0|}\Big)dx\end{aligned}$$

i.e.

$$\int_{B_\rho(x_0)} |Du|^2((n-2)\eta + R\eta')dx = 2\int_{B_\rho(x_0)} R\eta'\Big|\frac{\partial u}{\partial r}\Big|^2 dx.$$

Choosing a sequence η_j suitably approximating the characteristic function of $[0, \rho]$, and taking the limit, we get for almost every $\rho \in (0, \operatorname{dist}(x_0, \partial\Omega))$

$$\begin{aligned}(n-2)\int_{B_\rho(x_0)} |Du|^2 dx - \rho\frac{d}{d\rho}\Big(\int_{B_\rho(x_0)} |Du|^2 dx\Big)&\\ = -2\rho\frac{d}{d\rho}\Big(\int_{B_\rho(x_0)} \Big|\frac{\partial u}{\partial r}\Big|^2 dx\Big).&\end{aligned}$$

Dividing by ρ^{n-1} we obtain (10.7). □

Observing that the right-hand side of (10.8) is positive, we infer

Corollary 10.6 *In the hypothesis of the above proposition, the quantity*

$$\frac{1}{\rho^{n-2}}\int_{B_\rho(x_0)}|Du|^2dx$$

is monotone increasing with respect to ρ.

Definition 10.7 (Density) *We define the density of the harmonic function u at x_0 as*

$$\Theta_u(x_0) := \lim_{\rho\to 0^+}\frac{1}{\rho^{n-2}}\int_{B_\rho(x_0)}|Du|^2dx,$$

where the limit exists thanks to Corollary 10.6.

10.2 Giaquinta and Giusti's regularity results

We now study the regularity of locally energy minimizing harmonic maps. In this section we assume that the target manifold N is diffeomorphic to an open set of $\mathbb{R}^m$, or equivalently that there exists a global chart $\psi : N \to \mathbb{R}^m$.

10.2.1 The main regularity result

Let us generalize Definition 10.2 to the case in which Ω is an arbitrary Riemannian manifold.

Definition 10.8 *We define $W^{1,2}_{\mathrm{loc}}(M,N)$ to be the space of functions u such that for every chart $\varphi : U \subset M \to \mathbb{R}^n$,*

$$\psi\circ u\circ\varphi^{-1}\in W^{1,2}_{\mathrm{loc}}(\varphi(U),\mathbb{R}^m).$$

Definition 10.9 (Local minimizer) *A function $u \in W^{1,2}_{\mathrm{loc}}(M,N)$ will be called* local minimizer *of the Dirichlet energy if for every chart $\varphi : U \subset M \to \mathbb{R}^n$, $\overline{u} := \psi\circ u\circ\varphi^{-1}$ is a local minimizer in $W^{1,2}_{\mathrm{loc}}(\varphi(U),\mathbb{R}^m)$ of*

$$\mathcal{E}(\overline{u}) := \frac{1}{2}\int_{\varphi(U)}\gamma_{ij}(u(x))g^{\alpha\beta}(x)D_\alpha\overline{u}^i(x)D_\beta\overline{u}^j(x)\sqrt{g(x)}dx,$$

where $g(x) := \det(g_{\alpha\beta}(x))$.

Let now the chart $\varphi : U \subset M \to \mathbb{R}^n$ be fixed. Define

$$A^{\alpha\beta}_{ij}(x,\overline{u}) := \sqrt{g(x)}g^{\alpha\beta}(x)\gamma_{ij}(\overline{u}),$$

and observe that if u (or $\overline{u}$) is locally bounded, say $\sup_{\Omega_0} |\overline{u}| \leq C(\Omega_0)$ for every $\Omega_0 \Subset \varphi(U)$, then

$$\lambda|\xi|^2 \leq A_{ij}^{\alpha\beta}(x,\overline{u})\xi_\alpha^i\xi_\beta^j \leq \Lambda|\xi|^2, \quad \text{for all } x \in \Omega_0,$$

where λ and Λ depend on Ω_0, $M \cap \varphi^{-1}(\Omega_0)$, $N \cap B_{C(\Omega_0)}(0)$. Then Theorem 9.15 gives at once

Theorem 10.10 *A (locally) bounded local minimizer $\overline{u}$ of the Dirichlet energy is Hölder continuous except in the singular set*

$$\Sigma := \left\{ x \in M : \liminf_{R\to 0} \frac{1}{R^{n-2}} \int_{B_R(x)} |D\overline{u}|^2 dx > 0 \right\}.$$

More precisely, for every $M_0 \Subset M$ one has

$$\Sigma \cap M_0 := \left\{ x \in M_0 : \liminf_{R\to 0} \frac{1}{R^{n-2}} \int_{B_R(x)} |D\overline{u}|^2 dx > \varepsilon_0 \right\}.$$

where $\varepsilon_0 > 0$ depends only on M_0 and N. Moreover $\dim^{\mathcal{H}}(\Sigma) < n-2$.

In fact $u \in C^\infty(M \backslash \Sigma, N)$ by Theorem 9.7 and Schauder estimates.

10.2.2 The dimension reduction argument

Following [44], we now improve the estimate on the dimension of the singular set, using the *dimension reduction argument* of Federer. We shall prove

Theorem 10.11 *Let Σ be (as in the previous section) the singular set of a bounded local minimizer of the Dirichlet energy $\mathcal{E}$. Then*

1. *for $n = 3$, Σ contains only isolated points;*
2. *for $n \geq 4$, $\dim^{\mathcal{H}}(\Sigma) \leq n-3$.*

In fact we have

Theorem 10.12 *The same conclusions of Theorem 10.11 hold for locally bounded minimizers of*

$$\mathcal{J}(u) := \int_\Omega A_{ij}^{\alpha\beta}(x,u) D_\alpha u^i D_\beta u^j \, dx, \quad A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha},$$

where the coefficients $A_{ij}^{\alpha\beta}$

1. *are of the form $A_{ij}^{\alpha\beta} = g^{\alpha\beta}\gamma_{ij}$,*
2. *are bounded: $|A(x,u)| \leq M$ for some M,*

3. satisfy the Legendre condition: $A_{ij}^{\alpha\beta}\xi_\alpha^i\xi_\beta^j \geq |\xi|^2$ *for every* ξ,

4. are uniformly continuous: $|A(x,u) - A(y,v)| \leq \omega(|x-y|^2 + |u-v|^2)$, *where* $\omega(t)$ *is a bounded continuous and concave function with* $\omega(0) = 0$,

and

$$\int_0^1 \frac{\omega(t^2)}{t}dt < +\infty. \tag{10.9}$$

We shall need the following lemma on the convergence of minimizers, based on Caccioppoli's inequality and the higher integrability of the gradient.

Lemma 10.13 *Let* $A^{(\nu)}(x,u) = A_{ij}^{\alpha\beta(\nu)}(x,u)$ *be a sequence of continuous functions in* $B_1(0)\times\mathbb{R}^m$ *converging uniformly to* $A(x,u)$ *and satisfying hypothesis 2,3,4 of Theorem 10.12, uniformly with respect to* ν. *For each* $\nu\in\mathbb{N}$ *let* $u^{(\nu)}$ *be a minimizer in* $B_1(0)$ *of*

$$\mathcal{J}^{(\nu)}(w;B_1(0)) := \int_{B_1(0)} A_{ij}^{\alpha\beta(\nu)}(x,w)D_\alpha w^i D_\beta w^j\,dx,$$

and suppose that the sequence $(u^{(\nu)})$ *is uniformly bounded in* L^∞ *and converges weakly in* $L^2(B_1(0),\mathbb{R}^m)$ *to* v. *Then* v *is a minimizer of*

$$\mathcal{J}(w;B_1(0)) := \int_{B_1(0)} A_{ij}^{\alpha\beta}(x,w)D_\alpha w^i D_\beta w^j\,dx.$$

Moreover, if x_ν *is a singular point for* $u^{(\nu)}$ *and* $x_\nu\to x_0$, *then* x_0 *is a singular point of* v.

Proof.
Step 1. By Proposition 9.13, for every $B_r(x_0)\subset B_1(0)$ we have

$$\int_{B_{\frac{r}{2}}(x_0)} |Du^{(\nu)}|^2dx \leq \frac{c_1}{r^2}\int_{B_r(x_0)} |u^{(\nu)} - u^{(\nu)}_{x_0,r}|^2dx, \tag{10.10}$$

and by Lemma 9.14, there exists $p>2$ such that $Du^{(\nu)}\in L^p_{\mathrm{loc}}(B_1(0))$ and

$$\Big(\fint_{B_{\frac{r}{2}}(x_0)} |Du^{(\nu)}|^p dx\Big)^{\frac{1}{p}} \leq c_2\Big(\fint_{B_r(x_0)} |Du^{(\nu)}|^2dx\Big)^{\frac{1}{2}}, \tag{10.11}$$

where c_1, c_2 and p do not depend on ν. By the weak L^2-convergence, the L^2-norm of $u^{(\nu)}$ is equibounded with respect to ν and this implies, by (10.10) and (10.11), that the L^2 and L^p-norms of $Du^{(\nu)}$ are locally equibounded. In particular there exists a function $c(R)$ such that for $R<1$

$$\int_{B_R(0)} |Du^{(\nu)}|^p dx \le c(R). \tag{10.12}$$

By Rellich's theorem, up to a subsequence, $u^{(\nu)} \to v$ in $L^2_{\mathrm{loc}}(B_1(0))$.

Step 2. We now prove that for any $R \in (0,1)$

$$\mathcal{J}(v; B_R(0)) \le \liminf_{\nu\to+\infty} \mathcal{J}^{(\nu)}(u^{(\nu)}; B_R(0)). \tag{10.13}$$

In order to do that, write

$$\begin{aligned}
&\int_{B_R(0)} A^{(\nu)}(x,u^{(\nu)}) Du^{(\nu)} Du^{(\nu)} dx - \int_{B_R(0)} A(x,v) Dv Dv dx \\
&= \int_{B_R(0)} \Big[A^{(\nu)}(x,u^{(\nu)}) - A(x,u^{(\nu)})\Big] Du^{(\nu)} Du^{(\nu)} dx \\
&\quad + \int_{B_R(0)} \Big[A(x,u^{(\nu)}) - A(x,v)\Big] Du^{(\nu)} Du^{(\nu)} dx \\
&\quad + \int_{B_R(0)} A(x,v) Du^{(\nu)} Du^{(\nu)} dx - \int_{B_R(0)} A(x,v) Dv Dv dx.
\end{aligned}$$

As $\nu \to +\infty$ we have

$$\begin{aligned}
&\int_{B_R(0)} \Big[A^{(\nu)}(x,u^{(\nu)}) - A(x,u^{(\nu)})\Big] Du^{(\nu)} Du^{(\nu)} dx \\
&\qquad \le \sup_{B_R(0)} \Big[A^{(\nu)}(x,u^{(\nu)}) - A(x,u^{(\nu)})\Big] \|Du^{(\nu)}\|^2_{L^2(B_R(0))} \to 0
\end{aligned}$$

because of the uniform convergence of the coefficients and the equiboundedness of $Du^{(\nu)}$ in $L^2(B_R(0))$. By Hölder's inequality

$$\begin{aligned}
&\int_{B_R(0)} \Big[A(x,u^{(\nu)}) - A(x,v)\Big] Du^{(\nu)} Du^{(\nu)} dx \\
&\le \left(\int_{B_R(0)} \Big|A(x,u^{(\nu)}) - A(x,v)\Big|^q dx\right)^{\frac{1}{q}} \left(\int_{B_R(0)} |Du|^p dx\right)^{\frac{2}{p}} \\
&\to 0,
\end{aligned} \tag{10.14}$$

where $q := \frac{p}{p-2}$, because, up to a subsequence, $u^{(\nu)}$ converges a.e. to v, hence by continuity and uniform convergence of the coefficients,

$$A(x, u^{(\nu)}(x)) \to A(x, v(x)) \quad \text{a.e.},$$

and (10.14) follows by Lebesgue's dominated convergence theorem. Finally

$$\liminf_{\nu\to+\infty} \int_{B_R(0)} A(x,v) Du^{(\nu)} Du^{(\nu)} dx \ge \int_{B_R(0)} A(x,v) Dv Dv dx,$$

because the function

$$u \mapsto \int_{B_R(0)} A_{ij}^{\alpha\beta}(x,v) D_\alpha u^i D_\beta u^j \, dx$$

is continuous in $W^{1,2}(B_R(0))$, convex thanks to the Legendre condition, and therefore weakly lower semicontinuous in $W^{1,2}(B_R(0))$.

Step 3. Again fix $R \in (0,1)$. Let w be an arbitrary function matching v outside $B_R(0)$, and choose $\eta \in C^1(B_1(0))$ satisfying:

1. $0 \le \eta \le 1$;
2. $\eta \equiv 0$ in $B_r(0)$ for some $r < R$;
3. $\eta \equiv 1$ outside $B_R(0)$.

Then $v^{(\nu)} := w + \eta(u^{(\nu)} - v)$ equals $u^{(\nu)}$ outside $B_R(0)$, therefore

$$\mathcal{J}^{(\nu)}(u^{(\nu)}; B_R(0)) \le \mathcal{J}^{(\nu)}(v^{(\nu)}; B_R(0)). \tag{10.15}$$

By the boundedness of $A^{(\nu)}$ and by (10.12) we get

$$\begin{aligned} &\mathcal{J}^{(\nu)}(v^{(\nu)}; B_R(0)) \\ &\qquad \le \int_{B_R(0)} A^{(\nu)}(x, v^{(\nu)}) DwDw dx + c_3(R) \|\eta\|_{L^{\frac{p}{p-2}}(B_R(0))} \\ &\qquad + c_4(R,\eta) \|u^{(\nu)} - v\|_{L^2(B_R(0))} \Big(1 + \|u^{(\nu)} - v\|_{L^2(B_R(x_0))}\Big). \end{aligned}$$

Letting $\nu \to +\infty$, we deduce from (10.13) and (10.15)

$$\mathcal{J}(v; B_R(0)) \le \mathcal{J}(w; B_R(0)) + c_3 \|\eta\|_{L^{\frac{p}{p-2}}(B_R(0))}.$$

Taking r close to R, the last term can be made arbitrarily small, and that proves that v is a minimizer for $\mathcal{J}$.

Step 4. In order to prove the second part of the lemma, let us recall that, because of Caccioppoli's inequality (10.10) and Theorem 9.15, a point $\overline{x}$ is singular if and only if

$$\liminf_{\rho \to 0^+} \frac{1}{\rho^n} \int_{B_\rho(\overline{x})} |u - u_{\overline{x},\rho}|^2 dx \ge \varepsilon_0,$$

where ε_0 is independent of ν.

Suppose now that x_0 is a regular point of v and $x^{(\nu)} \to x_0$. Then for some ρ small enough we have

$$\frac{1}{\rho^n} \int_{B_\rho(x_0)} |v - v_{x_0,\rho}|^2 dx < \varepsilon_0,$$

and hence, by dominated convergence,

$$\lim_{\nu\to+\infty} \frac{1}{\rho^n} \int_{B_\rho(x^{(\nu)})} |u^{(\nu)} - u^{(\nu)}_{x^{(\nu)},\rho}|^2 = \frac{1}{\rho^n} \int_{B_\rho(x_0)} |v - v_{x_0,\rho}|^2 dx < \varepsilon_0,$$

which implies that x_ν is a regular point for $u^{(\nu)}$ for ν large enough. This completes the proof of the lemma. □

Lemma 10.14 (Monotonicity) *Let $A^{\alpha\beta}_{ij}$ satisfy the hypothesis of Theorem 10.12 and let u be a local minimizer of $\mathcal{J}$ on $B_1(0)$. Then, for a.e. ρ, R with $0 < \rho < R < 1$, we have*

$$\int_{\partial B_1(0)} |u(Rx) - u(\rho x)|^2 d\mathcal{H}^{n-1} \le c \log\left(\frac{R}{\rho}\right)[\Phi(R) - \Phi(\rho)], \qquad (10.16)$$

where

$$\Phi(t) := t^{n-2} \exp\left(c_1 \int_0^t \frac{\omega(s^2)}{s} ds\right) \int_{B_t(0)} A^{\alpha\beta}_{ij}(x,u) D_\alpha u^i D_\beta u^j \, dx.$$

Proof. For simplicity, we shall only consider the case of the Dirichlet integral, i.e. $A^{\alpha\beta}_{ij} = \delta^{\alpha\beta}\delta_{ij}$, referring to [44] for the general case. Then the expression for $\Phi(t)$ simplifies to

$$\Phi(t) = \frac{1}{t^{n-2}} \int_{B_t(0)} |Du|^2 dx.$$

For $0 < t < 1$ let $x_t := t\frac{x}{|x|}$ and $u_t(x) := u(x_t)$. We have

$$\mathcal{J}(u; B_t(0)) \le \mathcal{J}(u_t; B_t(0)) \qquad (10.17)$$

and since $u|_{\partial B_t} = u_t|_{\partial B_t}$

$$\mathcal{J}(u_t; B_t(0)) = \int_{B_t(0)} \frac{t^2}{|x|^2}\left(\delta_{\alpha h} - \frac{x_\alpha x_h}{|x|^2}\right)\left(\delta_{\alpha k} - \frac{x_\alpha x_k}{|x|^2}\right) D_h u^i(x_t) D_k u^i(x_t) dx.$$

Assume that $n \ge 3$ (the case $n = 2$ will not be treated). Observing that for every $f \in L^1_{\mathrm{loc}}(B_1(0))$ and a.e. $t < 1$ we have

$$\int_{B_t(0)} |x|^{-2} f(x_t) dx = \frac{1}{(n-2)t} \int_{\partial B_t(0)} f(x) d\mathcal{H}^{n-1},$$

we get

$$\mathcal{J}(u_t; B_t(0)) = \frac{t}{n-2} \int_{\partial B_t(0)} \left\{ |Du|^2 - \frac{x_\alpha x_h}{|x|^2}\left[2\delta_{\alpha k} - \frac{x_\alpha x_k}{|x|^2}\right] D_h u^i D_k u^i \right\} d\mathcal{H}^{n-1}.$$

Therefore

$$\mathcal{J}(u_t; B_t(0)) \le \frac{t}{n-2}\left\{\int_{\partial B_t(0)} |Du|^2 d\mathcal{H}^{n-1} - \int_{\partial B_t(0)} \frac{|\langle x, Du\rangle|^2}{|x|^2} d\mathcal{H}^{n-1}\right\}, \tag{10.18}$$

where $\langle x, Du\rangle := x^\alpha D_\alpha u$. Now for a.e. $t < 1$ we have

$$\frac{1}{t^{n-2}}\int_{\partial B_t(0)} |Du|^2 d\mathcal{H}^{n-1} = \Phi'(t) + (n-2)\frac{\Phi(t)}{t},$$

therefore, from (10.17), (10.18) we get

$$\Phi'(t) \ge \frac{1}{t^{n-2}}\int_{\partial B_t(0)} \frac{|\langle x, Du\rangle|^2}{|x|^2} d\mathcal{H}^{n-1},$$

and integrating

$$\Phi(R) - \Phi(\rho) \ge \int_\rho^R t^{2-n}\int_{\partial B_t(0)} \frac{|\langle x, Du\rangle|^2}{|x|^2} d\mathcal{H}^{n-1} dt.$$

On the other hand

$$\begin{aligned} |u(Rx) - u(\rho x)|^2 &\le \left(\int_\rho^R |\langle x, Du(tx)\rangle| dt\right)^2 \\ &\le \log\left(\frac{R}{\rho}\right)\int_\rho^R t|\langle x, Du(tx)\rangle|^2 dt, \end{aligned}$$

and the conclusion follows at once integrating over $\partial B_1(0)$. □

Proof of Theorem 10.11. Assume $n = 3$ and suppose that u has a sequence of singular points x_ν converging to x_0; up to translation we can assume $x_0 = 0$. We use a rescaling argument. Let $R_\nu := 2|x_\nu| < 1$ for ν large enough; the function $u^{(\nu)}(x) := u(R_\nu x)$ is a local minimizer in $B_1(0)$ for

$$\mathcal{J}^{(\nu)}(u^{(\nu)}; B_1(0)) := \int_{B_1(0)} A^{(\nu)}(x, u^{(\nu)}) Du^{(\nu)} Du^{(\nu)} dx,$$

where

$$A^{(\nu)}(x, v) := A(R_\nu x, v).$$

Each $u^{(\nu)}$ has a singular point y_ν with $|y_\nu| = \frac{1}{2}$. Since the $u^{(\nu)}$'s are uniformly bounded, up to a subsequence, they converge weakly in $L^2(B_1(0))$ to some function v.[2] By compactness, we may also assume that $y_\nu \to y_0$ for some y_0 with $|y_0| = \frac{1}{2}$.

[2] By the theorem of Banach-Alaoglu.

By Lemma 10.13 we conclude that v is a local minimizer of

$$\mathcal{J}^0(v; B_1(0)) := \int_{B_1(0)} A(0,v)DvDvdy$$

and that v is singular at y_0. Moreover by monotonicity we shall see that v is homogeneous of degree zero:

$$v(\tau y) = v(y), \quad \forall \tau \in (0,1),\ y \in B_1(0).$$

Indeed by Lemma 10.14 $\Phi(t)$ is increasing, and therefore tends to a finite limit as $t \to 0$; moreover, setting $\rho := \lambda R_\nu, R := \mu R_\nu$, $0 < \lambda < \mu < 1$, we have

$$\int_{\partial B_1(0)} |u^{(\nu)}(\lambda x) - u^{(\nu)}(\mu x)|^2 d\mathcal{H}^{n-1} \le c \log\left(\frac{\mu}{\lambda}\right)[\Phi(\mu R_\nu) - \Phi(\lambda R_\nu)],$$

hence, letting $\nu \to 0$, we conclude that

$$\int_{\partial B_1(0)} |v(\lambda x) - v(\mu x)|^2 d\mathcal{H}^{n-1} = 0$$

for a.e. λ and μ.

Since v is homogeneous of degree 0, we have that τy_0 lies in the singular set Σ for every $\tau \in (0,1)$, hence $\dim^{\mathcal{H}}(\Sigma) \ge 1$, contradicting Theorem 10.10, thus proving part 1.

To prove part 2, let us recall without proof that for any set A

$$\mathcal{H}^k(A) = 0 \Leftrightarrow \mathcal{H}^k_\infty(A) = 0,^3 \tag{10.19}$$

and that, given a Borel set Σ, then for $\mathcal{H}^k$-a.e. $x \in \Sigma$ we have

$$\limsup_{r \to 0^+} \frac{\mathcal{H}^k(\Sigma \cap B(x,r))}{r^k} \ge \frac{\omega_k}{2^k}. \tag{10.20}$$

Finally, if $Q, Q_\nu, \nu = 1, 2, \ldots$, are compact sets such that every open set $A \supset Q$ contains Q_ν for ν large enough, then

$$\mathcal{H}^k_\infty(Q) \ge \limsup_{\nu \to +\infty} \mathcal{H}^k_\infty(Q_\nu).^4 \tag{10.21}$$

Let Σ be the singular set of u and assume that for some $k > 0$ we have $\mathcal{H}^k(\Sigma) > 0$, so that also $\mathcal{H}^k_\infty(\Sigma) > 0$. Then there exists a point x_0, which we may take to be the origin, such that (10.20) holds. Let $R_\nu \to 0$ be a sequence such that

$$\frac{\mathcal{H}^k(\Sigma \cap B(0,R_\nu))}{R_\nu^k} \ge \frac{\omega_k}{2^{k+1}}, \tag{10.22}$$

[3] Compare Definition 9.20

[4] The same property is false for the measure $\mathcal{H}^k$, and this is why we work with $\mathcal{H}^k_\infty$.

and let $u^{(\nu)}(x) := u(2R_\nu x)$. Again by Lemma 10.13, up to a subsequence, $u^{(\nu)}$ converges to a 0-homogeneous local minimizer v. If $\Sigma_{(\nu)}$ denotes the singular set of $u^{(\nu)}$, from (10.22)

$$\mathcal{H}^k_\infty(\Sigma_{(\nu)} \cap B_{\frac{1}{2}}(0)) \geq \frac{\omega_k}{2^{2k+1}}.$$

Set Σ_0 to be the singular set of v; by (10.21) with $Q = \Sigma_0$, $Q_\nu = \Sigma_{(\nu)}$, we have

$$\mathcal{H}^k_\infty(\Sigma_0 \cap B_{\frac{1}{2}}(0)) \geq \frac{\omega_k}{2^{2k+1}}.$$

In particular there exists x_0 such that (10.20) holds with Σ_0 in place of Σ. Up to rotation, assume $x_0 = (0, 0, \dots, a)$ for some $a \neq 0$ and blow up at x_0 as before. We obtain a local minimizer w_1 in $\mathbb{R}^n$ independent of x^n, so that

$$\widetilde{w}_1(x_1, \dots, x_{n-1}) := w_1(x_1, \dots, x_{n-1}, 0)$$

is a local minimizer in $\mathbb{R}^{n-1}$. Moreover its singular set Σ_1 satisfies

$$\mathcal{H}^{k-1}(\Sigma_1) > 0,$$

as comes easily from the invariance of the singular set of w_1. Suppose now $k > n-3$ and apply the procedure $n-3$ times. We obtain

$$\mathcal{H}^{k-(n-3)}(\Sigma_{n-3}) > 0, \quad k-(n-3) > 0,$$

contradicting the fact that w_{n-3} is local minimizer in $\mathbb{R}^3$, and has only isolated singularities by step 1. $\square$

10.3 Schoen and Uhlenbeck's regularity results

Let us discuss the general results by Schoen and Uhlenbeck [95].

10.3.1 The main regularity result

The following theorem, an ε-regularity result, says that an energy minimizing harmonic map is regular in a neighborhood of every point with density suitably small. It is the analog of Theorem 10.10, and implies that the $(n-2)$-dimensional measure of the singular set is zero. We present here the original proof of Schoen and Uhlenbeck, which does not make use of Caccioppoli's inequality. In fact a Caccioppoli type inequality can be proved using a lemma of Luckhaus, leading to a different proof of Theorem 10.15 below, see [98], [99].

Theorem 10.15 (Schoen-Uhlenbeck [95]) *Let $u \in W^{1,2}_{\mathrm{loc}}(B_1(0), N)$ be a local minimizer of the Dirichlet integral. Assume also that the Riemannian manifold (N, γ) is closed (compact and without boundary). Then there exist $\varepsilon_0 > 0$, $\sigma \in (0,1)$ depending on (N, γ) such that if for some ball $B_R(x_0) \Subset B_1(x_0)$ we have*

$$\frac{1}{R^{n-2}} \int_{B_R(x_0)} |Du|^2 dx \le \varepsilon_0, \tag{10.23}$$

then $u \in C^{0,\alpha}(B_{\sigma R}(x_0), N)$, for some $\alpha \in (0,1)$.

The main step in the proof of the above theorem is the following decay estimate.

Proposition 10.16 *Under the assumptions of Theorem 10.15 there exist $\varepsilon_0 > 0$ and $\tau \in (0,1)$ such that if*

$$\int_{B_1(0)} |Du|^2 dx \le \varepsilon_0, \tag{10.24}$$

then

$$\frac{1}{\tau^{n-2}} \int_{B_\tau(0)} |Du|^2 dx \le \frac{1}{2} \int_{B_1(0)} |Du|^2 dx. \tag{10.25}$$

Proof of Theorem 10.15. Thanks to Morrey's Theorem 5.7, it is enough to prove that for any x in a neighborhood V of x_0 we have

$$\frac{1}{\rho^{n-2}} \int_{B_\rho(x)} |Du|^2 dx \le c\rho^{2\alpha}, \quad \forall \rho > 0 \tag{10.26}$$

for some $c > 0$, that is $Du \in L^{2,n-2+2\alpha}(V)$. Since for $x \in B_{\frac{R}{2}}(x_0)$ we have

$$\left(\frac{2}{R}\right)^{n-2} \int_{B_{\frac{R}{2}}(x)} |Du|^2 dx \le \left(\frac{2}{R}\right)^{n-2} \int_{B_R(x_0)} |Du|^2 dx \le 2^{n-2}\varepsilon_0,$$

we shall first prove (10.26) for $x = x_0$ and then, up to take $\frac{\varepsilon_0}{2^{n-2}}$ instead of ε_0, we have that (10.26) holds true in $V = B_{\frac{R}{2}}(x_0)$.

Let u satisfy (10.23), and assume without loss of generality that $x_0 = 0$. Then the rescaled map $u_R(x) := u(Rx)$ satisfies

$$\int_{B_1(0)} |Du_R|^2 dx = \frac{1}{R^{n-2}} \int_{B_R(0)} |Du|^2 dx.$$

This shows that we may also assume $R = 1$. Now apply Proposition 10.16 to $u : B_1(0) \to N$:

$$\frac{1}{\tau^{n-2}} \int_{B_\tau(0)} |Du|^2 dx \le \frac{1}{2} \int_{B_1(0)} |Du|^2 dx.$$

Also the rescaled map $u_\tau(x) := u(\tau x)$ satisfies the hypothesis of Proposition 10.16:

$$\int_{B_1(0)} |Du_\tau|^2 dx = \frac{1}{\tau^{n-2}} \int_{B_\tau(0)} |Du|^2 dx \le \frac{1}{2} \int_{B_1(0)} |Du|^2 dx \le \varepsilon_0,$$

Therefore

$$\frac{1}{\tau^{n-2}} \int_{B_\tau(0)} |Du_\tau|^2 dx \le \frac{1}{2} \int_{B_1(0)} |Du_\tau|^2 dx,$$

which is the same as

$$\frac{1}{(\tau^2)^{n-2}} \int_{B_{\tau^2}(0)} |Du|^2 dx \le \left(\frac{1}{2}\right)^2 \int_{B_1(0)} |Du|^2 dx,$$

and iterating

$$\frac{1}{(\tau^i)^{n-2}} \int_{B_{\tau^i}(0)} |Du|^2 dx \le \left(\frac{1}{2}\right)^i \int_{B_1(0)} |Du|^2 dx,$$

for any positive integer i. Given now $\rho \in (0,1)$, choose i such that $\tau^{i+1} \le \rho \le \tau^i$, and set $\alpha = \frac{\log 2}{2 \log \tau^{-1}}$. Then, since $2^{-i} = (\tau^i)^{\frac{\log 2}{\log \tau^{-1}}}$, we find

$$\left(\frac{1}{\tau^i}\right)^{n-2} \int_{B_\rho(0)} |Du|^2 dx \le (\tau^i)^{2\alpha} \int_{B_1(0)} Du^2 dx,$$

and finally

$$\begin{aligned}
\frac{1}{\rho^{n-2}} \int_{B_\rho(0)} |Du|^2 dx &\le \frac{1}{\tau^{n-2}} \frac{1}{(\tau^i)^{n-2}} \int_{B_{\tau^i}(0)} |Du|^2 dx \\
&\le \tau^{2-n} \left(\frac{1}{2}\right)^i \int_{B_1(0)} |Du|^2 dx \\
&\le \frac{1}{\tau^{n-2}} (\tau^i)^{2\alpha} \int_{B_1(0)} |Du|^2 dx \\
&\le \frac{1}{\tau^{n-2}} \left(\frac{\rho}{\tau}\right)^{2\alpha} \int_{B_1(0)} |Du|^2 dx \\
&\le \left(\tau^{2-n-2\alpha} \int_{B_1(0)} |Du|^2 dx\right) \rho^{2\alpha}.
\end{aligned}$$

□

Proof of Proposition 10.16. We shall first approximate u with a smooth function $u_{\overline{h}}$ (step 1), then prove a decay estimate similar to (10.25) for $u_{\overline{h}}$ (step 2), and finally show how to compare the Dirichlet integral of $u_{\overline{h}}$ with the Dirichlet integral of u using the minimining property of u (steps 3 and 4).

Step 1. Fix a smooth radial mollifier $\varphi : \mathbb{R}^n \to \mathbb{R}^+$, with $\operatorname{spt}\varphi \subset B_1(0)$ and $\int_{B_1(0)} \varphi(x)dx = 1$. Set

$$u^* := \int_{B_1(0)} \varphi(x)u(x)dx \in \mathbb{R}.$$

By a variant of Poincaré inequality[5] we have

$$\int_{B_1(0)} |u - u^*|^2 dx \le c_1 \int_{B_1(0)} |Du|^2 dx \le c_1\varepsilon_0.$$

In particular

$$\operatorname{dist}(u^*, N) \le c_2\sqrt{\varepsilon_0}. \tag{10.27}$$

By monotonicity, for the map $u_{x,h}(y) := u(x - hy)$, we have

$$\begin{aligned} \int_{B_1(0)} |Du_{x,h}(y)|^2 dy &= \frac{1}{h^{n-2}} \int_{B_h(x)} |Du(y)|^2 dy \\ &\le c_3 \int_{B_1(0)} |Du|^2 dx \le c_3\varepsilon_0 \end{aligned}$$

for every $x \in B_{\frac{1}{2}}(0)$ and $h \in (0, 1/4]$. Therefore, if we define for $x \in B_{\frac{1}{2}}(0)$ and $h \in \left(0, \frac{1}{4}\right]$ the h-mollified function

$$u^{(h)}(x) := \int_{B_1(0)} \varphi(y)u(x - hy)dy = \int_{B_h(x)} \varphi^{(h)}(x - z)u(z)dz,$$

where $\varphi^{(h)}(x) := \frac{1}{h^n}\varphi\left(\frac{x}{h}\right)$, we infer from (10.27)

$$\operatorname{dist}(u^{(h)}(x), N) \le c_4\sqrt{\varepsilon_0}, \quad \forall x \in B_{\frac{1}{2}}(0),\ h \in \left(0, \frac{1}{4}\right]. \tag{10.28}$$

Consequently, for ε_0 small enough, depending on N, $u^{(h)}$ lies in a tubolar neighborhood $N_\delta := \{x \in \mathbb{R}^p : \operatorname{dist}(x, N) < \delta\}$ of N, and can be smoothly projected onto N. If $\Pi : N_\delta \to N$ is the normal projection, define $u_h : B_{\frac{1}{2}}(0) \to N$

$$u_h := \Pi \circ u^{(h)}.$$

Next observe that

$$\int_{B_{\frac{1}{2}}(0)} |Du^{(h)}|^2 dx \le c_5 \int_{B_1(0)} |Du|^2 dx. \tag{10.29}$$

[5]Whose simple proof may be obtained using Rellich's theorem, as in the proof of Proposition 3.21.

Moreover, by Jensen's inequality

$$
\begin{aligned}
|Du^{(h)}(x)|^2 &= \Big| \int_{B_h(x)} \varphi^{(h)}(x-y) Du(y) dy \Big|^2 \\
&\le \int_{B_h(x)} \varphi^{(h)}(x-y) |Du(y)|^2 dy \\
&\le c_6 \frac{1}{h^n} \int_{B_h(x)} |Du|^2 dy \\
&\le c_7 \frac{\varepsilon_0}{h^n},
\end{aligned}
$$

hence $|Du^{(\overline{h})}(x)|^2 \le c_7 \sqrt{\varepsilon_0}$ if we choose $\overline{h} = \varepsilon_0^{\frac{1}{2n}}$. Consequently

$$
\sup_{x \in B_{\frac{1}{2}}(0)} |u^{(\overline{h})}(x) - u^{(\overline{h})}(0)|^2 \le c_8 \varepsilon_0^{\frac{1}{2}}, \quad \overline{h} := \varepsilon_0^{\frac{1}{2n}} \tag{10.30}
$$

Step 2. Let $v \in C^\infty\left(\overline{B_{\frac{1}{2}}(0)}, \mathbb{R}^p\right)$ be the solution of

$$
\begin{cases} \Delta v = 0 & \text{in } B_{\frac{1}{2}}(0), \\ v = u^{(\overline{h})} & \text{on } \partial B_{\frac{1}{2}}(0). \end{cases}
$$

Then by (10.30) and the maximum principle for v, we have

$$
\sup_{x \in B_{\frac{1}{2}}(0)} |v(x) - u^{(\overline{h})}(x)|^2 \le c_9 \varepsilon_0^{\frac{1}{2}}. \tag{10.31}
$$

By (5.13) we have

$$
\sup_{B_{\frac{1}{4}}(0)} |Dv|^2 \le c_{10} \int_{B_{\frac{1}{2}}(0)} |Dv|^2 dx,
$$

while (10.29) and the minimality of v give

$$
\int_{B_{\frac{1}{2}}(0)} |Dv|^2 dx \le \int_{B_{\frac{1}{2}}(0)} |Du^{(\overline{h})}|^2 dx \le c_5 \int_{B_1(0)} |Du|^2 dx.
$$

Hence we have

$$
\sup_{B_{\frac{1}{4}}(0)} |Dv|^2 \le c_{11} \int_{B_1(0)} |Du|^2 dx. \tag{10.32}
$$

Since $D\Pi$ is bounded, for any $\theta \in \left(0, \frac{1}{4}\right)$, we can estimate

$$\begin{aligned}\frac{1}{\theta^{n-2}}\int_{B_\theta(0)}|Du_{\overline{h}}|^2dx &\le c_{13}\frac{1}{\theta^{n-2}}\int_{B_\theta(0)}|Du^{(\overline{h})}|^2dx\\ &\le c_{14}\frac{1}{\theta^{n-2}}\int_{B_\theta(0)}\left(|D(u^{(\overline{h})}-v)|^2+|Dv|^2\right)dx\\ &\le c_{15}\frac{1}{\theta^{n-2}}\int_{B_\theta(0)}|D(u^{(\overline{h})}-v)|^2dx\\ &\quad + c_{15}\theta^2\int_{B_1(0)}|Du|^2dx,\end{aligned} \tag{10.33}$$

where we also used

$$\fint_{B_\theta(x)}|Dv|^2dx \le \fint_{B_1(0)}|Dv|^2dx \le \fint_{B_1(0)}|Du|^2dx,$$

coming from (5.13) (remember that Dv is harmonic). Integration by parts and (10.31) give

$$\begin{aligned}\int_{B_{\frac{1}{2}}(0)}|D(u^{(\overline{h})}-v)|^2dx &= -\int_{B_{\frac{1}{2}}(0)}(u^{(\overline{h})}-v)\cdot\Delta(u^{(\overline{h})}-v)dx\\ &\le c_{16}\varepsilon_0^{\frac{1}{4}}\int_{B_{\frac{1}{2}}(0)}|\Delta u^{(\overline{h})}|dx.\end{aligned}$$

On the other hand, from the Euler-Lagrange equation for u we have

$$\Delta u^{(\overline{h})}(x) = \int_{B_{\overline{h}}(x)}\varphi^{(\overline{h})}(x-y)\sum_{i=1}^n A_u(D_iu, D_iu)dy,$$

hence, since $|A_u(D_iu, D_iu)| \le c_{16}|Du|^2$, Jensen's inequality gives

$$\int_{B_{\frac{1}{2}}(0)}|\Delta u^{(\overline{h})}(x)|dx \le c_{17}\int_{B_1(0)}|Du|^2dx.$$

Therefore we deduce from (10.33)

$$\frac{1}{\theta^{n-2}}\int_{B_\theta(0)}|Du_{\overline{h}}|^2dx \le c_{18}\left(\frac{\varepsilon_0^{\frac{1}{4}}}{\theta^{n-2}}+\theta^2\right)\int_{B_1(0)}|Du|^2dx, \tag{10.34}$$

for any $\theta \in \left(0, \frac{1}{4}\right)$, with $\overline{h} := \varepsilon_0^{\frac{1}{2n}}$ and c_{18} depending on N but independent of ε_0 and θ.

Step 3. In order to compare $u_{\overline{h}}$ to u, we modify $u_{\overline{h}}$ so that it agrees with u on $\partial B_1(0)$, and then we use the minimality of u.

Set $\tau := \varepsilon_0^\gamma$ where $\gamma \in \left(0, \frac{1}{16}\right]$ will be chosen depending only on n. We assume, by taking ε_0 possibly smaller, that $\tau < \frac{1}{2}$ and let p be the greatest integer such that $p \leq \frac{\tau}{3\varepsilon_0^{\frac{1}{8}}}$, and write

$$\left[\tau, \tau + 3p\varepsilon_0^{\frac{1}{8}}\right] = \bigcup_{i=1}^{p} I_i, \quad I_i := \left[\tau + 3(i-1)\varepsilon_0^{\frac{1}{8}}, \tau + 3i\varepsilon_0^{\frac{1}{8}}\right].$$

Since $\gamma \leq \frac{1}{16}$, we have $p \geq \frac{\tau}{3\varepsilon_0^{\frac{1}{8}}} - 1 \geq \frac{1}{3}\varepsilon_0^{-\frac{1}{16}} - 1 \geq c_{19}\varepsilon_0^{-\frac{1}{16}}$. Since

$$\int_{\{|x|\in[\tau,\tau+3p\varepsilon_0^{\frac{1}{8}}]\}} |Du|^2 dx \leq \sum_{i=1}^{p} \int_{\{|x|\in I_i\}} |Du|^2 dx \leq \int_{B_1(0)} |Du|^2 dx,$$

we can choose j with $1 \leq j \leq p$ such that

$$\int_{\{|x|\in I_j\}} |Du|^2 dx \leq \frac{1}{p}\int_{B_1(0)} |Du|^2 dx \leq \frac{\varepsilon_0^{\frac{1}{16}}}{c_{19}} \int_{B_1(0)} |Du|^2 dx. \tag{10.35}$$

Let θ be such that $I_j = \left[\theta - \varepsilon_0^{\frac{1}{8}}, \theta + 2\varepsilon_0^{\frac{1}{8}}\right]$ and let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing smooth function such that (see Figure 10.1)

1. $h(r) = \overline{h}$ for $r \leq \theta$;
2. $h(r) = 0$ for $r \geq \theta + \varepsilon_0^{\frac{1}{8}}$;
3. $|h'(r)| \leq 2\overline{h}\varepsilon_0^{-\frac{1}{8}} = 2\varepsilon_0^{\frac{1}{8}}$.

Set

$$u^{(h(r))}(x) := \varphi^{(h(r))} * u(x) = \int_{B_1(0)} \varphi^{(h(r))}(x-y)u(y)dy, \quad r := |x|,$$

and by (10.28) we can also define

$$u_0(x) := \Pi \circ u^{(h(r))}(x), \quad r := |x|.$$

It is easily seen that $u_0 \in W^{1,2}\left(B_{\theta+2\varepsilon_0^{\frac{1}{8}}}(0), N\right)$ and that

$$u_0(x) = \begin{cases} u_{\overline{h}}(x) & \text{for } |x| \leq \theta, \\ u(x) & \text{for } |x| \geq \theta + \varepsilon_0^{\frac{1}{8}}. \end{cases}$$

Therefore from the minimality of u we have

$$\begin{aligned} \int_{\{|x|\leq\theta+\varepsilon_0^{1/8}\}} |Du|^2 dx &\leq \int_{\{|x|\leq\theta+\varepsilon_0^{1/8}\}} |Du_0|^2 dx \\ &= \int_{B_\theta(0)} |Du_{\overline{h}}|^2 dx + \int_{\{|x|\in[\theta,\theta+\varepsilon_0^{1/8}]\}} |Du_0|^2 dx. \end{aligned} \tag{10.36}$$

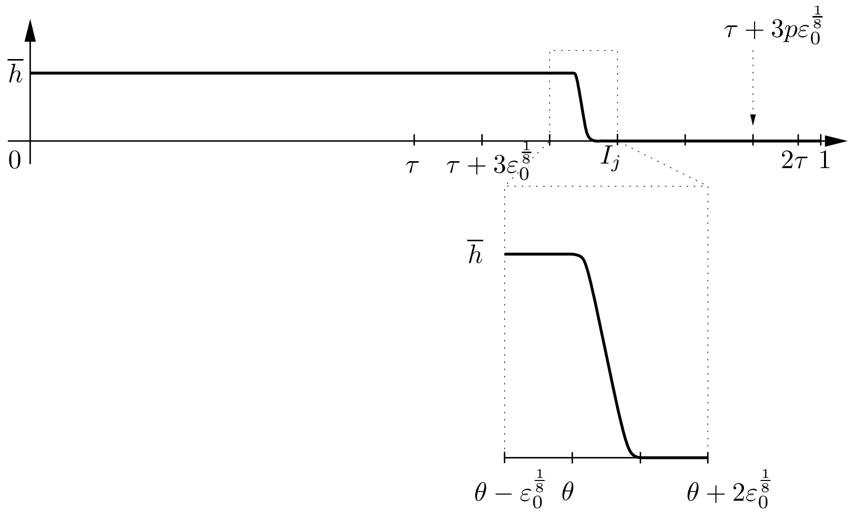

Figure 10.1: The cut-off function $h(r)$.

We now claim that

$$\int_{\{|x|\in[\theta,\theta+\varepsilon_0^{1/8}]\}} |Du_0|^2 dx \leq c_{20} \int_{I_j} |Du|^2 dx. \tag{10.37}$$

Combining (10.34), (10.35), (10.36) and (10.37) and using the fact that $\theta \in [\tau, 2\tau]$ we infer

$$\begin{aligned}\frac{1}{\tau^{n-2}} \int_{B_\tau(0)} |Du|^2 dx &\leq \frac{1}{\tau^{n-2}} \int_{B_\theta(0)} |Du_{\overline{h}}|^2 dx + \frac{c_{19}}{\tau^{n-2}} \varepsilon_0^{\frac{1}{16}} \int_{B_1(0)} |Du|^2 dx \\ &\leq c_{21}\Big(\frac{1}{\tau^{n-2}} \varepsilon_0^{\frac{1}{16}} + \tau^2\Big) \int_{B_1(0)} |Du|^2 dx \\ &\leq c_{21}(\varepsilon_0^{\frac{1}{16}-\gamma(n-2)} + \varepsilon_0^{2\gamma}) \int_{B_1(0)} |Du|^2 dx.\end{aligned}$$

Choosing $\gamma := \min\{[32(n-2)]^{-1}, 64^{-1}\}$, we get

$$\frac{1}{\tau^{n-2}} \int_{B_\tau(0)} |Du|^2 dx \leq c_{22}\varepsilon_0^{2\gamma} \int_{B_1(0)} |Du|^2 dx,$$

whence (10.25), provided ε_0 is small enough so that $c_{22}\varepsilon_0^{2\gamma} \leq \frac{1}{2}$.

Step 4. It remains to prove (10.37). By the boundedness of $D\Pi$ it is enough to prove it for $u^{(h(|x|))}(x)$ instead of $u_0(x)$. Remember that

$$u^{(h(|x|))}(x) = \int_{B_1(0)} \varphi(y) u(x - h(x)y) dy.$$

First assume u smooth, so that also $u^{(h(|x|))}(x)$ is smooth. Set

$$\Lambda := \{x : \theta \le |x| \le \theta + \varepsilon^{\frac{1}{8}}\}, \quad \Lambda_1 := \{x : \theta - \varepsilon^{\frac{1}{8}} < |x| < \theta + 2\varepsilon^{\frac{1}{8}}\}.$$

Next compute

$$D_\alpha u^{(h(|x|))}(x) = \int_{B_1(0)} \varphi(y)\big[D_\alpha u(x - h(x)y) - D_\alpha h(|x|)\cdot Du(x - h(x)y)\big]dy,$$

thus, observing that $h' \le 2$, we have

$$\int_\Lambda |Du^{(h(|x|))}(x)|^2 dx \le c_{23} \int_\Lambda \int_{B_1(0)} \varphi(y)^2 |Du(x - h(x)y)|^2 dy dx.$$

The map $x \mapsto x - h(x)y$ for each $y \in B_1(0)$ defines a diffeomorphism of Λ into Λ_1 with Jacobian close to 1, thus we have

$$\int_\Lambda |Du(x - h(x)y)|^2 dx \le 2 \int_{\Lambda_1} |Du|^2 dx;$$

therefore

$$\int_\Lambda |Du^{(h(|x|))}(x)|^2 dx \le c_{24} \int_{\Lambda_1} |Du|^2. \tag{10.38}$$

Now use (10.38) to prove that if $u_i \to u$ in $W^{1,2}(\Lambda_1, \mathbb{R}^p)$, then $u_i^{(h(x))}(x)$ is a Cauchy sequence in $W^{1,2}(\Lambda, \mathbb{R}^p)$ and converges to $u^{(h(x))}(x)$. Consequently (10.38) extends by density to any arbitrary $u \in W^{1,2}(B_1(0))$ and (10.37) is proved. $\square$

Corollary 10.17 *For any energy minimizing harmonic map*

$$u \in W^{1,2}_{\mathrm{loc}}(\Omega, N), \quad \Omega \subset \mathbb{R}^n,$$

we have

$$\mathcal{H}^{n-2}(\Sigma(u)) = 0,$$

where $\Sigma(u)$ is the singular set of u.

Proof. Apply Proposition 9.21 with $f = |Du|^2$. $\square$

10.3.2 The dimension reduction argument

Finally we prove

Theorem 10.18 *The singular set $\Sigma(u)$ of a locally energy minimizing harmonic map $u \in W^{1,2}_{\mathrm{loc}}(\Omega, N)$, $\Omega \subset \mathbb{R}^n$, N compact*

1. *contains only isolated points, if $n = 3$,*
2. *has dimension at most $n - 3$, if $n > 3$.*

As we shall see, essentially the proof of Theorem 10.11 works, if we can provide a compactness theorem replacing Lemma 10.13. This is done in Theorem 10.25 and Proposition 10.26.

Remark 10.19 Theorem 10.18 is sharp. For instance the map

$$u : \mathbb{R}^3 \to S^2, \qquad u(x) := \frac{x}{|x|}$$

is a locally minimizing harmonic map, compare [16] and [56].

The compactness theorem

In this section we shall prove the compactness theorem of Luckhaus; it generalizes earlier results of Schoen-Uhlenbeck [95] and Hardt-Lin [57]. Then we shall use it to prove Theorem 10.18.

Definition 10.20 *For any $v \in L^2(S^{n-1}, \mathbb{R}^p)$, set $\widetilde{v}(rx) := v(x)$ for every $r > 0$, $x \in S^{n-1}$, and define*

$$W^{1,2}(S^{n-1}, \mathbb{R}^p) := \Big\{ v \in L^2(S^{n-1}, \mathbb{R}^p) : \widetilde{v} \in W^{1,2}(U, \mathbb{R}^p) \\ \text{for some neighborhood } U \text{ of } S^{n-1} \Big\}.$$

By $W^{1,2}(S^{n-1}, N)$ we shall denote the maps $v \in W^{1,2}(S^{n-1}, \mathbb{R}^p)$ such that $v(x) \in N$ for a.e. $x \in S^{n-1}$.

Similarly, for any $v \in L^2(S^{n-1} \times [a,b], \mathbb{R}^p)$, set $\widetilde{v}(rx, t) := v(x,t)$ for every $r > 0$, $x \in S^{n-1}$, $t \in [a,b]$, and

$$W^{1,2}(S^{n-1} \times [a,b], \mathbb{R}^p) := \Big\{ v \in L^2(S^{n-1} \times [a,b], \mathbb{R}^p) : \\ \widetilde{v} \in W^{1,2}(U \times [a,b], \mathbb{R}^p) \text{ for some neighborhood } U \text{ of } S^{n-1} \Big\}$$

Lemma 10.21 (Luckhaus) *Let $N \subset \mathbb{R}^p$ be compact, $n \geq 2$, and consider $u, v \in W^{1,2}(S^{n-1}, N)$. Then there is a constant C such that for every $\varepsilon > 0$ there is a function $w \in W^{1,2}(S^{n-1} \times [0,\varepsilon], \mathbb{R}^p)$ such that $w\big|_{S^{n-1}\times\{0\}} = u$, $w\big|_{S^{n-1}\times\{\varepsilon\}} = v$,*

$$\int_{S^{n-1}\times[0,\varepsilon]} |\overline{\nabla} w|^2 d\mathcal{H}^n \leq C\varepsilon \int_{S^{n-1}} (|\nabla u|^2 + |\nabla v|^2) d\mathcal{H}^{n-1} \\ + \frac{C}{\varepsilon} \int_{S^{n-1}} |u - v|^2 d\mathcal{H}^{n-1} \tag{10.39}$$

and

$$\operatorname{dist}^2(w(x,s),N) \leq \frac{C}{\varepsilon^{n-1}}\Big(\int_{S^{n-1}}(|\nabla u|^2+|\nabla v|^2)d\mathcal{H}^{n-1}\Big)^{\frac{1}{2}} \times\Big(\int_{S^{n-1}}|u-v|^2 d\mathcal{H}^{n-1}\Big)^{\frac{1}{2}}+\frac{C}{\varepsilon^n}\int_{S^{n-1}}|u-v|^2 d\mathcal{H}^{n-1}, \tag{10.40}$$

for a.e. $(x,s)\in S^{n-1}\times[0,\varepsilon]$*. Here* ∇ *is the gradient on* S^{n-1} *and* $\overline{\nabla}$ *is the gradient on the product space* $S^{n-1}\times[0,\varepsilon]$*.*

Proof. In the case $n=2$ we choose the absolutely continuous representative for u and v on S^1. Then by 1-dimensional calculus on S^1 and Cauchy-Schwarz's inequality we have

$$\begin{aligned}\sup_{S^1}|u-v|^2 &\leq \int_{S^1}|\nabla|u-v|^2|d\mathcal{H}^1+\frac{1}{2\pi}\int_{S^1}|u-v|^2 d\mathcal{H}^1\\ &\leq C\Big(\int_{S^1}|\nabla(u-v)|^2 d\mathcal{H}^1\Big)^{\frac{1}{2}}\Big(\int_{S^1}|u-v|^2 d\mathcal{H}^1\Big)^{\frac{1}{2}}\\ &\quad+C\int_{S^1}|u-v|^2 d\mathcal{H}^1.\end{aligned} \tag{10.41}$$

Define now

$$w(\omega,s):=\Big(1-\frac{s}{\varepsilon}\Big)u(\omega)+\frac{s}{\varepsilon}v(\omega).$$

Then

$$|\overline{\nabla}w|\leq|\nabla u|+|\nabla(v-u)|+\frac{1}{\varepsilon}|v-u|,$$

hence

$$|\overline{\nabla}w|^2\leq 8(|\nabla u|^2+|\nabla v|^2)+\frac{2}{\varepsilon^2}|v-u|^2.$$

By integrating over $S^1\times[0,\varepsilon]$ we get at once (10.39). Moreover, as $u(S^1)\subset N$, (10.41) implies that for each $\omega\in S^1$, $s\in[0,\varepsilon]$ we have

$$\begin{aligned}\operatorname{dist}(w(\omega,s),N)&\leq C\Big(\int_{S^1}|\nabla(u-v)|^2 d\mathcal{H}^1\Big)^{\frac{1}{4}}\Big(\int_{S^1}|u-v|^2 d\mathcal{H}^1\Big)^{\frac{1}{4}}\\ &\quad+C\Big(\int_{S^1}|u-v|^2 d\mathcal{H}^1\Big)^{\frac{1}{2}},\end{aligned}$$

which is even stronger than (10.40), since there is no dependence on ε on the right-hand side.

For the case $n\geq 3$, we only show how to construct the function w, omitting the verification of (10.39) and (10.40). For a complete proof see [71], [98].

We first extend u and v to the cube $[-1,1]^n$ by $u(r\omega) := u(\omega)$ and $v(r\omega) := v(\omega)$ for every $\omega \in S^{n-1}$, $r > 0$, and then choose absolutely continuous representatives for u and v on $[-1,1]^n$.[6] Then, for any $\varepsilon \in \big(0, \frac{1}{8}\big)$, we partition $\mathbb{R}^n$ into cubes Q_i, $i = (i_1, \dots, i_n) \in \mathbb{Z}^n$,

$$Q_i := [i_1\varepsilon, (i_1+1)\varepsilon] \times \ldots \times [i_n\varepsilon, (i_n+1)\varepsilon],$$

and consider those cubes Q of the form $a + Q_i$ for any $Q_i \subset \big[-\frac{1}{2}, \frac{1}{2}\big]^n$, where a is a suitably chosen point inside $\big[-\frac{1}{2}, \frac{1}{2}\big]^n$. We then define w on $Q \times [0,\varepsilon]$ by an inductive procedure. First consider the 1-skeleton[7] Q^1 of Q and set

$$w(x,s) := \Big(1 - \frac{s}{\varepsilon}\Big)u(x) + \frac{s}{\varepsilon}v(x), \qquad x \in Q^1, s \in [0,\varepsilon].$$

For $k \geq 2$ we assume that w has been defined on $Q^{k-1} \times [0,\varepsilon]$ and extend it to $Q^k \times [0,\varepsilon]$ observing that for any k-dimensional face $F^k \subset Q^k$, w has already been defined on

$$\partial(F^k \times [0,\varepsilon]) \subset \Bigg(\bigcup_{F^{k-1}} F^{k-1} \times [0,\varepsilon]\Bigg) \cup F^k \times \{0,\varepsilon\}.$$

Hence we can use the homogeneous degree zero extension of $w\big|_{\partial(F^k\times[0,\varepsilon])}$ to $F^k \times [0,\varepsilon]$ with origin at $\big(q, \frac{\varepsilon}{2}\big)$, q being the center of F^k, to define w on $F^k \times [0,\varepsilon]$, thus on $Q^k \times [0,\varepsilon]$. This induction completes the definition of w on $Q \times [0,\varepsilon]$, and choosing several Q's, we can define w on all of $\big([-\frac{1}{4},\frac{1}{4}]^n \setminus [-\frac{1}{8},\frac{1}{8}]^n\big) \times [0,\varepsilon]$. By absolute continuity and Fubini's theorem, one can find $\rho \in \big[\frac{1}{8}, \frac{1}{4}\big]$, such that the restriction of w to $\partial([-\rho,\rho]^n) \times [0,\varepsilon]$ is a $W^{1,2}$-function. To obtain a function defined on $S^{n-1} \times [0,\varepsilon]$ it is enough to radially project $\partial([-\rho,\rho]^n)$ onto S^{n-1}, this being a bilipschitz tranformation. □

Remark 10.22 Given $g \geq 0$ integrable on $B_\rho(y)$, using the identity

$$\int_{B_\rho(y)\setminus B_{\frac{\rho}{2}}(y)} g\,dx = \int_{\frac{\rho}{2}}^{\rho} \Bigg(\int_{\partial B_\sigma(y)} g\,d\mathcal{H}^{n-1}\Bigg) d\sigma, \tag{10.42}$$

[6] An absolutely continuous representative u of a $W^{1,2}$-function on a cube $[a_1,b_1] \times \ldots \times [a_n,b_n]$ is an L^2-function such that the restrictions

$$u^{(j)}(x^j) := u(x^1, \dots, x^{j-1}, x^j, x^j+1, \dots, x^n)$$

are absolutely continuous as functions from $[a_j,b_j]$ into $\mathbb{R}$ for $\mathcal{H}^{n-1}$-a.e. $(x^1, \dots, x^{j-1}, x^{j+1}, \dots, x^n)$. A way to construct such a representative is to define $u(x) := \lambda_x$ at all points where there exists a λ_x such that

$$\lim_{\rho\to 0} \frac{1}{\rho^n} \int_{B_\rho(x)} |u(y) - \lambda_x|\,dy = 0,$$

and to define u arbitrarily at the points where such a λ_x cannot be found.

[7] The k-skeleton of the cube Q is the union of its k-dimensional faces.

we have that for all $\theta \in (0,1)$

$$\int_{\partial B_\sigma(y)} g d\mathcal{H}^{n-1} \leq \frac{2}{\theta\rho} \int_{B_\rho(y)\setminus B_{\frac{\rho}{2}}(y)} g dx \tag{10.43}$$

for all $\sigma \in \left(\frac{\rho}{2}, \rho\right)$ with the exception of a set of measure at most $\frac{\theta\rho}{2}$. Otherwise, integrating the reverse inequality on a set of measure greater than $\frac{\theta\rho}{2}$ would give

$$\int_{B_\rho(y)\setminus B_{\frac{\rho}{2}}(y)} g dx < \int_{\frac{\rho}{2}}^{\rho} \left(\int_{\partial B_\sigma(y)} g d\mathcal{H}^{n-1} \right) d\sigma,$$

contradicting (10.42).

Remark 10.23 Given $w \in W^{1,2}(\Omega, \mathbb{R})$, and a fixed ball $\overline{B_\rho(y)} \subset \Omega$, define $w_\sigma \in W^{1,2}(S^{n-1}, \mathbb{R})$ by $w_\sigma(\omega) := w(y + \sigma\omega)$, $\omega \in S^{n-1}$. Then it can be easily verified that for each $\theta \in (0,1)$, we have

$$\begin{aligned} \int_{S^{n-1}} |\nabla^{S^{n-1}} w_\sigma|^2 d\mathcal{H}^{n-1} &\leq \frac{1}{\sigma^{n-3}} \int_{\partial B_\sigma(y)} |Dw|^2 dx \\ &\leq \frac{2}{\theta}\left(\frac{2}{\rho}\right)^{n-2} \int_{B_\rho(y)\setminus B_{\frac{\rho}{2}}(y)} |Dw|^2 dx, \end{aligned} \tag{10.44}$$

for all $\sigma \in \left(\frac{\rho}{2}, \rho\right)$ with the exception of a set of measure at most $\frac{\theta\rho}{2}$.

Corollary 10.24 *Given a smooth compact manifold $N \subset \mathbb{R}^p$ and $\Lambda > 0$, there exist $\delta_0(n, N, \Lambda)$ and $C(n, N, \Lambda)$ such that the following holds:*

If $\varepsilon \in (0, \delta_0]$, and if $u, v \in W^{1,2}(B_{(1+\varepsilon)\rho}(y)\setminus B_\rho(y), N)$ satisfy

$$\begin{aligned} \frac{1}{\rho^{n-2}} \int_{B_{(1+\varepsilon)\rho}(y)\setminus B_\rho(y)} (|Du|^2 + |Dv|^2) dx &\leq \Lambda, \\ \frac{1}{\varepsilon^{2n}\rho^n} \int_{B_{(1+\varepsilon)\rho}(y)\setminus B_\rho(y)} |u - v|^2 &\leq \delta_0, \end{aligned}$$

then there is $w \in W^{1,2}(B_{(1+\varepsilon)\rho}(y)\setminus B_\rho(y), N)$ such that

$$w\big|_{\partial B_\rho(y)} = u\big|_{\partial B_\rho(y)}, \qquad w\big|_{\partial B_{(1+\varepsilon)\rho}(y)} = u\big|_{\partial B_{(1+\varepsilon)\rho}(y)}$$

and

$$\begin{aligned} \frac{1}{\rho^{n-2}} \int_{B_{(1+\varepsilon)\rho}(y)\setminus B_\rho(y)} |Dw|^2 dx \leq C \frac{1}{\rho^{n-2}} \int_{B_{(1+\varepsilon)\rho}(y)\setminus B_\rho(y)} (|Du|^2 + |Dv|^2) dx \\ + \frac{C}{\varepsilon^2 \rho^n} \int_{B_{(1+\varepsilon)\rho}(y)\setminus B_\rho(y)} |u - v|^2 dx. \end{aligned}$$

Proof. Up to translation, we can assume $y = 0$. To simplify the notation, we shall write $B_\rho := B_\rho(0)$ for every $\rho > 0$. By (10.43) and (10.44) there is a set of $\sigma \in \big(\rho, \big(1 + \frac{\varepsilon}{2}\big)\rho\big)$ of positive measure such that

$$\frac{1}{\sigma^{n-3}} \int_{\partial B_\sigma} (|Du|^2 + |Dv|^2) d\mathcal{H}^{n-1} \le \frac{C}{\varepsilon \rho^{n-2}} \int_{B_{(1+\varepsilon)\rho} \setminus B_\rho} (|Du|^2 + |Dv|^2) dx \tag{10.45}$$

and

$$\begin{aligned} \frac{1}{\sigma^{n-1}} \int_{\partial B_\sigma} |u - v|^2 d\mathcal{H}^{n-1} &\le \frac{C}{\varepsilon \rho^n} \int_{B_{(1+\varepsilon)\rho} \setminus B_\rho} |u - v|^2 dx \\ &\le C \delta_0^2 \varepsilon^{2n-1}. \end{aligned} \tag{10.46}$$

By (10.44) we know that, for almost all of these σ, $u, v \in W^{1,2}(\partial B_\sigma, \mathbb{R}^p)$. Now we can apply Luckhaus' lemma with $\frac{\varepsilon}{4}$ in place of ε to the functions $\widetilde{u}(\omega) := u(\sigma\omega)$ and $\widetilde{v}(\omega) := v(\sigma\omega)$, obtaining a function $\widetilde{w}$ on $S^{n-1} \times [0, \varepsilon/4]$ with $\widetilde{w} = \widetilde{u}$ on $S^{n-1} \times \{0\}$, $\widetilde{w} = \widetilde{v}$ on $S^{n-1} \times \{\varepsilon/4\}$ and

$$\begin{aligned} &\int_{S^{n-1} \times [0,\varepsilon/4]} |\overline{\nabla} \widetilde{w}|^2 d\mathcal{H}^n \\ &\le C\varepsilon \int_{S^{n-1}} (|\nabla \widetilde{u}|^2 + |\nabla \widetilde{v}|^2) d\mathcal{H}^{n-1} + \frac{C}{\varepsilon} \int_{S^{n-1}} |\widetilde{u} - \widetilde{v}|^2 d\mathcal{H}^{n-1} \\ &\le \frac{C\varepsilon}{\sigma^{n-3}} \int_{\partial B_\sigma} (|Du|^2 + |Dv|^2) d\mathcal{H}^{n-1} + \frac{C}{\varepsilon \sigma^{n-1}} \int_{\partial B_\sigma} |u - v|^2 d\mathcal{H}^{n-1} \\ &\le \frac{C}{\rho^{n-2}} \int_{B_{(1+\varepsilon)\rho} \setminus B_\rho} (|Du|^2 + |Dv|^2) dx + \frac{C}{\varepsilon^2 \rho^n} \int_{B_{(1+\varepsilon)\rho} \setminus B_\rho} |u - v|^2 dx, \end{aligned} \tag{10.47}$$

by (10.45) and (10.46). Moreover

$$\begin{aligned} \operatorname{dist}^2(\widetilde{w}, N) \le C &\Big(\int_{S^{n-1}} (|\nabla \widetilde{u}|^2 + |\nabla \widetilde{v}|^2) d\mathcal{H}^{n-1} \Big)^{\frac{1}{2}} \\ &\times \Big(\frac{1}{\varepsilon^{2n-2}} \int_{S^{n-1}} |\widetilde{u} - \widetilde{v}|^2 d\mathcal{H}^{n-1} \Big)^{\frac{1}{2}} \\ &+ \frac{1}{\varepsilon^n} \int_{S^{n-1}} |\widetilde{u} - \widetilde{v}|^2 d\mathcal{H}^{n-1}. \end{aligned} \tag{10.48}$$

Again by (10.45) and (10.46), the right-hand side of (10.48) is bounded by $C\delta_0$, where C depends only on n, N, Λ, hence for $\delta_0 = \delta_0(n, N, \Lambda)$ small enough we conclude that $\widetilde{w}$ maps into a small neighborhood N_α of N, where the closest point projection $\Pi : N_\alpha \to N$ is well defined. Now define $w \in W^{1,2}(B_{(1+\varepsilon/2)\sigma})$ by

$$w(x) := \begin{cases} u(x) & \text{if } |x| \leq \sigma \\ \Pi \circ \widetilde{w}\big(\omega, \frac{|x|}{\sigma} - 1\big) & \text{if } |x| \in (\sigma, (1+\varepsilon/4)\sigma) \\ v\big(\psi(|x|)\frac{x}{|x|}\big) & \text{if } |x| \in ((1+\varepsilon/4)\sigma, (1+\varepsilon/2)\sigma), \end{cases}$$

where $\psi \in C^1(\mathbb{R})$ satisfies

(i) $\psi((1+\varepsilon/4)\sigma) = \sigma$,

(ii) $\psi((1+\varepsilon/2)\sigma) = (1+\varepsilon/2)\sigma$,

(iii) $t|\psi'(t)| \leq 2$ for $t \in ((1+\varepsilon/4)\sigma, (1+\varepsilon/2)\sigma)$.

In view of (10.47) it is straightforward to verify that w satisfies the inequality stated in the corollary. □

Theorem 10.25 *Consider a sequence of energy minimizing harmonic maps $u_j \in W^{1,2}(\Omega, N)$ with* locally equibounded energies*, i.e. such that for every $\overline{B_R(x_0)} \subset \Omega$, we have*

$$\sup_{j\in\mathbb{N}} \int_{B_R(x_0)} |Du_j|^2 dx < +\infty.$$

Then a subsequence u_{j_k} converges in $W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^p)$ to an energy minimizing harmonic map $u \in W^{1,2}_{\mathrm{loc}}(\Omega, N)$.

Proof. By Rellich's and Banach-Alaoglu's theorems we can assume that, up to a subsequence,

$$u_j \to u \text{ strongly in } L^2_{\mathrm{loc}}(\Omega, N) \text{ and weakly in } W^{1,2}_{\mathrm{loc}}(\Omega, N),$$

for some $u \in W^{1,2}_{\mathrm{loc}}(\Omega, N)$. Let $\overline{B_{\rho_0}(y)} \subset \Omega$ and let $\delta > 0$ and $\theta \in (0,1)$ be given. Choose any positive integer M such that

$$\limsup_{j\to\infty} \frac{1}{\rho_0^{n-2}} \int_{B_{\rho_0}(y)} |Du_j|^2 dx < M\delta,$$

and note that if $\varepsilon \in \big(0, \frac{1-\varepsilon}{M}\big)$, then there is some integer l such that

$$\frac{1}{\rho_0^{n-2}} \int_{B_{\rho_0(\theta+l\varepsilon)}(y)\setminus B_{\rho_0(\theta+(l-2)\varepsilon)}(y)} |Du_j|^2 dx < \delta$$

for infinitely many j, because otherwise we could sum over l and get

$$\rho_0^{2-n} \int_{B_{\rho_0}(y)} |Du_j|^2 dx \geq M\delta$$

for all sufficiently large j, contrary to the definition of M. Now choose such an l and set $\rho := \rho_0(\theta + (l-2)\varepsilon)$; noting that $\rho(1+\varepsilon) \le \rho_0(\theta + l\varepsilon)$, we get

$$\frac{1}{\rho_0^{n-2}} \int_{B_{\rho(1+\varepsilon)}(y)\setminus B_\rho(y)} |Du_{j_k}|^2 dx < \delta \tag{10.49}$$

for some subsequence u_{j_k}. By weak convergence of Du_{j_k} to Du in L^2, we get

$$\frac{1}{\rho_0^{n-2}} \int_{B_{\rho(1+\varepsilon)}(y)\setminus B_\rho(y)} |Du|^2 dx \le \delta. \tag{10.50}$$

Since

$$\int_{B_{\rho_0}(y)} |u - u_{j_k}|^2 dx \to 0,$$

by Corollary 10.24 we can find a function $w_{j_k} \in W^{1,2}(B_{\rho(1+\varepsilon)}(y)\setminus B_\rho(y), N)$ with $w_{j_k} = u$ on $\partial B_\rho(y)$ and $w_{j_k} = u_{j_k}$ on $\partial B_{\rho(1+\varepsilon)}(y)$, and

$$\begin{aligned} \frac{1}{\rho^{n-2}} & \int_{B_{\rho(1+\varepsilon)}(y)\setminus B_\rho(y)} |Dw_{j_k}|^2 dx \\ & \le \frac{C}{\rho^{n-2}} \int_{B_{\rho(1+\varepsilon)}(y)\setminus B_\rho(y)} \left(|Du|^2 + |Du_{j_k}|^2 + \frac{|u-u_{j_k}|^2}{\varepsilon^2\rho^2}\right) dx, \end{aligned} \tag{10.51}$$

where C depends only on n and N. Now take any $v \in W^{1,2}(B_{\theta\rho_0}(y), N)$ with $v = u$ on $\partial B_{\theta\rho_0}(y)$, extend v to a function $\widetilde{v} \in W^{1,2}(B_{\rho_0}(y), N)$ by setting $\widetilde{v} = u$ on $B_{\rho_0}(y)\setminus B_{\theta\rho_0}(y)$, and define

$$\widetilde{u}_{j_k} := \begin{cases} u_{j_k} & \text{on } B_{\rho_0}(y)\setminus B_{(1+\varepsilon)\rho}(y) \\ w_{j_k} & \text{on } B_{(1+\varepsilon)\rho}(y)\setminus B_\rho(y) \\ \widetilde{v} & \text{on } B_\rho(y). \end{cases}$$

Then, by the minimizing property of u_j, we have

$$\begin{aligned} \int_{B_{(1+\varepsilon)\rho}(y)} |Du_{j_k}|^2 dx & \le \int_{B_{(1+\varepsilon)\rho}(y)} |D\widetilde{u}_{j_k}|^2 dx \qquad (10.52) \\ & \le \int_{B_\rho(y)} |D\widetilde{v}|^2 dx + \int_{B_{(1+\varepsilon)\rho}(y)\setminus B_\rho(y)} |Dw_{j_k}|^2 dx, \end{aligned}$$

hence, by (10.49), (10.50), (10.51),

$$\begin{aligned} \frac{1}{\rho^{n-2}} \int_{B_\rho(y)} |Du|^2 dx & \le \liminf_{k\to\infty} \frac{1}{\rho^{n-2}} \int_{B_\rho(y)} |Du_{j_k}|^2 dx \\ & \le \frac{1}{\rho^{n-2}} \int_{B_\rho(y)} |D\widetilde{v}|^2 dx + C\delta. \end{aligned} \tag{10.53}$$

Therefore

$$\frac{1}{\rho^{n-2}}\int_{B_{\theta\rho_0}(y)}|Du|^2dx \le \frac{1}{\rho^{n-2}}\int_{B_{\theta\rho_0}(y)}|Dv|^2dx.$$

Since $\delta > 0$ was arbitrary, this shows that u is minimizing on $B_{\theta\rho_0}(y)$, and in view of the arbitrariness of θ, ρ and y, u is a locally minimizing harmonic map.

To prove that the convergence is strong, we note that if we use (10.53) with $v = u$, then we can conclude

$$\liminf_{k\to\infty}\frac{1}{\rho^{n-2}}\int_{B_\rho(y)}|Du_{j_k}|^2dx \le \frac{1}{\rho^{n-2}}\int_{B_\rho(y)}|Du|^2dx + C\delta,$$

hence, by the arbitrariness of θ and δ,

$$\liminf_{k\to\infty}\frac{1}{\rho^{n-2}}\int_{B_{\rho_1}(y)}|Du_{j_k}|^2dx \le \frac{1}{\rho^{n-2}}\int_{B_{\rho_0}(y)}|Du|^2dx,$$

for each $\rho_1 < \rho_0$. It follows from this that

$$\liminf_{k\to\infty}\int_{B_\rho(y)}|Du_{j_k}|^2dx \le \int_{B_\rho(y)}|Du|^2dx, \tag{10.54}$$

for every ball $\overline{B_\rho(y)} \subset \Omega$. Now writing

$$\begin{aligned}\int_{B_\rho(y)}|Du_{j_k} - Du|^2dx = &\int_{B_\rho(y)}|Du_{j_k}|^2dx + \int_{B_\rho(y)}|Du|^2dx \\ &- 2\int_{B_\rho(y)}Du_{j_k}\cdot Dudx,\end{aligned}$$

and observing that the left-hand side is nonnegative, (10.54) implies strong convergence on $B_\rho(y)$, hence in $W^{1,2}_{\mathrm{loc}}(\Omega, N)$. □

An important consequence of the compactness theorem of Luckhaus is the semicontinuity of the density:

Proposition 10.26 *The density function $\Theta_u(y)$ is upper semicontinuous with respect to the joint variables y and u, meaning that if $y_j \to y$ and $\{u_j\} \subset W^{1,2}_{\mathrm{loc}}(\Omega)$ is a sequence of locally energy minimizing maps with locally equibounded energies and $u_j \rightharpoonup u$ in L^2, then*

$$\Theta_u(y) \ge \limsup \Theta_{u_j}(y_j).$$

Proof. By Luckhaus' compactness Theorem 10.25, $u_j \to u$ strongly in $W^{1,2}_{\mathrm{loc}}(\Omega)$. Let $y_j \to y$ and fix $\rho, \varepsilon > 0$ such that $\overline{B_{\rho+\varepsilon}(y)} \subset \Omega$. For j large enough $|y - y_j| < \varepsilon$, hence $B_\rho(y_j) \subset B_{\rho+\varepsilon}(y)$, implying

$$\Theta_{u_j}(y_j) \le \frac{1}{\rho^{n-2}}\int_{B_\rho(y_j)}|Du_j|^2dx \le \frac{1}{\rho^{n-2}}\int_{B_{\rho+\varepsilon}(y)}|Du_j|^2dx.$$

By the $W^{1,2}_{\text{loc}}$-convergence we get, for j large enough, that

$$\frac{1}{\rho^{n-2}}\int_{B_{\rho+\varepsilon}(y)}|Du_j|^2dx \le \frac{1}{\rho^{n-2}}\int_{B_{\rho+\varepsilon}(y)}|Du|^2dx+\varepsilon,$$

hence

$$\limsup_{j\to\infty}\Theta_{u_j}(y_j)\le\frac{1}{\rho^{n-2}}\int_{B_{\rho+\varepsilon}(y)}|Du|^2dx+\varepsilon.$$

Letting $\varepsilon\to 0$ first, and then $\rho\to 0$, completes the proof. □

Proof of Theorem 10.18. Let $n=3$ and assume that we have a sequence of singular points x_ν converging to x_0, and we can assume $x_0=0$. Rescaling as in the proof of Theorem 10.11, we find an equibounded sequence of harmonic maps $u^{(\nu)}(x):=u(2|x_\nu|x)$ with singular points $y_\nu=\frac{x_\nu}{2|x_\nu|}$, $|y_\nu|=\frac{1}{2}$; by Theorem 10.25 we may assume, up to a subsequence, that $u^{(\nu)}\to v$ in $W^{1,2}_{\text{loc}}(B_1(0),\mathbb{R}^p)$, where v is energy minimizing. We can also assume that $y_\nu\to y_0$, $|y_0|=\frac{1}{2}$, and by Theorem 10.15 and Proposition 10.26, we have that y_0 is a singular point.

We now claim that v is positively homogeneous of degree 0: by $W^{1,2}$-convergence, we have

$$\begin{aligned}\frac{1}{\rho^{n-2}}\int_{B_\rho(0)}|Dv|^2dx &= \lim_{\nu\to\infty}\frac{1}{\rho^{n-2}}\int_{B_\rho(0)}|Du^{(\nu)}|^2dx\\ &= \lim_{\nu\to\infty}\frac{1}{(2\rho|x_\nu|)^{n-2}}\int_{B_{2\rho|x_\nu|}(0)}|Du|^2dx\\ &= \Theta_u(0),\end{aligned}\tag{10.55}$$

hence the left-hand side does not depend on ρ. Then, by the monotonicity formula (10.8), $\left|\frac{\partial v}{\partial r}\right|=0$ a.e., and the claim follows. We now have that the whole segment $\{\lambda y_0:\lambda>0\}\cap B_1(0)$ is singular, contrary to Corollary 10.17.

The second part of the proof follows exactly as in Theorem 10.11. □

10.3.3 The stratification of the singular set

We now discuss the structure of the singular set of a locally energy minimizing harmonic map as related to the tangent maps at singular points. The techniques used here can also be found in the theory of minimal surfaces and mean curvature flow. We shall closely follow [99], to which we refer for a deeper discussion of the singular set of energy minimizing harmonic maps.

Tangent maps

Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, N)$ be a locally minimizing harmonic map, consider a ball $B_R(y) \subset \Omega$ and define the *rescaled* map $u_{y,\rho} \in W^{1,2}(B_{\frac{R}{\rho}}(0), N)$ by

$$u_{y,\rho}(x) := u(y + \rho x). \tag{10.56}$$

As we already saw in the proof of Theorem 10.18, we have

Proposition 10.27 (Blow-up) *There exist a sequence $\rho_j \to 0^+$ and a locally minimizing harmonic map $\varphi \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n, N)$ such that $u_{y,\rho_j} \to \varphi$ in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^p)$. Moreover φ is* positively homogeneous of degree zero*:*

$$\varphi(\lambda x) = \varphi(x), \quad \text{for all } x \in \mathbb{R}^n, \lambda > 0. \tag{10.57}$$

Remark 10.28 Tangent maps need not be unique, as shown by B. White [114].

Remark 10.29 By Theorem 10.15 and equation (10.55), for any $y \in \Omega$, the following facts concerning a locally minimizing harmonic map u are clearly equivalent:

1. u is regular at y;
2. $\Theta_u(y) = 0$;
3. there exists a *constant* tangent map φ for u at y.

Proposition 10.30 *Let $\varphi \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n, N)$ be a locally minimizing homogeneous of degree zero harmonic map. Then, for every $y \in \mathbb{R}^n$ we have $\Theta_\varphi(y) \le \Theta_\varphi(0)$. Set*

$$S(\varphi) := \{y \in \mathbb{R}^n : \Theta_\varphi(y) = \Theta_\varphi(0)\}. \tag{10.58}$$

Then $S(\varphi)$ is a linear subspace of $\mathbb{R}^n$ and

$$\varphi(x + y) = \varphi(x), \quad \text{for every } x \in \mathbb{R}^n, y \in S(\varphi). \tag{10.59}$$

Proof. As $\sigma \to 0^+$ in (10.8), we obtain

$$2\int_{B_\rho(y)} \frac{1}{R^{n-2}}\left|\frac{\partial \varphi}{\partial r}\right|^2 dx + \Theta_\varphi(y) = \frac{1}{\rho^{n-2}}\int_{B_\rho(y)} |D\varphi^2| dx, \tag{10.60}$$

where

$$R := |x - y|, r := \frac{x - y}{|x - y|}.$$

Since $B_\rho(y) \subset B_{\rho+|y|}(0)$ and

$$\frac{1}{\sigma^{n-2}}\int_{B_\sigma(0)}|D\varphi|^2 = \Theta_\varphi(0) \quad \text{for every } \sigma>0,$$

we have

$$\begin{aligned}\frac{1}{\rho^{n-2}}\int_{B_\rho(y)}|D\varphi|^2dx &\le \frac{1}{\rho^{n-2}}\int_{B_{\rho+|y|}(0)}|D\varphi|^2dx\\ &= \Big(1+\frac{|y|}{\rho}\Big)^{n-2}\frac{1}{(\rho+|y|)^{n-2}}\int_{B_{\rho+|y|}(0)}|D\varphi|^2dx\\ &= \Big(1+\frac{|y|}{\rho}\Big)^{n-2}\Theta_\varphi(0). \qquad (10.61)\end{aligned}$$

As we let $\rho\to+\infty$, we infer from (10.60) and (10.61)

$$2\int_{\mathbb{R}^n}\frac{1}{R^{n-2}}\Big|\frac{\partial\varphi}{\partial r}\Big|^2dx+\Theta_\varphi(y)\le\Theta_\varphi(0). \qquad (10.62)$$

This implies at once that $\Theta_\varphi(y)\le\Theta_\varphi(0)$. Moreover if $y\in S(\varphi)$, that is $\Theta_\varphi(y)=\Theta_\varphi(0)$, then $\frac{\partial\varphi}{\partial r}(x)=0$ for a.e. x, with $r:=\frac{x-y}{|x-y|}$. In particular

$$\varphi(y+\lambda x)=\varphi(y), \quad \forall x\in\mathbb{R}^n, y\in S(\varphi), \lambda>0.$$

Using homogeneity we infer

$$\begin{aligned}\varphi(x)=\varphi(\lambda x) &= \varphi(y+(\lambda x-y))=\varphi\Big(y+\frac{\lambda x-y}{\lambda^2}\Big)\\ &= \varphi\Big(\lambda\Big(y+\frac{\lambda x-y}{\lambda^2}\Big)\Big)=\varphi(x+ty), \qquad t:=\lambda-\lambda^{-1}.\end{aligned}$$

Since t may be chosen to be any real number, we have obtained

$$\varphi(x+ty)=\varphi(x), \quad \forall x\in\mathbb{R}^n, y\in S(\varphi), t\in\mathbb{R}.$$

For $y_1,y_2\in S(\varphi)$ and $t_1,t_2\in\mathbb{R}$ we then have

$$\varphi(x+t_1y_1+t_2y_2)=\varphi(x), \quad \forall x\in\mathbb{R}^n,$$

which implies that $\Theta_\varphi(t_1y_1+t_2y_2)=\Theta_\varphi(0)$, hence $t_1y_1+t_2y_2\in S(\varphi)$ and the proof is complete. □

Remark 10.31 If φ is non-constant, then by homogeneity it is discontinuous at 0, hence 0 lies in $\Sigma(\varphi)$, the singular set of φ. Then by (10.59) we have

$$S(\varphi)\subset\Sigma(\varphi), \qquad (10.63)$$

for any non-constant local minimizer homogeneous of degree 0.

Definition 10.32 *Given a locally minimizing harmonic map*

$$u \in W^{1,2}_{\mathrm{loc}}(\Omega, N)$$

define

$$S_j(u) := \{x \in \Sigma(u) : \dim S(\varphi) \le j \text{ for every tangent map } \varphi \text{ of } u \text{ at } x\},$$

where $\Sigma(u)$ *is the singular set of* u.

Lemma 10.33 *For a locally minimizing harmonic map* $u \in W^{1,2}_{\mathrm{loc}}(\Omega, N)$ *we have*

$$S_0(u) \subset S_1(u) \subset \ldots \subset S_{n-3}(u) = S_{n-2}(u) = S_{n-1}(u) = \Sigma(u).$$

Proof. The inclusions are obvious. $S_{n-1}(u) = \Sigma(u)$ is a consequence of Remark 10.29. Since we also know that

$$S_{n-3}(u) \subset S_{n-2}(u) \subset S_{n-1}(u) = \Sigma(u),$$

to conclude the proof it is enough to show that $\Sigma(u) \subset S_{n-3}$. Consider $x \in \Sigma(u)$. Any tangent map φ for u at x is a non-constant locally minimizing harmonic map. By Remark 10.31 $S(\varphi) \subset \Sigma(\varphi)$. Therefore if $\dim S(\varphi) \ge n-2$, then $\mathcal{H}^{n-2}(\Sigma(\varphi)) = +\infty$, contradicting Corollary 10.17. Hence $\dim S(\varphi) \le n-3$, and $x \in S_{n-3}(u)$. □

Proposition 10.34 *For any* $j = 0, 1, \ldots, n-3$ *we have* $\dim^{\mathcal{H}} S_j(u) \le n-3$. *For any* $\alpha > 0$, *and* $n = 3$ $S_0(u) \cap \{x \in \Omega : \Theta_u(x) > \alpha\}$ *is a* discrete *set.*

A straightforward corollary of this proposition is an alternative proof of Theorem 10.18.

Corollary 10.35 *For any energy minimizing harmonic map*

$$u \in W^{1,2}_{\mathrm{loc}}(\Omega, N)$$

we have

$$\dim^{\mathcal{H}} \Sigma(u) \le n-3.$$

For $n = 3$, $\Sigma(u)$ *is a discrete set.*

Proof. The first assertion is a direct consequence of Lemma 10.33 and Proposition 10.34. For $n = 3$ take $\alpha = \varepsilon_0$, and apply Theorem 10.15: □

To prove Proposition 10.34, we need the following

Lemma 10.36 *Set* $\eta_{y,\rho}(x) := \frac{1}{\rho}(x-y)$. *Then for each* $y \in S_j(u)$ *and each* $\delta > 0$, *there is an* $\varepsilon = \varepsilon(u, y, \delta) > 0$ *such that for every* $\rho \in (0, \varepsilon]$ *we have*

$$\eta_{y,\rho}\{x \in B_\rho(y) : \Theta_u(x) \geq \Theta_u(y) - \varepsilon\} \subset \{x \in \mathbb{R}^n : \mathrm{dist}(x, L_{y,\rho}) < \delta\},$$

for some j*-dimensional subspace* $L_{y,\rho}$.

Proof. Were the lemma false, we could find $\delta > 0$, $y \in S_j$ and two sequences $\rho_k \to 0$, $\varepsilon_k \to 0$ such that

$$\{x \in B_1(0) : \Theta_{u_{y,\rho_k}}(x) \geq \Theta_u(y) - \varepsilon_k\} \not\subseteq \{x \in \mathbb{R}^n : \mathrm{dist}(x, L) < \delta\}, \tag{10.64}$$

for every j-dimensional subspace L of $\mathbb{R}^n$, where $u_{y,\rho_k}(x) := u(y+\rho x)$. Up to a subsequence $u_{y,\rho_k} \to \varphi$ for some tangent map φ, and $\Theta_u(y) = \Theta_\varphi(0)$. Since $y \in S_j$ we have $\dim S(\varphi) \leq j$, and we can set L_0 to be any j-dimensional subspace containing $S(\varphi)$. Since for $x \in B_1(0)\backslash S(\varphi)$ we have $\Theta_\varphi(x) < \Theta_\varphi(0)$, we conclude by compactness and upper semicontinuity of the density that there exists $\alpha > 0$ such that

$$\Theta_\varphi(x) < \Theta_\varphi(0) - \alpha, \quad \text{for all } x \in \overline{B_1(0)},\ \mathrm{dist}(x, L_0) \geq \delta. \tag{10.65}$$

We want to show that for k large enough, we have

$$\{x \in B_1(0) : \Theta_{u_{y,\rho_k}}(x) \geq \Theta_\varphi(0) - \alpha\} \subset \{x \in \mathbb{R}^n : \mathrm{dist}(x, L_0) < \delta\}. \tag{10.66}$$

Indeed assume the inclusion false for arbitrarily large k. Up to a subsequence, we can find points $x_k \to x_0$ in $\overline{B_1(0)}$, with $\mathrm{dist}(x_k, L_0) \geq \delta$, such that

$$\Theta_{u_{y,\rho_k}}(x_k) \geq \Theta_\varphi(0) - \alpha.$$

By upper semicontinuity, this implies

$$\Theta_\varphi(x_0) \geq \Theta_\varphi(0) - \alpha,$$

contradicting (10.65). Therefore we have proved (10.66), which implies that (10.64) cannot be true for every j-dimensional subspace $L \subset \mathbb{R}^n$. □

Proof of Proposition 10.34. Fix $\delta > 0$ and for any integer $i \geq 1$ set $S_{j,i}(u)$ to be the set of points in $S_j(u)$ such that the statement of Lemma 10.36 holds with $\varepsilon = \frac{1}{i}$. By Lemma 10.36 we have $S_j(u) = \cup_{i=1}^\infty S_{j,i}(u)$. Next for any integer $q \geq 1$ define

$$S_{j,i,q}(u) := \Big\{x \in S_{j,i}(u) : \Theta_u(x) \in \Big(\frac{q-1}{i}, \frac{q}{i}\Big]\Big\}.$$

Clearly

$$S_j(u) = \bigcup_{i,q=1}^{+\infty} S_{j,i,q}(u).$$

For any $y \in S_{j,i,q}(u)$ we trivially have

$$S_{j,i,q}(u) \subset \left\{x \in S_{j,u}(u) : \Theta_u(x) > \Theta_u(y) - \frac{1}{i}\right\},$$

hence, by Lemma 10.36 with $\varepsilon = \frac{1}{i}$, we have that for any $\rho \le \frac{1}{i}$

$$\eta_{y,\rho}(S_{j,i,q}(u) \cap B_\rho(y)) \subset \{x \in \mathbb{R}^n : \mathrm{dist}(x, L_{y,\rho}) < \delta\},$$

for some j-dimensional subspace $L_{y,\rho} \subset \mathbb{R}^n$. Thus every set $S_{j,i,q}$ has the δ-*approximation property*, as defined below, with $\rho_0 = \frac{1}{i}$, and for every $\delta > 0$, hence

$$\dim^{\mathcal{H}} S_{j,i,q}(u) \le j$$

by Lemma 10.38 below. Since the Hausdorff dimension does not increase under countable union, the first part of the propostition is proved.

For the second part, assume that $x_k \in S_0(u) = \Sigma(u)$ and $x_k \to x_0$, $x_k \neq x_0$. We may assume without loss of generality that $x_0 = 0$. Consider the rescaled maps $u_k(x) := u(|x_k|x)$. Up to a subsequence, u_k converges in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^p)$ to a tangent map φ and $\frac{x_k}{|x_k|} \to \xi \in S^{n-1}$. By upper semicontinuity of the density, $\Theta_\varphi(\xi) > \alpha$, thus ξ is singular for φ. By 0-homogeneity of φ, the half line $\{\lambda\xi : \lambda > 0\}$ lies in the singular set $\Sigma(\varphi)$, hence $\mathcal{H}^{n-2}(\Sigma(\varphi)) = +\infty$, contradicting Corollary 10.17. $\square$

Definition 10.37 *A set $A \subset \mathbb{R}^n$ is said to satisfy the δ-*approximation property *if there is $\rho_0 > 0$ such that for every $y \in A$ and every $\rho \in (0, \rho_0]$*

$$\eta_{y,\rho}(A \cap B_\rho(y)) \subset \{x \in \mathbb{R}^n : \mathrm{dist}(x, L_{y,\rho}) < \delta\},$$

for some j-dimensional subspace $L_{y,\rho} \subset \mathbb{R}^n$.

Lemma 10.38 *There is a function $\beta : (0, +\infty) \to (0, +\infty)$ with*

$$\lim_{\delta \to 0^+} \beta(\delta) = 0$$

such that, if $\delta > 0$ and $A \subset \mathbb{R}^n$ satisfies the δ-approximation property above, then $\mathcal{H}^{j+\beta(\delta)}(A) = 0$.

Proof. For $\delta \ge \frac{1}{8}$, fix $\beta(\delta) = n - j + 1$, so that $\mathcal{H}^{j+\beta(\delta)}(A) = \mathcal{H}^{n+1}(A) = 0$. From now on let $\delta \in \left(0, \frac{1}{8}\right)$. It is easy to see that there is a constant c_j such that for every $\sigma \in \left(0, \frac{1}{2}\right)$ we can cover the closed unit ball $\overline{B_1(0)} \subset \mathbb{R}^j$ with a finite collection of balls $\{B_\sigma(y_k)\}_{k=1,\dots,Q}$, $y_k \in \overline{B_1(0)}$, such that $Q \le \frac{c_j}{\sigma^j}$. For any σ we can find $\beta = \beta(\sigma)$ such that $Q\sigma^{j+\beta(\sigma)} \le \frac{1}{2}$, and $\lim_{\sigma \to 0^+} \beta(\sigma) = 0$.

Let now $L \subset \mathbb{R}^n$ be a j-dimensional subspace. From the discussion above it follows that for any $\delta \in \left(0, \frac{1}{8}\right)$, $\sigma := 4\delta$, we can find balls $B_\sigma(y_k) \subset \mathbb{R}^n$ such that

$$\{x \in \mathbb{R}^n \cap \overline{B_1(0)} : \mathrm{dist}(x, L) < \delta\} \subset \bigcup_{k=1}^{Q} B_\sigma(y_k), \quad Q\sigma^{j+\beta(\delta)} \le \frac{1}{2}.$$

By scaling we obtain that, for a suitable choice of the centers $y_k \in L \cap B_R(0)$,

$$\{x \in \mathbb{R}^n \cap \overline{B_R(0)} : \mathrm{dist}(x, L) < \delta R\} \subset \bigcup_{k=1}^{Q} B_{\sigma R}(y_k),$$

and

$$Q(\sigma R)^{j+\beta(\delta)} \leq \frac{1}{2} R^{j+\beta(\delta)}.$$

Now let A satisfy the δ-approximation property for some ρ_0. We assume without loss of generality that A is bounded, we cover it by balls $B_{\frac{\rho_0}{2}}(y_k)$, $k = 1, \dots, Q$, $A \cap B_{\frac{\rho_0}{2}}(y_k) \neq \emptyset$, and set $T_0 := Q\big(\frac{\rho_0}{2}\big)^{j+\beta(\delta)}$. For each k pick $z_k \in A \cap B_{\frac{\rho_0}{2}}(y_k)$; by the δ-approximation property with $\rho = \rho_0$, $A \cap B_{\frac{\rho_0}{2}}(y_k)$ is contained in the $2\rho_0\delta$-neighborhood of some j-dimensional subspace L_k. Since $L_k \cap B_{\frac{\rho_0}{2}}(y_k)$ is a j-dimensional disk, by the discussion above, its $2\delta\rho_0$-neighborhood (and so also $A \cap B_{\frac{\rho_0}{2}}(y_k)$) can be covered by balls $B_{\frac{\sigma\rho_0}{2}}(z_{k,l})$, $l = 1, \dots, P$ such that

$$P\Big(\frac{\sigma\rho_0}{2}\Big)^{j+\beta(\delta)} \leq \frac{1}{2}\Big(\frac{\rho_0}{2}\Big)^{j+\beta(\delta)}.$$

Therefore A can be covered by balls $B_{\frac{\sigma\rho_0}{2}}(w_l)$, $l = 1, \dots, M$ such that

$$M\Big(\frac{\sigma\rho}{2}\Big)^{j+\beta(\delta)} \leq \frac{1}{2} T_0.$$

Proceeding iteratively, for every $q \in \mathbb{N}$, we can find a cover of balls

$$B_{\frac{\sigma^q\rho_0}{2}}(w_l), \quad l = 1, \dots, M_q, \qquad R_q\Big(\frac{\sigma^q\rho_0}{2}\Big)^{j+\beta(\delta)} \leq \frac{T_0}{2^q}.$$

As $Q \to \infty$, this proves that $\mathcal{H}^{j+\beta(\delta)}(A) = 0$. □

10.4 Regularity of 2-dimensional weakly harmonic maps

A direct consequence of Schoen-Uhlenbeck's theorem (Theorem 10.15) is that a 2-dimensional locally energy minimizing harmonic map is in fact smooth. Whether this also holds for 2-dimensional weakly harmonic maps (i.e. $W^{1,2}$-functions weakly solving (10.5)) is far from obvious, but still true, as proven by F. Hélein.

Theorem 10.39 (Hélein [59]) *Let $u \in W^{1,2}(\Omega, N)$ be a weakly harmonic map from the 2-dimensional domain $\Omega \subset \mathbb{R}^2$ into the closed (and smooth) manifold N. Then $u \in C^\infty(\Omega, N)$.*

We will present the elegant and simple proof of Hélein when the target manifold N is the round sphere S^n. Also the general case was obtained by Hélein, with the moving frame technique, but we will present a more recent proof, due to T. Rivière [88] (see also [89]), whose interest actually goes beyond the case of harmonic maps.

Let us also remark that continuous weakly harmonic maps are in fact smooth:

Proposition 10.40 *Let $u \in W^{1,2}_{\mathrm{loc}} \cap C^0_{\mathrm{loc}}(\Omega, N)$ be a weakly harmonic map, where N is a closed manifold and $\Omega \subset \mathbb{R}^n$. Then u is smooth.*

Proof. Since u satisfies (10.5), and

$$\left| \sum_{\alpha=1}^{n} A_u(D_\alpha u, D_\alpha u) \right| \le c|Du|^2$$

we can apply Theorem 9.10 to show that u is Hölder continuous, and then bootstrap regularity using Schauder's theory. □

Then, as in the case of energy minimizing harmonic maps, the problem is to prove continuity. Notice that $u \in W^{1,2}_{\mathrm{loc}}(\Omega, N)$ (with N closed) implies that

$$|\Delta u| \le c|Du|^2 \in L^1_{\mathrm{loc}}(\Omega).$$

When Ω is 2-dimensional, proving that $D^2 u \in L^1_{\mathrm{loc}}(\Omega)$ would suffice, since

$$W^{2,1}_{\mathrm{loc}}(\Omega) \hookrightarrow C^0_{\mathrm{loc}}(\Omega) \quad \text{for } \Omega \Subset \mathbb{R}^2.$$

On the other hand, as seen in Example 7.5, the L^p-estimates fail for $p = 1$.

In the following sections we shall see how the lack of L^p-estimates for $p = 1$ can be "compensated" by the special structure of the right-hand side of (10.5).

10.4.1 Hélein's proof when the target manifold is S^n

Since the result is local we can assume

$$\Omega = D^2 := \{x \in \mathbb{R}^2 : |x| < 1\}.$$

Moreover we will set

$$S^n := \{x \in \mathbb{R}^{n+1} : |x| = 1\}.$$

Let $u \in W^{1,2}_{\mathrm{loc}}(D^2; S^n)$ be a weakly harmonic map. In this case (10.5) reduces to the system

$$-\Delta u = u|\nabla u|^2, \tag{10.67}$$

where $u = (u^1, \dots, u^{m+1})$. As already discussed, the right-hand side of (10.67) lies in $L^1_{\mathrm{loc}}(D^2, \mathbb{R}^m)$, but as seen in Example 7.5, this information does not guarantee that u is continuous. On the other hand, we can recast (10.67) in such a way that the right-hand side has a particular structure, yielding a better integrability of u than the one given by the L^p-estimates. Let us first write (10.67) as

$$-\Delta u^i = \sum_{j=1}^{n+1} u^i \nabla u^j \cdot \nabla u^j. \tag{10.68}$$

Then notice that $|u| \equiv 1$ implies

$$\sum_{j=1}^{n+1} u^j \nabla u^i \cdot \nabla u^j = \nabla u^i \cdot \nabla \left(\frac{1}{2} \sum_{j=1}^{n+1} |u^j|^2 \right) = \nabla u^i \cdot \nabla \left(\frac{1}{2} |u|^2 \right) = 0. \tag{10.69}$$

Then, subtracting (10.69) from (10.68) we get a new version of (10.67), namely

$$\begin{aligned} -\Delta u^i &= \sum_{j=1}^{n+1} \left(u^i \nabla u^j \cdot \nabla u^j - u^j \nabla u^i \cdot \nabla u^j \right) \\ &= \sum_{j=1}^{n+1} \left(u^i \nabla u^j - u^j \nabla u^i \right) \cdot \nabla u^j. \end{aligned} \tag{10.70}$$

Now observe that for each i and j the vector field

$$A^i_j := u^i \nabla u^j - u^j \nabla u^i \in L^2_{\mathrm{loc}}(D^2, \mathbb{R}^2)$$

is divergence-free: formally (if $u \in C^2$)

$$\operatorname{div} A^i_j = \sum_{\ell=1}^{2} \left(D_\ell (u^i D_\ell u^j) - D_\ell (u^j D_\ell u^i) \right) = u^i \Delta u^j - u^j \Delta u^i = 0,$$

where the last identity follows from (10.67). To be more precise we write for $\varphi \in C^\infty_c(D^2)$

$$\begin{aligned} \int_{D^2} \operatorname{div}(A^i_j) \varphi dx &= -\int_{D^2} A^i_j \cdot \nabla \varphi dx \\ &= \int_{D^2} (-u^i \nabla u^j \cdot \nabla \varphi + u^j \nabla u^i \cdot \nabla \varphi) dx \\ &= \int_{D^2} (-\nabla u^j \cdot \nabla (u^i \varphi) + \nabla u^i \cdot \nabla (u^j \varphi)) dx \\ &= 0, \end{aligned}$$

since by (10.67)

$$-\int_{D^2} \nabla u^j \cdot \nabla (u^i \varphi) dx = \int_{D^2} -u^j |\nabla u|^2 u^i \varphi dx = -\int_{D^2} \nabla u^i \cdot \nabla (u^j \varphi) dx.$$

Since for every i, j the vector field A^i_j is divergence-free, by Corollary 10.70 we can find a vector field $B^i_j \in W^{1,2}_{\mathrm{loc}}(D^2, \mathbb{R}^2)$ whose curl is A^i_j, i.e.

$$\nabla^\perp B^i_j := (-D_2 B^i_j, D_1 B^i_j) = A^i_j.$$

Then we can rewrite (10.70) has

$$-\Delta u^i = \sum_{j=1}^{n+1} \nabla^\perp B^i_j \cdot \nabla u^j = \sum_{j=1}^{n+1} \left(-D_2 B^i_j D_1 u^j + D_1 B^i_j D_2 u^j\right). \quad (10.71)$$

Notice that the right-hand side of (10.71) is still no better than L^1_{loc} as far as integrability is concerned, but now it presents the same Jacobian structure as in Wente's theorem, Theorem 7.8, which then yields $u \in C^0_{\mathrm{loc}}(D^2; S^n)$, hence (together with Proposition 10.40) completing the proof of Hélein's theorem in this case.

10.4.2 Rivière's proof for arbitrary target manifolds

The regularity of 2-dimensional weakly harmonic maps follows from Theorem 10.41 below (thanks to Proposition 10.44), whose interest goes beyond harmonic maps, since it allows to prove Hildebrandt's conjecture, namely that every critical point of a 2-dimensional conformally invariant functional is continuous.

Theorem 10.41 (Rivière [88]) *Consider a vector field*

$$\Omega \in L^2(D^2, \wedge^1 \mathbb{R}^2 \otimes so(m)),$$

and suppose that $u \in W^{1,2}_{\mathrm{loc}}(D^2, \mathbb{R}^m)$ *satisfies*

$$-\Delta u = \Omega \cdot \nabla u. \quad (10.72)$$

Then $u \in W^{1,p}_{\mathrm{loc}}(D^2, \mathbb{R}^m) \cap C^{0,\alpha}_{\mathrm{loc}}(D^2, \mathbb{R}^m)$ *for every* $p \in [1, \infty)$ *and every* $\alpha \in [0, 1)$.

Notation The vector field Ω can be seen as a tensor $\Omega^i_{\ell j}$ anti-symmetric with respect to i and j. The scalar product $\Omega \cdot \nabla u$ is the vector given by

$$(\Omega \cdot \nabla u)^i = \sum_{j=1}^{m} \sum_{\ell=1}^{2} \Omega^i_{\ell j} \frac{\partial u^j}{\partial x_\ell}.$$

Similarly, for matrix-valued functions $A, B \in W^{1,2}(D^2, gl(m))$,

$$\nabla A \cdot \nabla B \in L^1(D^2, gl(m))$$

is the matrix-valued function with components

$$(\nabla A \cdot \nabla B)^i_j = \sum_{k=1}^{m} \sum_{\ell=1}^{2} \frac{\partial A^i_k}{\partial x_\ell} \frac{\partial B^k_j}{\partial x_\ell}.$$

We will also often use the curl operator on D^2:

$$\nabla^\perp := \left(-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1} \right).$$

We will also use the following groups of $m \times m$ matrices:

$$\begin{aligned} gl(m) &= \text{arbitrary } m \times m \text{ matrices,} \\ GL(m) &= \text{invertible } m \times m \text{ matrices,} \\ SO(m) &= \text{orthogonal } m \times m \text{ matrices with positive determinant,} \\ so(m) &= \text{anti-symmetric } m \times m \text{ matrices.} \end{aligned}$$

Remark 10.42 While in Hélein's proof one uses that A^i_j is *divergence-free*, such condition is too restrictive for general manifolds (which is the motivation for introducing the moving-frame technique). Theorem 10.41 shows on the other hand that the *anti-symmetry* of A^i_j is in fact sufficient to give regularity. Shifting the attention from the divergence-free property to the anti-symmetry is natural because, as shown by Rivière in the same paper [88], weakly harmonic maps into arbitrary manifolds satisfy (10.72) for some $\Omega \in L^2$ anti-symmetric. This is the content of Proposition 10.44 below.

Remark 10.43 Theorem 10.41 is sharp for what concerns the regularity of u. Indeed B. G. Sharp [92, Section 4.3] recently constructed a solution u of (10.72), under the same assumptions as in Theorem 10.41 but with $u \notin W^{1,\infty}_{\mathrm{loc}}(D^2, \mathbb{R}^m) \cong C^{0,1}_{\mathrm{loc}}(D^2, \mathbb{R}^m)$.

Proposition 10.44 ([88]) *Let $u : D^2 \to N \subset \mathbb{R}^m$ be a weakly harmonic map. Then u satisfies the hypothesis of Theorem 10.41. In particular u is continuous, hence smooth.*

Proof. We have

$$-\Delta u^i = \sum_{j,k=1}^{m} A^i_{jk}(u) \nabla u^k \cdot \nabla u^j = \sum_{j,k=1}^{m} A^i_{jk}(u) \sum_{\ell=1}^{n} \frac{\partial u^k}{\partial x_\ell} \frac{\partial u^j}{\partial x_\ell},$$

where we extended the second fundamental form $A(u)$ to a bilinear form on $T_u \mathbb{R}^m$. But since $A(u)$ is orthogonal to $T_u(N)$, we have

$$\sum_{j=1}^{m} A^j_{ik} \nabla u^j = 0,$$

hence

$$-\Delta u^i = \sum_{j,k=1}^{m} (A^i_{jk}(u) - A^j_{ik}(u))\nabla u^k \cdot \nabla u^j = \Omega^i_j \cdot \nabla u^j,$$

where

$$\Omega^i_j(x) := (A^i_{jk}(u(x)) - A^j_{ik}(u(x)))\nabla u^k(x) \in L^2(\wedge^1 D^2 \otimes so(m)).$$

Then by Theorem 10.41 we have $u \in C^{0,\alpha}_{\mathrm{loc}}(D^2, N)$ for $\alpha \in [0,1)$, hence u is smooth by Theorem 9.8. $\square$

Proof of Theorem 10.41

The proof is split in several steps. By Proposition 10.45 below, Equation (10.72) can be recast in the form of a conservation law (Equation (10.74) below) if one can find suitable matrix-valued functions A and B satisfying (10.73). In the following Theorem 10.46 we see that the solutions (in $W^{1,2}_{\mathrm{loc}}$) of such a conservation law are Hölder continuous. Theorems 10.47 and 10.48 will deal with the proof of the existence of the matrix-valued functions A and B needed in Proposition 10.45, at least locally. This completes the proof of the Hölder continuity of u. But in fact Theorem 10.46 below also gives $Du \in L^{2,2\alpha}_{\mathrm{loc}}(D^2)$ [8] for $\alpha \in (0,1)$, which then implies

$$-\Delta u = \Omega \cdot \nabla u \in L^{1,\alpha}_{\mathrm{loc}}(D^2, \mathbb{R}^m).$$

This also implies

$$Du \in L^q_{\mathrm{loc}}(D^2) \quad \text{for some } q > 2,$$

by a result of Adams [1], and by L^p-theory and a simple bootstrap argument we finally infer

$$Du \in L^q_{\mathrm{loc}}(D^2) \quad \text{for every } q \in [1,\infty).$$

We will not give the details of this part.

Proposition 10.45 *Let $\Omega \in L^2(D^2, \wedge^1\mathbb{R}^2 \otimes so(m))$ and*

$$A \in W^{1,2}(D^2, GL(m)), \quad B \in W^{1,2}(D^2, gl(m)), \quad A, A^{-1} \in L^\infty.$$

Assume that

$$\nabla A + \nabla^\perp B = A\Omega. \tag{10.73}$$

Then (10.72) *is equivalent to*

$$\operatorname{div}(A\nabla u - B\nabla^\perp u) = 0. \tag{10.74}$$

[8] $L^{2,2\alpha}_{\mathrm{loc}}(D^2)$ denotes the Morrey space, see Chapter 5.

Proof. It is enough to observe that, at least formally, (10.73) implies

$$\operatorname{div}(A\nabla u - B\nabla^{\perp}u) = \nabla A\cdot\nabla u + A\Delta u + \nabla^{\perp}B\cdot\nabla u = A\Omega\cdot\nabla u + A\Delta u.$$

To be rigorous one actually computes for $\varphi\in C_c^\infty(D^2)$

$$\begin{aligned}0 &= \int_{D^2}\operatorname{div}(A\nabla u - B\nabla^{\perp}u)\varphi dx\\ &= \int_{D^2}(A\nabla u - B\nabla^{\perp}u)\cdot\nabla\varphi dx\\ &= \int_{D^2}(\nabla(A\varphi)\cdot\nabla u - \nabla A\cdot\nabla u\,\varphi - \nabla(B\varphi)\cdot\nabla^{\perp}u + \nabla B\cdot\nabla^{\perp}u\,\varphi)dx\\ &= \int_{D^2}\nabla(A\varphi)\cdot\nabla u dx - \int_{D^2}A\Omega\cdot\nabla u\varphi dx.\end{aligned}$$

□

Theorem 10.46 (Rivière [88]) *Let $u\in W^{1,2}(D^2,\mathbb{R}^m)$ be a solution to*

$$\operatorname{div}(A\nabla u - B\nabla^{\perp}u) = 0.$$

with

$$A\in W^{1,2}(D^2,GL(m)),\quad B\in W^{1,2}(D^2,gl(m)),$$
$$A,A^{-1}\in L^\infty(D^2,GL(m)).$$

Then $Du\in L^{2,2\alpha}_{\mathrm{loc}}(D^2,\mathbb{R}^m)$ for every $\alpha\in(0,1)$. In particular $u\in C^{0,\alpha}_{\mathrm{loc}}(D^2,\mathbb{R}^m)$.

Proof. First notice that

$$\operatorname{div}(A\nabla u) = \operatorname{div}(B\nabla^{\perp}u) = \nabla B\cdot\nabla^{\perp}u.$$

Observe that if A is the identity matrix, then we are in the hypothesis of Wente's theorem 7.8, which would immediately give $u\in C^0_{\mathrm{loc}}(D^2,\mathbb{R}^m)$. Fix now $B_R(x_0)\Subset D^2$ and define $C\in W^{1,1}_0(D^2,\mathbb{R}^m)$ to be the unique solution to

$$\begin{cases}\Delta C = \operatorname{div}(A\nabla u) = \nabla B\cdot\nabla^{\perp}u & \text{in } B_R(x_0)\\ C=0 & \text{on } \partial B_R(x_0).\end{cases}\tag{10.75}$$

By Wente's theorem (Theorem 7.8) we have $u\in C^0\cap W^{1,2}(B_R(x_0),\mathbb{R}^m)$ and

$$\int_{B_R(x_0)}|\nabla C|^2dx\le C_0\int_{B_R(x_0)}|\nabla B|^2dx\int_{B_R(x_0)}|\nabla u|^2dx.\tag{10.76}$$

Now observe that

$$\operatorname{div}(A\nabla u-\nabla C)=0,$$

hence by the Hodge theory (see Corollary 10.70) there exists a function $D \in W^{1,2}(B_R(x_0), \mathbb{R}^m)$ such that

$$\nabla^\perp D = A\nabla u - \nabla C. \tag{10.77}$$

Now we bound

$$\begin{aligned}\int_{B_R(x_0)} |\nabla D|^2 dx &\leq 2\int_{B_R(x_0)} |A\nabla u|^2 dx + 2\int_{B_R(x_0)} |\nabla C|^2 dx \\ &\leq 2\|A\|_{L^\infty} \int_{B_R(x_0)} |\nabla u|^2 dx \\ &\quad + 2C_0 \int_{B_R(x_0)} |\nabla B|^2 dx \int_{B_R(x_0)} |\nabla u|^2 dx.\end{aligned} \tag{10.78}$$

Let $v \in W^{1,2}(B_R(x_0), \mathbb{R}^m)$ be the solution to

$$\Delta v = 0 \quad \text{in } B_R(x_0), \qquad v = D \quad \text{on } \partial B_R(x_0).$$

Since v is harmonic we get by (5.13)

$$\int_{B_r(x_0)} |\nabla v|^2 dx \leq c\left(\frac{r}{R}\right)^2 \int_{B_R(x_0)} |\nabla v|^2 dx, \quad 0 < r \leq R,$$

hence as in (5.22) and setting $w := D - v$

$$\int_{B_r(x_0)} |\nabla D|^2 dx \leq c\left(\frac{r}{R}\right)^2 \int_{B_R(x_0)} |\nabla D|^2 dx + c\int_{B_R(x_0)} |\nabla w|^2 dx, \tag{10.79}$$

for $r \in (0, R)$. On the other hand by (10.77)

$$\begin{cases} \Delta w = \Delta D = -\nabla A \cdot \nabla^\perp u & \text{in } B_R(x_0) \\ w = 0 & \text{on } \partial B_R(x_0), \end{cases}$$

and again by Wente's theorem

$$\int_{B_R(x_0)} |\nabla w|^2 dx \leq C_0 \int_{B_R(x_0)} |\nabla A|^2 dx \int_{B_R(x_0)} |\nabla u|^2 dx. \tag{10.80}$$

Then, collecting (10.76), (10.77), (10.78), (10.79), (10.80) and further assuming that R is so small that

$$\int_{B_R(x_0)} \big(|\nabla A|^2 + |\nabla B|^2\big) dx < \varepsilon$$

for some $\varepsilon > 0$ to be chosen, we bound

$$\begin{aligned}\int_{B_r(x_0)} |A\nabla u|^2 dx \le & 2\int_{B_r(x_0)} |\nabla C|^2 dx + 2\int_{B_r(x_0)} |\nabla D|^2 dx \\ \le & 2C_0 \int_{B_R(x_0)} |\nabla B|^2 dx \int_{B_R(x_0)} |\nabla u|^2 dx \\ & + c\Big(\frac{r}{R}\Big)^2 \int_{B_R(x_0)} |\nabla D|^2 dx + \int_{B_R(x_0)} |\nabla w|^2 dx \\ \le & c_1 \|A\|_{L^\infty} \Big(\frac{r}{R}\Big)^2 \int_{B_R(x_0)} |\nabla u|^2 dx + c_1\varepsilon \int_{B_R(x_0)} |\nabla u|^2 dx.\end{aligned}$$

In fact, using that $A^{-1} \in L^\infty$, we bound, with a constant c_2 depending on $\|A\|_{L^\infty}$ and $\|A^{-1}\|_{L^\infty}$

$$\int_{B_r(x_0)} |\nabla u|^2 dx \le c_2 \Big(\frac{r}{R}\Big)^2 \int_{B_R(x_0)} |\nabla u|^2 dx + c_2\varepsilon \int_{B_R(x_0)} |\nabla u|^2 dx. \tag{10.81}$$

Now by Lemma 5.13 for every $\alpha \in (0, 1)$ we can choose ε so that (10.81) implies

$$\int_{B_r(x_0)} |\nabla u|^2 dx \le c_3 \Big(\frac{r}{R}\Big)^{2\alpha} \int_{B_R(x_0)} |\nabla u|^2 dx, \tag{10.82}$$

where now c_3 also depends on α. As usual (10.82) now holds for x in a neighborhood of x_0 (up to modifying the constants slightly), hence we have actually proven

$$\nabla u \in L^{2,2\alpha}_{\mathrm{loc}}(D^2) \quad \text{for every } \alpha \in (0, 1),$$

hence by Theorem 5.7 $u \in C^{0,\alpha}_{\mathrm{loc}}(D^2, \mathbb{R}^m)$. □

Theorem 10.47 (Rivière [88]) *There exists $\varepsilon_0 = \varepsilon_0(m) > 0$ such that if*

$$\Omega \in L^2(D^2, \wedge^1\mathbb{R}^2 \otimes so(m))$$

satisfies $\|\Omega\|^2_{L^2} \le \varepsilon_0$, then there exists

$$A \in W^{1,2}(D^2, GL(m)), \quad B \in W^{1,2}_0(D^2, gl(m))$$

solving

$$\nabla A + \nabla^\perp B = A\Omega$$

and

$$\|\nabla A\|^2_{L^2} + \|\nabla B\|^2_{L^2} + \|\operatorname{dist}(A, SO(m))\|_{L^\infty} \le C(m)\|\Omega\|^2_{L^2}. \tag{10.83}$$

In particular $A \in L^\infty(D^2, GL(m))$, and if ε_0 is small enough we also have $A^{-1} \in L^\infty(D^2, GL(m))$.

The proof of Theorem 10.47 will require the following result, whose proof will be presented later.

Theorem 10.48 (Rivière [88]) *For* $\Omega \in L^2(D^2, \wedge^1\mathbb{R}^2 \otimes so(m))$ *there exists* $\xi \in W_0^{1,2}(D^2, gl(m))$ *and* $P \in W^{1,2}(D^2, SO(m))$ *such that*

$$P^{-1}\nabla P + P^{-1}\Omega P = \nabla^\perp \xi \tag{10.84}$$

and

$$\|\nabla \xi\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 \le 3\|\Omega\|_{L^2}^2.$$

Proof of Theorem 10.47. We first show that for ε_0 sufficiently small the following system, defined for

$$\tilde{A} \in L^\infty \cap W^{1,2}(D^2, gl(m)), \quad B \in W^{1,2}(D^2, gl(m))$$

has a unique solution:

$$\begin{cases} \Delta \tilde{A} = \nabla \tilde{A} \cdot \nabla^\perp \xi - \nabla^\perp B \cdot \nabla P & \text{in } D^2 \\ \Delta B = \nabla^\perp \tilde{A} \cdot \nabla P^{-1} + \operatorname{div}(\tilde{A}\nabla\xi P^{-1}) + \operatorname{div}(\nabla \xi P^{-1}) & \text{in } D^2 \\ \dfrac{\partial \tilde{A}}{\partial \nu} = 0, \;\; B = 0 & \text{on } \partial D^2 \\ \int_{D^2} \tilde{A} dx = 0, \end{cases} \tag{10.85}$$

where ξ and P are as in Theorem 10.48. Indeed consider the operator

$$T : X \to X,$$

where

$$X := (L^\infty \cap W^{1,2}(D^2, gl(m))) \times W^{1,2}(D^2, gl(m)),$$

associating to $(\hat{A}, \hat{B}) \in X$ the unique solution $(\tilde{A}, B) \in X$ to

$$\begin{cases} \Delta \tilde{A} = \nabla \hat{A} \cdot \nabla^\perp \xi - \nabla^\perp \hat{B} \cdot \nabla P & \text{in } D^2 \\ \Delta B = \nabla^\perp \hat{A} \cdot \nabla P^{-1} + \operatorname{div}(\hat{A}\nabla\xi P^{-1}) + \operatorname{div}(\nabla \xi P^{-1}) & \text{in } D^2 \\ \dfrac{\partial \tilde{A}}{\partial \nu} = 0, \;\; B = 0 & \text{on } \partial D^2 \\ \int_{D^2} \tilde{A} dx = 0. \end{cases} \tag{10.86}$$

In fact $(\tilde{A}, B) \in X$ by Wente's theorem (Theorem 7.10), which together with the Poincaré inequality gives

$$\begin{aligned} \|\tilde{A}\|_{W^{1,2}} + \|\tilde{A}\|_{L^\infty} &\le C\|\nabla \xi\|_{L^2}\|\nabla \hat{A}\|_{L^2} + c\|\nabla P\|_{L^2}\|\nabla \hat{B}\|_{L^2} \\ &\le c\|\Omega\|_{L^2}(\|\nabla \hat{A}\|_{L^2} + \|\nabla \hat{B}\|_{L^2}) \\ &\le c\varepsilon_0 \|(\hat{A}, \hat{B})\|_X. \end{aligned}$$

Similarly, using Theorem 7.8, the L^2-theory and noticing that

$$\|\hat{A}\nabla\xi P^{-1}\|_{L^2} \le \|\hat{A}\|_{L^\infty}\|\nabla\xi\|_{L^2}, \quad \|\nabla\xi P^{-1}\|_{L^2} \le \|\nabla\xi\|_{L^2}$$

we obtain

$$\begin{aligned}\|B\|_{W^{1,2}} &\le c(\|\nabla\hat{A}\|_{L^2}\|\nabla P^{-1}\|_{L^2} + \|\hat{A}\|_{L^\infty}\|\nabla\xi\|_{L^2} + \|\nabla\xi\|_{L^2})\\ &\le c\|\Omega\|_{L^2}(\|\nabla A\|_{L^2} + \|\hat{A}\|_{L^\infty} + 1)\\ &\le c\varepsilon_0(\|\nabla A\|_{L^2} + \|\hat{A}\|_{L^\infty} + 1).\end{aligned}$$

Then if ε_0 is small enough T is a contraction,[9] hence it has a fixed point $(\tilde{A}, B) \in X$ which solves (10.85) and satisfies

$$\|\tilde{A}\|_{W^{1,2}} + \|\tilde{A}\|_{L^2} + \|B\|_{W^{1,2}} \le C\|\Omega\|.$$

Finally, setting

$$A = (\tilde{A} + I_m)P^{-1}, \quad (I_m)_i^j := \delta_i^j,$$

we conclude as follows. Notice that with $A' := \tilde{A} + I_m$ we find

$$\mathrm{div}(\nabla A' - A'\nabla^\perp\xi + \nabla^\perp B\, P) = 0$$

by the first equation in (10.85). Then according to Corollary 10.70 we can write

$$\nabla A' - A'\nabla^\perp\xi + \nabla^\perp B\, P = \nabla^\perp D,$$

where D solves

$$\mathrm{div}(\nabla D\, P^{-1}) = 0,$$

and using that $\xi = 0$ on ∂D^2 and the boundary conditions in (10.85) we can choose $D = 0$ on ∂D^2.

We now claim that if ε_0 is chosen possibly smaller, then $D \equiv 0$. Indeed we can write, again by Corollary 10.70,

$$\nabla D\, P^{-1} = -\nabla^\perp E \tag{10.87}$$

with $E \in W^{1,2}(D^2, gl(m))$, and Neumann boundary conditions (since the tangential derivative of D on ∂D^2 is 0). In particular, applying $\nabla^\perp$ to (10.87) we see that E solves

$$\begin{cases}\Delta E = -\nabla D \cdot \nabla^\perp P^{-1} & \text{in } D^2\\ \dfrac{\partial E}{\partial \nu} = 0 & \text{on } \partial D^2.\end{cases}$$

[9] i.e.

$$\|T(\hat{A}_1, \hat{B}_1) - T(\hat{A}_2, \hat{B}_2)\|_X \le \alpha\|(\hat{A}_1 - \hat{A}_2, \hat{B}_1 - \hat{B}_2)\|_X,$$

for some $\alpha < 1$.

By Theorem 7.10 we then infer

$$\int_{D^2} |\nabla E|^2 dx \leq C_0 \int_{D^2} |\nabla D|^2 dx \int_{D^2} |\nabla P^{-1}|^2 dx.$$

On the other hand, from (10.87) we infer $|\nabla D| \leq |\nabla E|$, hence

$$\int_{D^2} |\nabla E|^2 dx \leq C_0 \varepsilon_0 \int_{D^2} |\nabla E|^2 dx,$$

which implies $E \equiv 0$ if $\varepsilon_0 < C_0^{-1}$. Then from (10.87) we conclude $\nabla D \equiv 0$, hence $D \equiv 0$, as claimed.

Therefore we have proven

$$\nabla A' - A' \nabla^\perp \xi + \nabla^\perp B\, P = 0.$$

This, together with

$$P^{-1} \nabla P + P^{-1} \Omega P = \nabla^\perp \xi$$

finally shows that $A := A' P^{-1}$ solves

$$\begin{aligned} \nabla A + \nabla^\perp B &= \nabla(A' P^{-1}) + \nabla^\perp B \\ &= (\nabla A' + A' \nabla P^{-1}\, P + \nabla^\perp B\, P) P^{-1} \\ &= (A' \nabla^\perp \xi + A' \nabla P^{-1} P) P^{-1} \\ &= (A' P^{-1} \nabla P + A' P^{-1} \Omega P + A' \nabla P^{-1} P) P^{-1} \\ &= A\Omega, \end{aligned}$$

as wished. □

Proof of Theorem 10.48

The proof we present here is due to A. Schikorra [93]. It essentially follows from the two lemmas below.

Lemma 10.49 *For any regular domain $D \subset \mathbb{R}^n$ and vector field*

$$\Omega \in L^2(D, \wedge^1 \mathbb{R}^n \otimes so(m)),$$

there exists $P \in W^{1,2}(D, SO(m))$ minimizing the variational functional

$$E(Q) = \int_D |Q^{-1} \nabla Q - Q^{-1} \Omega Q|^2 dx, \quad Q \in W^{1,2}(D, SO(m)).$$

Furthermore,

$$\|\nabla P\|_{L^2(D)} \leq 2 \|\Omega\|_{L^2(D)}, \quad \|P^{-1} \nabla P - P^{-1} \Omega P\|_{L^2(D)} \leq \|\Omega\|_{L^2(D)}.$$

Proof. The function $Q \equiv I := (\delta_{ij})_{ij}$ is clearly admissible. Thus, there exists a minimizing sequence $Q_k \in W^{1,2}(D, SO(m))$ such that

$$E(Q_k) \le E(I) = \|\Omega\|_{L^2}^2, \quad k \in \mathbb{N}.$$

By a.e. orthogonality of $Q_k(x) \in SO(m)$ we know that $Q_k(x)$ is bounded and

$$|\nabla Q_k| = |Q_k^{-1}\nabla Q_k| \le |Q_k^{-1}\nabla Q_k - Q_k^{-1}\Omega Q_k| + |\Omega| \quad \text{a.e. in } D,$$

hence

$$\|\nabla Q_k\|_{L^2(D)}^2 \le 2(E(Q_k) + \|\Omega\|_{L^2(D)}^2) \le 4\|\Omega\|_{L^2(D)}^2.$$

Up to choosing a subsequence, we can assume that Q_k converges weakly in $W^{1,2}$ to $P \in W^{1,2}(D, gl(m))$. At the same time it shall converge strongly in L^2, and pointwise almost everywhere. The latter implies

$$P^{-1}P = \lim_{k\to\infty} Q_k^{-1}Q_k = I \quad \text{a.e.},$$

and $\det(P) = 1$ a.e., that is $P \in SO(m)$ almost everywhere. Denoting

$$\Omega^P := P^{-1}\nabla P - P^{-1}\Omega P$$

and recalling that

$$0 = \nabla(PP^{-1}) = \nabla P\, P^{-1} + P\nabla P^{-1},$$

we obtain

$$Q_k^{-1}\nabla Q_k - Q_k^{-1}\Omega Q_k = (P^{-1}Q_k)^{-1}\nabla(P^{-1}Q_k) + (P^{-1}Q_k)^{-1}\Omega^P(P^{-1}Q_k),$$

and consequently

$$\begin{aligned} |Q_k^{-1}\nabla Q_k - Q_k^{-1}\Omega Q_k|^2 &= |\nabla(P^{-1}Q_k) + \Omega^P P^{-1}Q_k|^2 \\ &= |\nabla(P^{-1}Q_k)|^2 \\ &\quad + 2\langle \nabla(P^{-1}Q_k), \Omega^P P^{-1}Q_k\rangle + |\Omega^P|^2, \end{aligned}$$

where in this case $\langle\cdot,\cdot\rangle$ is just the Hilbert-Schmidt scalar product for tensors: in this case

$$\langle A, B\rangle = \sum_{\ell=1}^{n}\sum_{i,j=1}^{m} A_{\ell j}^i B_{\ell j}^i.$$

This implies

$$\begin{aligned} E(Q_k) &= \int_D \left(|\nabla(P^{-1}Q_k)|^2 + 2\langle \nabla(P^{-1}Q_k), \Omega^P P^{-1}Q_k\rangle\right) dx + E(P) \\ &\ge \int_D \left|\nabla(P^{-1}Q_k)\right|^2 dx \\ &\quad + 2\int_D \langle \nabla(P^{-1}Q_k), \Omega^P P^{-1}Q_k\rangle dx + \inf_Q E(Q). \end{aligned}$$

The middle part of the right-hand side converges to zero as $k \to \infty$. To see this, one can check that $\Omega^P P^{-1} Q_k$ converges to Ω^P almost everywhere and Lebesgue's dominated convergence theorem implies strong convergence in L^2. On the other hand

$$\nabla(P^{-1}Q_k) \rightharpoonup 0 \quad \text{weakly in } L^2.$$

Now using

$$\lim_{k\to\infty} E(Q_k) = \inf_Q E(Q),$$

we have strong $W^{1,2}$-convergence of $P^{-1}Q_k$ to I. Then Q_k converges strongly to P, which readily implies minimality of P. □

Lemma 10.50 *Critical points $P \in W^{1,2}(D, SO(m))$ of*

$$E(Q) = \int_D |Q^{-1}\nabla Q - Q^{-1}\Omega Q|^2 dx, \quad Q \in W^{1,2}(D, SO(m)),$$

with $\Omega \in L^2(D, \wedge^1\mathbb{R}^n \otimes so(m))$ satisfy

$$\operatorname{div}(P^{-1}\nabla P - P^{-1}\Omega P) = 0 \quad \text{in } D \tag{10.88}$$

and calling ν the exterior unit normal to ∂D,

$$\nu \cdot (P^{-1}\nabla P - P^{-1}\Omega P) = 0, \quad \text{on } \partial D. \tag{10.89}$$

If $P^{-1}\nabla P - P^{-1}\Omega P$ is not regular enough (10.88) *and* (10.89) *mean*

$$\int_D (P^{-1}\nabla P - P^{-1}\Omega P) \cdot \nabla\varphi dx = 0 \quad \text{for } \varphi \in C^\infty(\overline{D}, gl(m)). \tag{10.90}$$

Proof. Let P be a critical point of $E(Q)$. A valid perturbation P_ε is the following

$$P_\varepsilon := Pe^{\varepsilon\varphi\alpha} = P + \varepsilon\varphi P\alpha + o(\varepsilon) \in W^{1,2}(D, SO(m))$$

for any $\varphi \in C^\infty(\overline{D})$, $\alpha \in so(m)$ and with

$$\lim_{\varepsilon\to 0} \frac{o(\varepsilon)}{\varepsilon} = 0.$$

Indeed the exponential function applied to a skew-symmetric matrix is an orthogonal matrix; in fact the space of skew-symmetric matrices is the tangential space to the manifold $SO(m) \subset \mathbb{R}^{n\times n}$ at the identity matrix. Now

$$P_\varepsilon^{-1} = P^{-1} - \varepsilon\varphi\alpha P^{-1} + o(\varepsilon),$$

$$\nabla P_\varepsilon = \nabla P + \varepsilon\varphi\nabla P\ \alpha + \varepsilon\nabla\varphi\ P\alpha + o(\varepsilon).$$

Thus, denoting again $\Omega^P := P^{-1}\nabla P - P^{-1}\Omega P$, we obtain

$$\Omega^{P_\varepsilon} = \Omega^P + \varepsilon\varphi(\Omega^P\alpha - \alpha\Omega^P) + \varepsilon\nabla\varphi\alpha + o(\varepsilon).$$

Using the anti-symmetry of Ω^P and α one verifies that

$$\sum_{i,j=1}^m (\Omega^P)^i_{\ell j}(\Omega^P\alpha - \alpha\Omega^P)^i_{\ell j} = 0 \quad \text{a.e. for } 1 \le \ell \le n,$$

since, ignoring the index ℓ, we see that

$$\langle\Omega^P, \Omega^P\alpha - \alpha\Omega^P\rangle = 2\langle\Omega^P\Omega^P, \alpha\rangle = 0,$$

the last identity coming from that fact that $\Omega^P\Omega^P$ is symmetric, while α is anti-symmetric and where only in this occasion $\langle\ ,\ \rangle$ denotes the product of matrices in $gl(m)$ (i.e. ignoring the index ℓ). It follows that,

$$|\Omega^{P_\varepsilon}|^2 = |\Omega^P|^2 + 2\varepsilon\langle\Omega^P, \alpha\nabla\varphi\rangle + o(\varepsilon),$$

which implies

$$0 = \frac{d}{d\varepsilon}E(P_\varepsilon)\Big|_{\varepsilon=0} = \int_D \langle\Omega^P, \alpha\nabla\varphi\rangle dx. \tag{10.91}$$

This is true for any $\varphi \in C^\infty(\overline{D})$ and $\alpha \in so(m)$. Now for arbitrary $1 \le s, t \le m$ setting $\alpha^i_j := \delta^i_s\delta^t_j - \delta^s_j\delta^i_t$ we obtain (10.90). If Ω^P is also regular enough, then integrating by parts gives (10.88) and (10.89). $\square$

Proof of Theorem 10.48. Considering Lemma 10.49 and Lemma 10.50, we find $P \in W^{1,2}(D^2, SO(m))$ such that

$$\operatorname{div}(P^{-1}\nabla P + P^{-1}\Omega P) = 0,$$

hence by Corollary 10.70

$$\Omega^P = P^{-1}\nabla P + P^{-1}\Omega P = \nabla^\perp\xi,$$

for some $\xi \in W^{1,2}(D^2, gl(m))$. Assume that Ω^P is smooth. Then since $\Omega^P \cdot \nu = 0$, we also have that the tangential derivative of ξ on ∂D^2 vanishes, hence $\xi|_{\partial D^2}$ is a constant, which we can choose to be 0. If Ω^P is only square-summable, we use (10.90), getting with an integration by parts

$$\begin{aligned} 0 &= \int_{D^2} \Omega^P \cdot \nabla\varphi dx \\ &= \int_{D^2} \nabla^\perp\xi \cdot \nabla\varphi dx \\ &= \int_{\partial D^2} \xi\nabla\varphi \cdot \tau d\mathcal{H}^1, \quad \text{for every } \varphi \in C^\infty(\overline{D^2}), \end{aligned}$$

where τ is the unit tangent vector to ∂D^2, taken with the appropriate orientation. This implies that ξ is constant on ∂D^2, and again we can choose this constant to be 0. $\square$

10.4.3 Irregularity of weakly harmonic maps in dimension 3 and higher

It is natural to wonder whether in dimension 3 and higher weakly harmonic maps are regular, at least on large regions of their domains. This is in general not the case, as shown by Rivière:

Theorem 10.51 (Rivière [86]) *Let B^3 denote the unit ball in $\mathbb{R}^3$. For any non-constant map $\varphi \in C^\infty(\partial B^3, S^2)$ there exists an everywhere discontinuous weakly harmonic map $u \in W^{1,2}(B^3, S^2)$ with $u = \varphi$ on ∂B^3.*

Remark 10.52 Clearly Theorem 10.51 implies the existence of everywhere discontinuous weakly harmonic maps from the unit ball $B^n \subset \mathbb{R}^n$ with values into S^p for every $n \geq 3$ and $p \geq 2$.

Remark 10.53 Actually in [86] a more general version of Theorem 10.51 is proved. In fact the target manifold S^2 can be replaced by an arbitrary manifold Σ homeomorphic to S^2. Without any assumption of the target manifold, things are different. For instance a weakly harmonic map from B^3 into a 2-dimensional torus of revolution is always smooth, see [87].

Remark 10.54 We will not prove Theorem 10.51, but only remark that it is based on the relaxed Dirichlet energy introduced by Bethuel, Brézis and Coron [10] (also recast in terms of Cartesian currents by Giaquinta, Modica and Souček [48]), and on a subtle dipole construction which extends previous dipole constructions by Brézis, Coron and Lieb [16] and Hardt, Lin and Poon [58].

10.5 Regularity of stationary harmonic maps

As already discussed, a map $u \in W^{1,2}_{\mathrm{loc}}(D, N)$ from a domain $D \subset \mathbb{R}^n$ into a manifold $N \subset \mathbb{R}^m$ is called stationary harmonic map if it is a weakly harmonic map which is also critical for inner variations. In other words if it satisfies (10.5) and (10.6). In this section we shall prove, using an approach of Rivière and Struwe, that for stationary harmonic maps an ε-regularity theorem similar to Theorem 10.15 holds.

Theorem 10.55 *Given a closed manifold $N \subset \mathbb{R}^m$ and a domain $D \subset \mathbb{R}^n$ there exists $\varepsilon = \varepsilon(n, N)$ such that if $u \in W^{1,2}_{\mathrm{loc}}(D, N)$ is a stationary harmonic maps and for some ball $B_R(x_0) \Subset D$ we have*

$$\frac{1}{R^{n-2}} \int_{B_R(x_0)} |\nabla u|^2 dx < \varepsilon_0,$$

then u is smooth in a neighborhood of x_0.

Theorem 10.55 was first proven by C. Evans [31] when the target manifold is a round sphere. Then, using the moving frame technique of F. Hélein, F. Bethuel [9] generalized the result to the case of an arbitrary (say closed) target manifolds N. The proof of T. Rivière and M. Struwe which we present here uses the conservation laws technique of Rivière and avoids the moving frames.[10]

Theorem 10.55 follows easily from the more general Theorem 10.56 below, since we have already seen in the proof of Proposition 10.44 that a weakly harmonic map $u : D \to N \hookrightarrow \mathbb{R}^m$ solves

$$-\Delta u = \Omega \cdot \nabla u$$

for a vector field $\Omega \in L^2(D, \wedge^1\mathbb{R}^n \otimes so(m))$ with

$$|\Omega| \le C|\nabla u| \quad \text{a.e. in } D.$$

Moreover the condition (10.94) in Theorem 10.56 follows at once from the monotonicity formula for maps which are stationary harmonic maps in $B_2(0)$, Proposition 10.5.

Theorem 10.56 ([31], [9], [90]) *There is $\varepsilon \equiv \varepsilon(n,m) \in (0,1)$ such that for $B := B_1(0) \subset \mathbb{R}^n$ and $u \in W^{1,2}(B, \mathbb{R}^m)$ a solution of*

$$\Delta u = \Omega \cdot \nabla u \quad \textit{in } B \tag{10.92}$$

with $\Omega \in L^2(B, \wedge^1\mathbb{R}^n \otimes so(m))$ such that

$$\sup_{B_r(x)\subset B} \frac{1}{r^{n-2}} \int_{B_r(x)} |\Omega|^2 \, dx \le \varepsilon \tag{10.93}$$

and

$$\sup_{B_r(x)\subset B} \frac{1}{r^{n-2}} \int_{B_r(x)} |\nabla u|^2 \, dx < \infty, \tag{10.94}$$

then $u \in C^{0,\alpha}_{\mathrm{loc}}(B, \mathbb{R}^m)$ for some $\alpha \in (0,1)$.

In the proof of Theorem 10.56 we shall use the following gauge construction, which is the equivalent of Theorem 10.48 in higher dimension, and which follows at once from Lemma 10.49 and Lemma 10.50.

Theorem 10.57 *Let $D \Subset \mathbb{R}^n$ be a regular domain and consider*

$$\Omega \in L^2(D, \wedge^1\mathbb{R}^n \otimes so(m)).$$

[10] In addition to being relatively simple, this technique works also for target manifolds of class C^2 (a minimal requirement), while the moving-frame technique used by Bethuel requires N to be of class C^5.

Then there exists $P \in W^{1,2}(D, SO(m))$ *such that*

$$\mathrm{div}(P^{-1}dP + P^{-1}\Omega P) = 0 \quad \textit{in } D$$

and

$$\|\nabla P\|_{L^2} + \|P^{-1}dP + P^{-1}\Omega P\|_{L^2} \leq 3\|\Omega\|_{L^2}$$

Proof of Theorem 10.56. The following proof is a slight modification, due to A. Schikorra [93], of the original proof of T. Rivière and M. Struwe.

Let $z \in B$, $0 < r < R < \frac{1}{2}\,\mathrm{dist}(z, \partial B)$. By Theorem 10.57 there exists $P \in W^{1,2}(B_R(z), SO(m))$ such that

$$\mathrm{div}(\Omega^P) = \mathrm{div}(P^{-1}\nabla P - P^{-1}\Omega P) = 0 \quad \text{weakly in } B_R(z), \tag{10.95}$$

with the estimate

$$\|\nabla P\|_{L^2(B_R(z))} + \|\Omega^P\|_{L^2(B_R(z))} \leq 3\|\Omega\|_{L^2(B_R(z))}. \tag{10.96}$$

We have weakly

$$\mathrm{div}(P^{-1}\nabla u) = \Omega^P \cdot P^{-1}\nabla u := \sum_{\ell=1}^{n}\sum_{j,k=1}^{m} (\Omega^P)^i_{\ell j}(P^{-1})^j_k \frac{\partial u^k}{\partial x^\ell}, \quad \text{in } B_R(z). \tag{10.97}$$

By the Hodge decomposition (see Theorem 10.66), and identifying the vector field $P^{-1}\nabla u$ (with coefficients in $\mathbb{R}^m$) with the 1-form $P^{-1}du$, we can find

$$f \in W^{1,2}_0(B_R(z), \mathbb{R}^m), \quad g \in W^{1,2}_N(B_R(z), \wedge^2\mathbb{R}^n \otimes \mathbb{R}^m),$$

and

$$h \in C^\infty(B_R(z), \wedge^1\mathbb{R}^n \otimes \mathbb{R}^m)$$

such that

$$P^{-1}du = df + \delta g + h \quad \text{a.e. in } B_R(z), \tag{10.98}$$

where f satisfies

$$\begin{cases} \Delta f = \delta(P^{-1}du) = \mathrm{div}(P^{-1}\nabla u) = \Omega^P \cdot P^{-1}\nabla u & \text{in } B_R(z), \\ f = 0 & \text{on } \partial B_R(z), \end{cases} \tag{10.99}$$

g satisfies

$$\begin{cases} dg = 0 & \text{in } B_R(z), \\ \Delta g = d\delta g = d(P^{-1}du) & \text{in } B_R(z), \\ g_N = 0 & \text{on } \partial B_R(z), \end{cases}$$

and h is harmonic. Fix $1 < p < \frac{n}{n-1}$. One estimates with (10.98)

$$\int_{B_r(z)} |\nabla u|^p \, dx = \int_{B_r(z)} |P^{-1}\nabla u|^p dx$$
$$\leq C_p \left(\int_{B_r(z)} |h|^p \, dx + \int_{B_R(z)} \left(|\nabla f|^p + |\nabla g|^p \right) dx \right).$$

Here and in the following C_p denotes a generic constant depending on p. Since h is harmonic we have by (5.13)[11]

$$\int_{B_r(z)} |h|^p \, dx \leq C_p \left(\frac{r}{R}\right)^n \int_{B_R(z)} |h|^p \, dx, \quad 0 < r < R.$$

Consequently, again by (10.98),

$$\int_{B_r(z)} |\nabla u|^p \, dx \leq C_p \left(\frac{r}{R}\right)^n \int_{B_R(z)} |\nabla u|^p \, dx$$
$$+ C_p \int_{B_R(z)} (|\nabla f|^p + |\delta g|^p) dx. \tag{10.100}$$

In order to estimate $\int_{B_R(z)} |\nabla f|^p dx$ note that since $f = 0$ on $\partial B_R(z)$, by duality

$$\|\nabla f\|_{L^p(B_R(z))} \leq C_p \sup_{\varphi \in C_0^\infty(B_R(z), \mathbb{R}^m), \|\nabla \varphi\|_{L^q} \leq 1} \int_{B_R(z)} \nabla f \cdot \nabla \varphi dx, \tag{10.101}$$

where

$$\nabla f \cdot \nabla \varphi := \sum_{\ell=1}^n \sum_{i=1}^m \frac{\partial f^i}{\partial x^\ell} \frac{\partial \varphi^i}{\partial x^\ell}$$

and $q = p' = \frac{p}{p-1}$.[12] If

$$\|\nabla \varphi\|_{L^q(B_R(z))} \leq 1$$

[11] Actually the L^p-version of (5.13), which can be easily obtained with the same proof.

[12] By duality

$$\|\nabla f\|_{L^p(B_R(z))} = \sup_{X \in L^q(B_R(z), \mathbb{R}^n \otimes \mathbb{R}^m), \|X\|_{L^q} \leq 1} \int_{B_R(z)} \nabla f \cdot X dx$$
$$= \sup_{\omega \in L^q(B_R(z), \wedge^1 \mathbb{R}^n \otimes \mathbb{R}^m), \|\omega\|_{L^q} \leq 1} \int_{B_R(z)} df \cdot \omega dx.$$

On the other hand, applying the Hodge decomposition to ω (Theorem 10.66 and Remark 10.67), we write $\omega = d\alpha + \delta\beta + h$ and integration by parts (actually using Stokes' theorem) implies

$$\int_{B_R(z)} (df \cdot \omega) dx = \int_{B_R(z)} (df \cdot d\alpha) dx,$$

one calculates with the Sobolev embedding and a scaling argument

$$\|\varphi\|_{L^\infty(B_R(z))} \le C_p R^{1+\frac{n}{p}-n}, \quad \|\nabla\varphi\|_{L^2(B_R(z))} \le C_p R^{\frac{n}{p}-\frac{n}{2}}. \qquad (10.102)$$

By (10.99),

$$\int_{B_R(z)} \nabla f \cdot \nabla\varphi dx = \int_{B_R(z)} (\Omega^P \cdot P^{-1}\nabla u) \cdot \varphi dx.$$

Taking (10.95) into account, we can apply Lemma 10.58 below by choosing

$$\Gamma = (\Omega^P)^i_{\ell j}, \quad c = (P^{-1})^j_k \varphi^i, \quad a = u^k, \quad \text{for } 1 \le i,j,k \le m,\ 1 \le \ell \le n,$$

where i,j,k are fixed. Then (10.101) yields

$$\begin{aligned}
\|\nabla f\|_{L^p} &\le C_p \|\Omega^P\|_{L^2} \big(\|\nabla P\|_{L^2}\|\varphi\|_{L^\infty} + \|\nabla\varphi\|_{L^2}\big)\|\nabla u\|_{L^{p,n-p}(B_{2R}(z))}\\
&\le C_p \|\Omega\|_{L^2} \big(\|\Omega\|_{L^2}\|\varphi\|_{L^\infty} + \|\nabla\varphi\|_{L^2}\big)\|\nabla u\|_{L^{p,n-p}(B_{2R}(z))}\\
&\le C_p \varepsilon R^{\frac{n}{p}-1}\|\nabla u\|_{L^{p,n-p}(B_{2R}(z))},
\end{aligned}$$

where we also used (10.96), (10.93) and (10.102) and the above norms are taken in $B_R(z)$, except for the Morrey norm of ∇u. By a similar argument we will also bound

$$\|\delta g\|_{L^p(B_R(z))} \le C_p \varepsilon R^{\frac{n}{p}-1}\|\nabla u\|_{L^{p,n-p}(B_{2R}(z))}.$$

Indeed we have[13]

$$\|\delta g\|_{L^p} \le C_p \sup_{\beta \in W^{1,q}_N(B_R(z),\wedge^2\mathbb{R}^n\otimes\mathbb{R}^m),\|\delta\beta\|_{L^q}=1} \int_{B_R(z)} (\delta g \cdot \delta\beta) dx,$$

i.e. given $X \in L^q(B_R(z), \mathbb{R}^n \otimes \mathbb{R}^m)$

$$\int_{B_R(z)} \nabla f \cdot X dx = \int_{B_R(z)} \nabla f \cdot \nabla\varphi dx,$$

for some $\varphi \in W^{1,q}_0(B_R(z), \mathbb{R}^m)$ with $\|\nabla\varphi\|_{L^q} \le C\|X\|_{L^q}$. This proves (10.101).

[13] Again writing

$$\|\delta g\|_{L^p} = \sup_{\omega \in L^q(B_R(z),\wedge^1\mathbb{R}^n\otimes\mathbb{R}^m),\|\omega\|_{L^q}=1} \int_{B_R(z)} (\delta g \cdot \omega) dx,$$

using the Hodge decomposition to write $\omega = d\alpha + \delta\beta + h$ and

$$\int_{B_R(z)} (\delta g \cdot \omega) dx = \int_{B_R(z)} (\delta g \cdot \delta\beta) dx,$$

since $g_N = 0$ on $\partial B_R(z)$.

and for s satisfying $\frac{1}{2}+\frac{1}{q}+\frac{1}{s}=1$ we bound (up to signs)

$$\begin{aligned}\int_{B_R(z)}(\delta g\cdot\delta\beta)dx &= \int_{B_R(z)}(\Delta g\cdot\beta)dx\\ &= \int_{B_R(z)}(dP^{-1}\wedge du)\cdot\beta dx\\ &= \int_{B_R(z)}(dP^{-1}\cdot\delta\beta)\big(u-u_{z,R}\big)dx\\ &\le C\|dP\|_{L^2(B_R(z))}\|\delta\beta\|_{L^q(B_R(z))}\|u-u_{z,R}\|_{L^s(B_R(z))}\\ &\le C\varepsilon R^{n/p-1}\|\nabla u\|_{L^{p,n-p}(B_{2R}(z))},\end{aligned}$$

where we also used Corollary 6.22 and the Poincaré inequality to bound

$$\|u-u_{z,R}\|_{L^s(B_R(z))}\le CR^{n/s}|u|_*\le CR^{n/s}\|\nabla u\|_{L^{p,n-p}(B_{2R}(z))},$$

and (10.93), (10.96), $|\nabla P|=|\nabla P^{-1}|$ to bound

$$\|dP\|_{L^2(B_R(z))}\le C\varepsilon R^{n/2-1}.$$

Plugging these estimates into (10.100) we arrive at

$$\begin{aligned}\int_{B_r(z)}|\nabla u|^p\,dx\le C_p\left(\frac{r}{R}\right)^n\int_{B_R(z)}|\nabla u|^p\,dx\\ +C_p\varepsilon R^{n-p}\|\nabla u\|^p_{L^{p,n-p}(B_{2R}(z))}.\end{aligned}$$

The right-hand side of this estimate is finite by (10.94). We divide by r^{n-p} to get

$$\begin{aligned}\frac{1}{r^{n-p}}\int_{B_r(z)}|\nabla u|^2\,dx\le C_p\ \left(\frac{r}{R}\right)^p\frac{1}{R^{n-p}}\int_{B_R(z)}|\nabla u|^p\,dx\\ +C_p\ \varepsilon\ \left(\frac{R}{r}\right)^{n-p}\ \|\nabla u\|^p_{L^{p,n-p}(B_{2R}(z))}.\end{aligned}$$

Hence,

$$\frac{1}{r^{n-p}}\int_{B_r(z)}|\nabla u|^p dx\le C_p\bigg(\Big(\frac{r}{R}\Big)^p+\varepsilon\Big(\frac{R}{r}\Big)^{n-p}\bigg)\|\nabla u\|^p_{L^{p,n-p}(B_{2R}(z))}.$$

Choose $\gamma\in(0,\frac{1}{2})$ such that $C_p\gamma^p\le\frac{1}{4}$ and set $\varepsilon:=\gamma^n$. Then for $r:=\gamma R$ we have shown

$$\frac{1}{(\gamma R)^{n-p}}\int_{B_{\gamma R}(z)}|\nabla u|^p dx\le\frac{1}{2}\|\nabla u\|^p_{L^{p,n-p}(B_{2R}(z))}.$$

This is valid for any $R>0$, $z\in D$ such that $B_{2R}(z)\subset B$. For arbitrary $\rho\in(0,1)$, $y\in B$, such that $B_{2R}(z)\subset B_\rho(y)$ and $B_\rho(y)\Subset B$ this implies

$$\frac{1}{(\gamma R)^{n-p}}\int_{B_{\gamma R}(z)}|\nabla u|^p dx\le\frac{1}{2}\|\nabla u\|^p_{L^{p,n-p}(B_\rho(y))}.$$

that yields

$$\|\nabla u\|^p_{L^{p,n-p}(B_{\gamma\rho/2}(y))} \leq \frac{1}{2}\|\nabla u\|^p_{L^{p,n-p}(B_\rho(y))}.$$

As in the proof of Theorem 10.15, iterating we obtain

$$\|\nabla u\|_{L^{p,n-p}(B_\rho(y))} \leq c\rho^\alpha$$

for some $\alpha > 0$, hence

$$\nabla u \in L^{p,n-p+\alpha p}_{\mathrm{loc}}(B)$$

which yields $u \in C^{0,\alpha}_{\mathrm{loc}}(B)$ by Theorem 5.7. □

Lemma 10.58 *For any $p > 1$, there is a uniform constant $C = C(n,p)$ such that for any triple of functions*

$$a \in W^{1,2}(\mathbb{R}^n), \quad \Gamma \in L^2(\mathbb{R}^n, \mathbb{R}^n), \quad c \in W^{1,2}_0 \cap L^\infty(\mathbb{R}^n)$$

with $\operatorname{div}\Gamma = 0$ *in the support of c, we have*

$$\left|\int_{\mathbb{R}^n} (\nabla a \cdot \Gamma) c dx\right| \leq C\|\Gamma\|_{L^2}\ \|\nabla c\|_{L^2}\|\nabla a\|_{L^{p,n-p}},$$

whenever the right-hand side is finite, where

$$\|\nabla a\|^p_{L^{p,n-p}} := \sup_{B_\rho(x)\subset\mathbb{R}^n} \frac{1}{\rho^{n-p}}\int_{B_\rho(x)} |\nabla a|^p dx.$$

Proof. First assume that $c \in C^\infty_c(\mathbb{R}^n)$, the general case following by density. We have

$$\int_{\mathbb{R}^n} (\nabla a \cdot \Gamma) c dx = -\int_{\mathbb{R}^n} a\Gamma \cdot \nabla c dx.$$

Notice that $\mathrm{curl}(\nabla c) = 0$, hence by Theorem 6.33[14] we have

$$\Gamma \cdot \nabla c \in \mathcal{H}^1(\mathbb{R}^n), \quad \|\Gamma \cdot \nabla c\|_{\mathcal{H}^1} \leq \|\Gamma\|_{L^2}\|\nabla c\|_{L^2}.$$

Observe that by the Poincaré inequality (Proposition 3.12) we have $a \in BMO(\mathbb{R}^n)$ with

$$|a|_* \leq c\|\nabla a\|_{L^{p,n-p}},$$

and the conclusion follows from the duality $\mathcal{H}^1 - BMO$, Theorem 6.35. □

The original proof of Rivière and Struwe instead of Lemma 10.58, uses the following lemma, which is similar in many respects. It also follows from the theorems of Coifman-Lions-Meyer-Semmes and Fefferman-Stein. We state it because it is interesting to compare it with Wente's Theorem, see Remark 10.60.

[14] Actually Theorem 6.33 would require $\operatorname{div}\Gamma = 0$ in all of $\mathbb{R}^n$, but from its proof it is clear than only the behavior of Γ over the support of ∇c matters.

Lemma 10.59 *Given two differential forms*

$$\alpha \in W^{1,2}(\mathbb{R}^n, \Lambda^{n-2}\mathbb{R}^n), \quad \beta \in W^{1,2}(\mathbb{R}^n, \Lambda^0\mathbb{R}^n) \simeq W^{1,2}(\mathbb{R}^n)$$

and a function $v \in BMO(\mathbb{R}^n)$, we have

$$d\alpha \wedge d\beta \in \mathcal{H}^1(\mathbb{R}^n, \Lambda^n\mathbb{R}^n) \simeq \mathcal{H}^1(\mathbb{R}^n)$$

and

$$\int_\Omega d\alpha \wedge d\beta\, v \le C\|d\alpha\|_{L^2}\|d\beta\|_{L^2}|v|_*,$$

where the above integral is defined by density as in the duality $\mathcal{H}^1 - BMO$ (Theorem 6.35), and $|v|_*$ denotes the BMO seminorm of v.

Proof. We see the form $d\beta$ as a vector field $B := \nabla\beta$. Then $\operatorname{curl} B = 0$. Similarly, to the $(n-1)$-form

$$\omega := d\alpha = \sum_{i=1}^n \omega_i \widehat{dx^i}$$

we associate the vector field $E = (E_1, \ldots, E_n)$, with $E_i := (-1)^{i+1}\omega_i$. Then, the condition $d\omega = 0$ yields $\operatorname{div} E = 0$. Then

$$d\beta \wedge d\alpha = (E \cdot B)dx^1 \wedge \cdots \wedge dx^n,$$

with $E \cdot B \in \mathcal{H}^1(\mathbb{R}^n)$ thanks to Theorem 6.33 and

$$\|E \cdot B\|_{\mathcal{H}^1} \le C\|E\|_{L^2}\|B\|_{L^2} = C\|d\alpha\|_{L^2}\|d\beta\|_{L^2}.$$

The conclusion follows from Theorem 6.35. $\square$

Remark 10.60 Theorem 10.59 can be seen as an extension of Wente's theorem (Theorem 7.8). In fact Wente's theorem can be equivalently stated for 2-forms $\psi \in W^{1,2}(D^2, \Lambda^2\mathbb{R}^2)$ solving

$$\Delta\psi = d\alpha \wedge d\beta \text{ in } D^2, \quad \psi = 0 \text{ on } \partial D^2,$$

with

$$\alpha \in W^{1,2}(D^2), \quad \beta \in W^{1,2}(D^2).$$

Then the core of the proof of Wente's theorem was (in this formulation with differential forms) the bound

$$\int_{D^2} \log\left(\frac{1}{r}\right) d\alpha \wedge d\beta \le C\|d\alpha\|_{L^2}\|d\beta\|_{L^2},$$

which follows from Theorem 7.9 observing that $\log(1/r)\chi \in BMO(\mathbb{R}^2)$, where χ is a smooth cut-off function supported in $B_{1/2}(0)$.

In fact $\Delta\psi \in \mathcal{H}^1$ (locally, in a suitable sense) implies that $D^2\psi \in W^{1,1}$, which in turn implies $D\psi \in W^{1,2}$ and $\psi \in L^\infty$.

10.6 The Hodge-Morrey decomposition

In the regularity theory for stationary harmonic maps we used the Hodge-Morrey decomposition of a k-form with coefficients in L^2. Before stating it let us recall some definitions.

Definition 10.61 (Hodge dual) *The Hodge dual, also called Hodge-$*$ is the linear isomorphism*

$$*: \wedge^k\mathbb{R}^n \to \wedge^{n-k}\mathbb{R}^n$$

defined on a basis of $\wedge^k\mathbb{R}^n$ by

$$*dx^{i_1}\wedge\cdots\wedge dx^{i_k} = dx^{i_{k+1}}\wedge\cdots\wedge dx^{i_n},$$

whenever $\{i_1,\ldots,i_n\}$ is an even permutation of $\{1,\ldots,n\}$. Similarly one can define the Hodge dual on differential forms, for instance

$$*: W^{1,2}(\Omega,\wedge^k\mathbb{R}^n) \to W^{1,2}(\Omega,\wedge^{n-k}\mathbb{R}^n).$$

Definition 10.62 (Codifferential) *Given a differential form*

$$\omega\in W^{1,2}(\Omega,\wedge^k\mathbb{R}^n)\quad (k\geq 1),$$

we define

$$\delta\omega := *d*\omega \in W^{1,2}(\Omega,\wedge^{k-1}\mathbb{R}^n).$$

If $k=0$, then $\delta\omega := 0$.

Remark 10.63 There exist definitions of the Hodge-$*$ and of the codifferential δ which differ from ours by a sign, which anyway is irrelevant for our purposes.

Definition 10.64 *For $\omega\in W^{1,2}(\Omega,\wedge^k\mathbb{R}^n)$ let $\omega_T\in L^2(\partial\Omega,\wedge^k\mathbb{R}^n)$ be the tangent part of $\omega|_{\partial\Omega}$. This can be defined as $i^*\omega$, where $i:\partial\Omega\to\mathbb{R}^n$ is the trivial inclusion ($i(x)=x$ for every $x\in\partial\Omega$). Equivalently, given an orthonormal basis $\{\tau_1,\ldots,\tau_n\}$ of $\mathbb{R}^n$ at $x\in\partial\Omega$ with $\{\tau_1,\ldots,\tau_{n-1}\}$ spanning $T_x\partial\Omega$ and τ_n orthogonal to $\partial\Omega$ at x we set*

$$\omega_T(x)(\tau_{i_1},\ldots,\tau_{i_k}) = \omega(x)(\tau_{i_1},\ldots,\tau_{i_k})\qquad \text{if } 1\leq i_1<\cdots<i_k<n,$$

and

$$\omega_T(x)(\tau_{i_1},\ldots,\tau_{i_k}) = 0\qquad \text{if } 1\leq i_1<\cdots<i_k=n.$$

We also define

$$\omega_N := \omega|_{\partial\Omega}-\omega_T\in L^2(\partial\Omega,\wedge^k\mathbb{R}^n).$$

Finally we set

$$W^{1,2}_T(\Omega,\wedge^k\mathbb{R}^n) = \{\omega\in W^{1,2}(\Omega,\wedge^k\mathbb{R}^n):\omega_T=0\},$$

$$W^{1,2}_N(\Omega,\wedge^k\mathbb{R}^n) = \{\omega\in W^{1,2}(\Omega,\wedge^k\mathbb{R}^n):\omega_N=0\}.$$

Example 10.65 We consider 3 special cases.
1) When $k = 0$, $\omega_T = \omega|_{\partial\Omega}$, $\omega_N = 0$, hence

$$W_T^{1,2}(\Omega, \wedge^0\mathbb{R}^n) = W_0^{1,2}(\Omega), \quad W_N^{1,2}(\Omega, \wedge^0\mathbb{R}^n) = W^{1,2}(\Omega).$$

2)Similarly when $k = n$, $\omega_T = 0$, $\omega_N = \omega|_{\partial\Omega}$. In particual

$$W_T^{1,2}(\Omega, \wedge^n\mathbb{R}^n) \cong W^{1,2}(\Omega), \quad W_N^{1,2}(\Omega, \wedge^n\mathbb{R}^n) \cong W_0^{1,2}(\Omega).$$

3)When $k = 1$, if we identify $\omega = \omega_1 dx^1 + \cdots + \omega_n dx^n$ with the vector field $X = (\omega_1, \ldots, \omega_n)$, then ω_T at $x \in \partial\Omega$ corresponds to the orthogonal projection of X onto $T_x\partial\Omega$ and ω_N at $x \in \partial\Omega$ corresponds to the orthogonal projection onto the normal line $N_x\partial\Omega$.

10.6.1 Decomposition of differential forms

We are now ready to state the Hodge-Morrey decomposition of a k-form in L^2. We will assume the the domain Ω is bounded and has smooth boundary.

Theorem 10.66 (Hodge-Morrey decomposition) *For a domain $\Omega \Subset \mathbb{R}^n$ (with smooth boundary) and $0 \le k \le n$, consider $\omega \in L^2(\Omega, \wedge^k\mathbb{R}^n)$. Then there exist*

$$\alpha \in W_T^{1,2}(\Omega, \wedge^{k-1}\mathbb{R}^n), \quad \beta \in W_N^{1,2}(\Omega, \wedge^{k+1}\mathbb{R}^n), \quad h \in L^2 \cap C^\infty(\Omega, \wedge^k\mathbb{R}^n)$$

such that

$$\omega = d\alpha + \delta\beta + h,$$

where h is harmonic, i.e. $dh = 0$, $\delta h = 0$ (and it is understood that if $k = 0$ then $\alpha = 0$, and if $k = n$ then $\beta = 0$). There also exist

$$\Gamma_1 \in W_T^{2,2}(\Omega, \wedge^k\mathbb{R}^n), \quad \Gamma_2 \in W_N^{2,2}(\Omega, \wedge^k\mathbb{R}^n)$$

such that $\alpha = \delta\Gamma_1$, $\beta = d\Gamma_2$. In particular $\delta\alpha = 0$, $d\beta = 0$. Moreover $d\alpha$, $\delta\beta$ and h are mutually orthogonal with respect to the L^2-product, and

$$\|\Gamma_1\|_{W^{2,2}} + \|\Gamma_2\|_{W^{2,2}} + \|h\|_{L^2} \le C\|\omega\|_{L^2},$$

with C depending on Ω.

We will not prove Theorem 10.66 but refer the interested reader to [49] (Volume 1) or [23]. We only remark that the core of the proof is the existence of Γ_1 and Γ_2, which can be shown with variational methods, in the spirit e.g. of Theorem 3.29, i.e. minimizing suitable functionals, which are coercive thanks to the so called Gaffney inequality.

Remark 10.67 Theorem 10.66 is still valid if we assume $\omega \in L^p(\Omega, \wedge^k\mathbb{R}^n)$ for some $p \in (1, \infty)$. In this case

$$\Gamma_1 \in W_T^{2,p}(\Omega, \wedge^k\mathbb{R}^n), \quad \Gamma_2 \in W_N^{2,p}(\Omega, \wedge^k\mathbb{R}^n), \quad h \in L^p \cap C^\infty(\Omega, \wedge^k\mathbb{R}^n),$$

and one has the bound

$$\|\Gamma_1\|_{W^{2,p}} + \|\Gamma_2\|_{W^{2,p}} + \|h\|_{L^p} \leq C\|\omega\|_{L^p}.$$

for some constant C depending on Ω and p.

The Poincaré lemma, stating that on a topologically trivial set a closed form is also exact, can be very useful in conjunction with the Hodge decomposition.

Lemma 10.68 (Poincaré) *Let $\omega \in W^{\ell,p}(\Omega, \wedge^k\mathbb{R}^n)$ with $\Omega \subset \mathbb{R}^n$ contractible (i.e. homotopic to a point), $\ell \geq 0$, $p \in (1, \infty)$ and $1 \leq k \leq n$, satisfy $d\omega = 0$. Then*

$$\omega = d\eta \quad \textit{for some } \eta \in W^{\ell+1,p}_{\mathrm{loc}}(\Omega, \wedge^{k-1}\mathbb{R}^n).$$

Similarly, if $0 \leq k \leq n-1$ and $\delta\omega = 0$ we have

$$\omega = \delta\eta \quad \textit{for some } \eta \in W^{\ell+1,p}_{\mathrm{loc}}(\Omega, \wedge^{k+1}\mathbb{R}^n).$$

If Ω is also bounded then we actually have in both cases

$$\|\eta\|_{W^{\ell+1,p}} \leq C\|\omega\|_{W^{\ell,p}}$$

for a constant depending on ℓ, p and Ω.

10.6.2 Decomposition of vector fields

Using the Hodge decomposition and the Poincaré lemma we easily obtain the following well-known decomposition results for vector fields.

Corollary 10.69 *Let $\Omega \subset \mathbb{R}^3$ be contractible. Then any vector field $X \in L^2(\Omega, \mathbb{R}^3)$ can be decomposed as*

$$X = \nabla p + \operatorname{curl} Y,$$

for a function $p \in W_0^{1,2}(\Omega)$ and a vector field $Y \in W^{1,2}(\Omega, \mathbb{R}^3)$. If in addition $\operatorname{div} X = 0$*, then $p \equiv 0$.*

Proof. We can associate to $X = (X_1, X_2, X_3)$ the 1-form

$$\omega = \sum_{i=1}^{3} X_i dx^i \in L^2(\Omega, \wedge^1\mathbb{R}^3).$$

By Theorem 10.66
$$\omega = d\alpha + \delta\beta + h$$
with $\alpha \in W_0^{1,2}(\Omega)$, $\beta \in W^{1,2}(\Omega, \wedge^2\mathbb{R}^3)$, $h \in C^\infty \cap L^2(\Omega, \wedge^1\mathbb{R}^3)$ and h harmonic. By the Poincaré lemma we can also write $h = \delta H$, hence
$$\omega = d\alpha + \delta(\beta + H). \tag{10.103}$$
Write
$$\beta + H =: \gamma = \gamma_1 dx^2 \wedge dx^3 + \gamma_2 dx^3 \wedge dx^1 + \gamma_3 dx^1 \wedge dx^2.$$
We obtain
$$\delta\gamma = \left(\frac{\partial\gamma^3}{\partial x^2} - \frac{\partial\gamma^2}{\partial x^3}\right)dx^1 + \left(\frac{\partial\gamma^1}{\partial x^3} - \frac{\partial\gamma^3}{\partial x^1}\right)dx^2 + \left(\frac{\partial\gamma^2}{\partial x^1} - \frac{\partial\gamma^1}{\partial x^2}\right)dx^3,$$
hence setting
$$p = \alpha, \qquad \text{and} \qquad Y = (-\gamma_1, -\gamma_2, -\gamma_3)$$
and switching back to vector fields in (10.103) we obtain
$$X = \nabla p + \operatorname{curl} Y.$$
If $\operatorname{div} X = 0$, then since $\operatorname{div}(\operatorname{curl} Y) = 0$ we also infer $\Delta p = 0$, hence since $p = 0$ on $\partial\Omega$ we have $p \equiv 0$. □

Corollary 10.70 *Let $\Omega \subset \mathbb{R}^2$ be contractible. Then any vector field $X \in L^2(\Omega)$ can be decomposed as*
$$X = \nabla p + \nabla^\perp \xi,$$
for functions $p \in W_0^{1,2}(\Omega)$ and $\xi \in W^{1,2}(\Omega)$. If in addition $\operatorname{div} X = 0$, then $p \equiv 0$.

Proof. As in the proof of Corollary 10.69 we write
$$\omega := X_1 dx^1 + X_2 dx^2 = d\alpha + \delta\beta + h = d\alpha + \delta\gamma$$
with
$$\alpha \in W_0^{1,2}(\Omega), \quad \gamma = \gamma_0 dx^1 \wedge dx^2 \in W^{1,2}(\Omega, \wedge^2\mathbb{R}^2).$$
Then
$$\delta\gamma = -\frac{\partial\gamma_0}{\partial x^2}dx^1 + \frac{\partial\gamma_0}{\partial x^1}dx^2.$$
Switching back to vector fields we can choose $p = \alpha$ and $\xi = \gamma_0$.

As before, if $\operatorname{div} X = 0$, then $\Delta p = 0$, hence $p \equiv 0$. □

Corollary 10.71 *Assume that* $X \in L^p(\Omega, \mathbb{R}^n)$ *for some* $p \in (1, \infty)$ *is a vector field in a contractible domain* $\Omega \subset \mathbb{R}^n$ *(not necessarily bounded), and that* $\operatorname{curl} X = 0$, *i.e*

$$\frac{\partial X^i}{\partial x^j} - \frac{\partial X^j}{\partial x^i} = 0 \quad \textit{weakly for } 1 \leq i < j \leq n.$$

Then we can write $X = \nabla \alpha$ *for some function* $\alpha \in W^{1,p}_{\mathrm{loc}}(\Omega)$.

Proof. Writing

$$\omega := \sum_{i=1}^n X^i dx^i \in L^p(\Omega, \wedge^1 \mathbb{R}^n),$$

the condition $\operatorname{curl} X = 0$ is equivalent to $d\omega = 0$. Then by the Poincaré lemma we can find $\alpha \in W^{1,p}_{\mathrm{loc}}(\Omega)$ such that $\omega = d\alpha$, i.e. $X = \nabla \alpha$. $\square$

Chapter 11
A survey of minimal graphs

In this chapter we shall discuss minimal graphs, i.e. graphs whose area has vanishing first variation. After introducing the variational equations, we shall work on the existence of minimal graphs with prescribed boundary (Plateau problem), their uniqueness, stability and regularity in codimension 1.

In higher codimension we shall only discuss the regularity theory. In particular we shall prove that any *area-decreasing* Lipschitz minimal graph is smooth. This result of M-T. Wang is based on the regularity theorem of Allard and a Bernstein-type theorem for area-decreasing minimal graphs.

Some relevant facts in the theory of abstract and rectifiable varifolds shall be collected in the last section.

11.1 Geometry of the submanifolds of $\mathbb{R}^{n+m}$

Let us recall a few facts about submanifolds of $\mathbb{R}^{n+m}$, and introduce the concepts of minimal submanifold and minimal graph.

11.1.1 Riemannian structure and Levi-Civita connection

Given a Riemannian manifold (M, g), a Levi-Civita connection on M is an application

$$\nabla : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M),$$

where $\mathcal{T}(M)$ is the space of tangent vector fields on M, such that

1. $\nabla_X Y$ is C^∞-linear in X: for every $f, g \in C^\infty(M)$, $X_1, X_2, Y \in \mathcal{T}(M)$

$$\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y;$$

2. $\nabla_X Y$ is $\mathbb{R}$-linear in Y: for every $a, b \in \mathbb{R}$, $X, Y_1, Y_2 \in \mathcal{T}(M)$

$$\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2;$$

3. it satisfies the Leibniz rule for the product: for every $f \in C^\infty(M)$, $X, Y \in \mathcal{T}(M)$

$$\nabla_X(fY) = f\nabla_X(Y) + X(f)Y;$$

4. it is torsion free: if $[X, Y] := XY - YX$, then

$$\nabla_X Y - \nabla_Y X = [X, Y], \quad \text{for every } X, Y \in \mathcal{T}(M);$$

5. it is compatible with the metric: for every $X, Y, Z \in \mathcal{T}(M)$

$$D_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

We recall without proof:

Theorem 11.1 *Every Riemannian manifolds admits exactly one Levi-Civita connection.*

In what follows $\mathbb{R}^{n+m}$ has the usual Riemannian structure, in which the scalar product of two vectors $X, Y \in \mathbb{R}^{n+m}$ is denoted by $X \cdot Y$ or $\langle X, Y\rangle$. We identify $\mathbb{R}^{n+m}$ with its tangent space at any of its points. The Levi-Civita connection ∇ of $\mathbb{R}^{n+m}$ is the *flat connection.* Let

$$\{e_1, \ldots, e_{n+m}\}$$

be an orthonormal basis of $\mathbb{R}^{m+n}$; then $\nabla_{e_i} e_j = 0$, for any i, j.

An n-dimensional submanifold $\Sigma \subset \mathbb{R}^{n+m}$ of class C^r, $r \geq 2$, will be seen as a Riemannian manifold with the metric induced by the ambient space $\mathbb{R}^{n+m}$. In other words the metric g on Σ is simply the restriction of the metric of $\mathbb{R}^{n+m}$ on each tangent plane.

We denote by $T\Sigma$ the tangent bundle of Σ, of class C^{r-1}, and for each $p \in \Sigma$, $T_p\Sigma$ will be the tangent space to Σ at p. Similarly $N\Sigma$ and $N_p\Sigma$ will denote respectively the normal bundle and the normal space at p. In the following $\{\tau_1, \ldots \tau_n\}$ will always denote an orthonormal basis of $T_p\Sigma$, while $\{\nu_1, \ldots, \nu_m\}$ will denote an orthonormal basis of $N_p\Sigma$.

The Levi-Civita connection ∇^Σ of Σ is simply the projection on $T\Sigma$ of the flat connection ∇ of $\mathbb{R}^{n+m}$. More precisely:

Proposition 11.2 *Let $X, Y \in \mathcal{T}(\Sigma)$ be tangent vector fields on Σ; given $\widetilde{X}$ and $\widetilde{Y}$, arbitrary extensions to a neighborhood of Σ in $\mathbb{R}^{n+m}$ of X and Y, we have*

$$\nabla^\Sigma_X Y = (\nabla_{\widetilde{X}} \widetilde{Y})^T, \tag{11.1}$$

where $(\nabla_{\widetilde{X}}\widetilde{Y})^T$ *is the orthogonal projection of* $\nabla_{\widetilde{X}}\widetilde{Y}$ *onto the tangent bundle* $T\Sigma$*.*

Proof. Notice that $\nabla_{\widetilde{X}}\widetilde{Y}$ does not depend on the extensions $\widetilde{X}$ and $\widetilde{Y}$, see Exercise 11.3. Then we shall simply write $\nabla_X Y$ instead of $\nabla_{\widetilde{X}}\widetilde{Y}$. Thanks to the uniqueness of the Levi-Civita connection it suffices to prove that the map $(X,Y)\mapsto(\nabla_X Y)^T$ is a Levi-Civita connection. The C^∞-linearity in X, the $\mathbb{R}$-linearity in Y and the Leibniz rule are obvious. Let's show that there is no torsion:

$$(\nabla_X Y)^T-(\nabla_Y X)^T=(\nabla_X Y-\nabla_Y X)^T=[X,Y]^T=[X,Y].$$

Finally, let's verify the compatibility with the metric:

$$D_X g(Y,Z)=g(\nabla_X Y,Z)+g(X,\nabla_Y Z)=g((\nabla_X Y)^T,Z)+g(X,(\nabla_Y Z)^T).$$

□

Exercise 11.3 Show that given $X,Y\in\mathcal{T}(\Sigma)$, $\nabla_X Y$ does not depend on the choice of the extensions $\widetilde{X}$ and $\widetilde{Y}$.
[Hint: prove that $\nabla_{\widetilde{X}}\widetilde{Y}(p)$ depends only on $X(p)$ and the value of Y on the image of any curve $\gamma:(-\varepsilon,\varepsilon)\to\mathbb{R}^{n+m}$ with $\gamma(0)=p$, $\dot\gamma(0)=X$.]

11.1.2 The gradient, divergence and Laplacian operators

In what follows Σ will be a submanifold of class C^1 at least, although assuming that Σ is a rectifiable set would suffice in most cases, see Section 11.4.1.

Given a C^1 function $f:\Sigma\to\mathbb{R}$ and $X\in T_p\Sigma$, define

$$D_X f(p)=\frac{d}{dt}\bigg|_{t=0} f(\gamma(t)),$$

where $\gamma:(-\varepsilon,\varepsilon)\to\Sigma$ is any C^1-curve satisfying $\gamma(0)=p$ and $\dot\gamma(0)=X$. The *gradient* on Σ of f in p is defined by

$$\nabla^\Sigma f(p):=\sum_{j=1}^n(D_{\tau_j}f(p))\tau_j,$$

for any orthonormal basis $\{\tau_j\}_{j=1,\dots,n}$ of $T_p\Sigma$.

Exercise 11.4 Prove that $\nabla^\Sigma f(p)$ does not depend on the particular orthonormal basis $\{\tau_j\}_{j=1,\dots,n}$ of $T_p\Sigma$ chosen.

Exercise 11.5 Prove that if f is defined in a neighborhood of p in $\mathbb{R}^{n+m}$, then

$$\nabla^\Sigma f(p) = (\nabla f(p))^T,$$

where $\nabla f(p) = \sum_{j=1}^{n+m} \frac{\partial f}{\partial x^j}(p)e_j$, and $\{e_j\}_{j=1,\dots,n+m}$ is the orthonormal basis of $\mathbb{R}^{n+m}$ corresponding to the coordinates $\{x_j\}_{j=1,\dots,n+m}$, i.e. $e_j = \frac{\partial}{\partial x^j}$.

Given a chart $\varphi : V \subset \Sigma \to \mathbb{R}^n$, and the corresponding local parametrization $F = \varphi^{-1}$, we have

$$\nabla^\Sigma f = \sum_{i,j=1}^{n} g^{ij} \frac{\partial f}{\partial x^i}\frac{\partial F}{\partial x^j}, \tag{11.2}$$

where

$$\frac{\partial}{\partial x^i} f(p) = \frac{\partial(f\circ\varphi^{-1})}{\partial x^i}(\varphi(p)), \quad g_{ij} = \frac{\partial F}{\partial x^i}\cdot\frac{\partial F}{\partial x^j}$$

and (g^{ij}) is the inverse matrix of (g_{ij}).

The *divergence on* Σ of a vector field (not necessarily tangent to Σ)

$$X = \sum_{j=1}^{n+m} X^j e_j \in C^1(\Sigma, \mathbb{R}^{n+m})$$

is defined by

$$\operatorname{div}^\Sigma X := \sum_{j=1}^{n+m} e_j \cdot (\nabla^\Sigma X^j) = \sum_{i=1}^{n} (D_{\tau_i} X)\cdot\tau_i. \tag{11.3}$$

In local coordinates, setting $g := \det(g_{ij})$, we have

$$\operatorname{div}^\Sigma X = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}X^i), \tag{11.4}$$

where we wrote

$$X = \sum_{i=1^n} X^i \frac{\partial F}{\partial x^i}.$$

The *Laplacian* on Σ of a function $f \in C^2(\Sigma)$ is defined as

$$\Delta_\Sigma f := \operatorname{div}^\Sigma(\nabla^\Sigma f);$$

this may be written in local coordinates inserting (11.2) into (11.4):

$$\Delta_\Sigma f = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\Big(\sqrt{g}g^{ij}\frac{\partial f}{\partial x^j}\Big). \tag{11.5}$$

11.1.3 Second fundamental form and mean curvature

Let $\mathcal{N}(\Sigma)$ denote the vector space of normal vector fields on Σ, $N\Sigma$ the normal bundle of Σ and $N_p\Sigma$ the normal space to Σ at p.

Definition 11.6 (Second fundamental form) *The second fundamental form of* Σ

$$h : \mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma) \to \mathcal{N}(\Sigma)$$

is the normal part of the connection of $\mathbb{R}^{n+m}$ *in the following sense: given* $X, Y \in \mathcal{T}(\Sigma)$

$$h(X, Y) := (\nabla_X Y)^N.$$

As before, we need to extend X and Y to a neighborhood of Σ, compare Exercise 11.3.

Proposition 11.7 *The second fundamental form h:*

(i) *is symmetric:* $h(X, Y) = h(Y, X)$;

(ii) *is* C^∞*-linear in both variables;*

(iii) $h(X, Y)(p)$ *depends only on* $X(p)$ *and* $Y(p)$.

In particular h is well defined as a family of bilinear applications

$$h_p : T_p\Sigma \times T_p\Sigma \to N_p\Sigma.$$

Proof. Since for $X, Y \in \mathcal{T}(\Sigma)$ we have $[X, Y] \in \mathcal{T}(\Sigma)$, we have

$$h(X, Y) - h(Y, X) = (\nabla_X Y - \nabla_Y X)^N = [X, Y]^N = 0,$$

hence h is symmetric.

To prove (ii), observe that h is the difference of two connections:

$$h(X, Y) = \nabla_X Y - \nabla^\Sigma_X Y,$$

hence it is C^∞-linear in X. By symmetry, h is also C^∞-linear in Y.

Finally, both $\nabla_X Y(p)$ and $\nabla^\Sigma_X Y(p)$ depend only on Y and $X(p)$, hence also (iii) follows by symmetry. □

Definition 11.8 (Mean curvature) *The mean curvature H of* Σ *is the trace of the second fundamental form:*

$$H(p) := \sum_{i=1}^n h_p(\tau_i, \tau_i)$$

for any orthonormal basis $\{\tau_i\}$ *of* $T_p\Sigma$.

Exercise 11.9 If $\{v_1, \dots, v_n\}$ is an arbitrary basis of $T_p\Sigma$, then

$$H(p) = \sum_{i,j=1}^{n} g^{ij}(p) h_p(v_i, v_j), \tag{11.6}$$

where, as usual $g_{ij}(p) := g(v_i, v_j)$ and $(g^{ij}) = (g_{ij})^{-1}$.

Let $F : \Omega \to \Sigma$ be a local parametrization of Σ at p, that is a diffeomorphism of Ω (as regular as Σ) onto a neighborhood of p. Then F induces a basis of $T_p\Sigma$, given by

$$\left\{ \frac{\partial F}{\partial x^i} \right\}_{i=1,\dots,n}.$$

Since

$$\nabla_{\frac{\partial F}{\partial x^i}} \frac{\partial F}{\partial x^j} = \frac{\partial^2 F}{\partial x^i \partial x^j},$$

from (11.6) we infer

$$H(p) = \left(\sum_{i,j=1}^{n} g^{ij}(p) \frac{\partial^2 F}{\partial x^i \partial x^j}(F^{-1}(p)) \right)^N, \tag{11.7}$$

where $g_{ij} := \frac{\partial F}{\partial x^i} \cdot \frac{\partial F}{\partial x^j}$.

Lemma 11.10 (Derivative of a determinant) *Let*

$$g(s) := \det(g_{ij}(s)),$$

where $(g_{ij}(s))$ is a family of square matrices differentiable with respect to s. Then

$$\frac{\partial g}{\partial s} = g g^{ij} \frac{\partial g_{ij}}{\partial s}. \tag{11.8}$$

The proof is left for the reader.

Proposition 11.11 *Let $F : \Omega \to \Sigma$ be a local parametrization at p of class C^2. Then*

$$H(p) = \Delta_\Sigma F(p), \tag{11.9}$$

where the Laplacian of $F = (F^1, \dots, F^{n+m})$ is defined componentwise.

Proof. We first prove that $\Delta_\Sigma F(p)$ is orthogonal to $T_p\Sigma$. Thanks to (11.5) we have

$$\Delta_\Sigma F \cdot \frac{\partial F}{\partial x^k} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \big(\sqrt{g} g^{ij} \frac{\partial F}{\partial x^j} \cdot \frac{\partial F}{\partial x^k} \big) - g^{ij} \frac{\partial F}{\partial x^j} \cdot \frac{\partial^2 F}{\partial x^i \partial x^k}.$$

On the other hand, (11.8) and the symmetry of g yield

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\big(\sqrt{g}g^{ij}\frac{\partial F}{\partial x^j}\cdot\frac{\partial F}{\partial x^k}\big) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\Big(\sqrt{g}g^{ij}g_{jk}\Big) = \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^k}$$
$$= \frac{1}{2}g^{ij}\frac{\partial}{\partial x^k}\Big(\frac{\partial F}{\partial x^i}\cdot\frac{\partial F}{\partial x^j}\Big) = g^{ij}\frac{\partial^2 F}{\partial x^k\partial x^i}\cdot\frac{\partial F}{\partial x^j}.$$

Therefore

$$\Delta_\Sigma F\cdot\frac{\partial F}{\partial x^k} = 0.$$

Since k is arbitrary we conclude that $\Delta_\Sigma F$ is orthogonal to Σ.

To prove (11.9) write the Laplacian in coordinates and differentiate:

$$\Delta_\Sigma F = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\Big(\sqrt{g}g^{ij}\frac{\partial F}{\partial x^j}\Big) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\Big(\sqrt{g}g^{ij}\Big)\frac{\partial F}{\partial x^j} + g^{ij}\frac{\partial^2 F}{\partial x^i\partial x^j}.$$

The first term on the right-hand side is tangent, and by (11.7) we get

$$\Delta_\Sigma F = (\Delta_\Sigma F)^N = \Big(g^{ij}\frac{\partial^2 F}{\partial x^i\partial x^j}\Big)^N = H.$$

□

11.1.4 The area and its first variation

We define the *area* $\mathcal{A}(\Sigma)$ of an n-dimensional submanifold Σ as its n-dimensional Hausdorff measure, i.e. $\mathcal{A}(\Sigma) := \mathcal{H}^n(\Sigma)$, compare Definition 9.20. It can be expressed in terms of local parametrizations thanks to the *area formula*, compare [32], [49].

Theorem 11.12 (Area formula) *Let $F : \Omega \to \mathbb{R}^{n+m}$ be a* locally Lipschitz *and injective map of an open set $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^{n+m}$. Let Σ be the image of F; then*

$$\mathcal{H}^n(\Sigma) = \int_\Omega \sqrt{\det(dF^*(x)dF(x))}dx, \tag{11.10}$$

where $dF^* : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is the transposed of the matrix $dF = \Big(\frac{\partial F^i}{\partial x^\alpha}\Big)$.

If $g_{ij} := \frac{\partial F}{\partial x^i}\cdot\frac{\partial F}{\partial x^j}$, we have

$$(dF^*dF)_{ij} = \sum_{\alpha=1}^{n+m}\frac{\partial F^\alpha}{\partial x^i}\frac{\partial F^\alpha}{\partial x^j} = g_{ij}.$$

Therefore if $g := \det g_{ij}$, we have

$$\mathcal{A}(\Sigma) = \int_\Omega \sqrt{g(x)}dx. \tag{11.11}$$

In particular, $\sqrt{g}dx$ is the area element of Σ, hence, given an $\mathcal{H}^n \llcorner \Sigma$-integrable function f, we have

$$\int_\Sigma f d\mathcal{H}^n = \int_\Omega f(F(x))\sqrt{g(x)}dx.$$

Exercise 11.13 (Area of a graph in codimension 1) Show that if

$$F(x) = (x, u(x)),$$

with $x \in \Omega \subset \mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ locally Lipschitz continuous, then

$$dF^* dF = (\delta_{ij} + u_{x_i} u_{x_j}),$$

and

$$\det(dF^* dF) = 1 + |\nabla u|^2,$$

so that for $\Sigma := F(\Omega)$ we have

$$\mathcal{H}^n(\Sigma) = \int_\Omega \sqrt{1 + |\nabla u|^2} dx.$$

Exercise 11.14 (Area of a graph in dimension and codimension 2) Show that if

$$F(x, y) = (x, y, u(x, y), v(x, y)), \quad (x, y) \in \Omega \subset \mathbb{R}^2,$$

with $u, v : \Omega \to \mathbb{R}$ locally Lipschitz continuous, then

$$dF^* dF = \begin{pmatrix} 1 + u_x^2 + v_x^2 & u_x u_y + v_x v_y \\ u_x u_y + v_x v_y & 1 + u_y^2 + v_y^2 \end{pmatrix}, \tag{11.12}$$

and use (11.12) to prove that

$$\det(dF^* dF) = 1 + |dU|^2 + (\det(dU))^2,$$

where $U := (u, v) : \Omega \to \mathbb{R}^2$, so that for $\Sigma := F(\Omega)$ we have

$$\mathcal{H}^2(\Sigma) = \int_\Omega \sqrt{1 + |dU|^2 + (\det(dU))^2} dx.$$

Definition 11.15 *Given a Lipschitz and injective map*

$$F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+m},$$

a variation of $\Sigma := F(\Omega)$ *is a family of diffeomorphisms*

$$\varphi_t : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}, \quad t \in (-1, 1)$$

such that

1. $\varphi(t, x) := \varphi_t(x)$ *is of class* C^2 *in* $(-1, 1) \times \mathbb{R}^{n+m}$*;*

2. *there exists a compact set K non intersecting $\partial\Sigma := F(\partial\Omega)$ (possibly $\partial\Sigma = \emptyset$) such that $\varphi_t(x) = x$ for each $x \notin K$ and $t \in (-1,1)$;*

3. *$\varphi_0(x) = x$ for each $x \in \mathbb{R}^{n+m}$.*

We shall set $\Sigma_t := \varphi_t(\Sigma)$ and $X := \frac{\partial \varphi_t}{\partial t}\Big|_{t=0} \in C^1(\mathbb{R}^{n+m}, \mathbb{R}^{n+m})$.

Proposition 11.16 *Let F, Σ and Σ_t be as in the preceding definition. Then*

$$\frac{d}{dt}\mathcal{A}(\Sigma_t)\Big|_{t=0} = -\int_\Sigma \Delta_\Sigma F \cdot X d\mathcal{H}^n, \tag{11.13}$$

where, since F can be differentiated only once, the Laplacian is intended in the weak sense:

$$\begin{aligned} -\int_\Sigma \Delta_\Sigma F \cdot X d\mathcal{H}^n &= -\int_\Omega \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\Big(\sqrt{g}g^{ij}\frac{\partial F^\alpha}{\partial x^j}\Big)X^\alpha \circ F\sqrt{g}dx \\ &:= \int_\Omega \sqrt{g}g^{ij}\frac{\partial F^\alpha}{\partial x^j}\frac{\partial (X^\alpha \circ F)}{\partial x^i}dx \\ &= \int_\Sigma g^{ij}\frac{\partial F^\alpha}{\partial x^j}\frac{\partial (X^\alpha \circ F)}{\partial x^i}d\mathcal{H}^n. \end{aligned}$$

Proof. Write the Taylor expansion

$$\varphi_t(y) = y + tX(y) + o(t). \tag{11.14}$$

To differentiate the area formula (11.11) we set

$$F_t(x) = \varphi_t(F(x)), \quad g^t_{ij} = \frac{\partial F_t}{\partial x^i}\cdot\frac{\partial F_t}{\partial x^i},$$

and obtain with (11.8)

$$\begin{aligned} \frac{d}{dt}\mathcal{A}(\Sigma_t) &= \frac{d}{dt}\int_\Omega \sqrt{g^t}dx \\ &= \int_\Omega \frac{\partial\sqrt{g^t}}{\partial t}dx \\ &= \int_\Omega \frac{1}{2\sqrt{g}}\Big(gg^{ij}\frac{\partial g^t_{ij}}{\partial t}\Big)dx, \end{aligned} \tag{11.15}$$

where the derivatives with respect to t are evaluated at $t = 0$ and $g = g^0$. To compute $\frac{\partial g^t_{ij}}{\partial t}$ we observe that, thanks to (11.14),

$$\frac{\partial \varphi^t(F(x))}{\partial x^i} = \frac{\partial F}{\partial x^i} + t\frac{\partial X}{\partial x^i} + o(t)$$

and substituting into (11.15) yields

$$\begin{aligned}
&\int_\Omega \frac{1}{2\sqrt{g}}\left(g g^{ij} \frac{\partial g^t_{ij}}{\partial t}\bigg|_{t=0}\right) dx \\
&= \int_\Omega \frac{1}{2}\sqrt{g} g^{ij} \frac{\partial}{\partial t}\left\{\left(\frac{\partial F}{\partial x^i} + t\frac{\partial X}{\partial x^i} + o(t)\right)\cdot\left(\frac{\partial F}{\partial x^j} + t\frac{\partial X}{\partial x^j} + o(t)\right)\right\}\bigg|_{t=0} dx \\
&= \int_\Omega \frac{1}{2}\sqrt{g} g^{ij}\left\{\frac{\partial X(F(x))}{\partial x^i}\cdot\frac{\partial F}{\partial x^j} + \frac{\partial F}{\partial x^i}\cdot\frac{\partial (X(F(x))}{\partial x^j}\right\} dx.
\end{aligned}$$

Due to the symmetry of g^{ij} the last term becomes

$$\int_\Omega \sqrt{g} g^{ij} \frac{\partial X(F(x))}{\partial x^i}\cdot\frac{\partial F}{\partial x^j} dx = -\int_\Sigma \Delta_\Sigma F\cdot X d\mathcal{H}^n, \tag{11.16}$$

and we conclude. □

Proposition 11.17 *Let Σ, φ_t and X be as in Proposition 11.16. Then*

$$\frac{d}{dt}\mathcal{A}(\Sigma_t)\bigg|_{t=0} = \int_\Sigma \operatorname{div}^\Sigma X d\mathcal{H}^n.$$

Proof. Let $\{v_1, \dots, v_n\}$ be a basis of $T_p\Sigma$, and set $g_{ij} = v_i \cdot v_j$. By linearity

$$\operatorname{div}^\Sigma X = g^{ij}\nabla_{v_i} X \cdot v_j.$$

Consequently, choosing a local parametrization F in p, and setting $v_i := \frac{\partial F}{\partial x^i}$ and using $\nabla_{\frac{\partial F}{\partial x^i}} X = \frac{\partial (X\circ F)}{\partial x^i}$ we obtain

$$\operatorname{div}^\Sigma X = g^{ij}\frac{\partial X(F(x))}{\partial x^i}\cdot\frac{\partial F}{\partial x^j}.$$

We conclude by comparison with (11.16). □

In fact we have proven that

$$\int_\Sigma \operatorname{div}^\Sigma X d\mathcal{H}^n = -\int_\Sigma \Delta_\Sigma F\cdot X d\mathcal{H}^n$$

whenever $F : \Omega \to \mathbb{R}^{n+m}$ parametrizes Σ and X vanishes in a neighborhood of $\partial\Sigma$.

First variation and mean curvature

The first variation of the area of a submanifold Σ and the mean curvature of Σ are closely related, as the following proposition shows.

Proposition 11.18 *Let Σ be a C^2 submanifold and let φ_t be as in Definition 11.15; set $X := \frac{\partial\varphi}{\partial t}\Big|_{t=0}$, $\Sigma_t := \varphi_t(\Sigma)$. Then the* first variation of the area *of Σ with respect to φ is*

$$\frac{d}{dt}\mathcal{A}(\Sigma_t)\Big|_{t=0} = -\int_\Sigma H\cdot X d\mathcal{H}^n. \tag{11.17}$$

Proof. Insert (11.9) into Proposition 11.16. □

Definition of minimal surface

In the following an n-dimensional Lipschitz submanifold Σ of $\mathbb{R}^{n+m}$ will be the image of a Lipschitz maps $F:\Omega\subset\mathbb{R}^n\to\mathbb{R}^{n+m}$ which is injective and such that the rank of $dF(x)$ is n for a.e. $x\in\Omega$. By $\partial\Sigma := F(\partial\Omega)$ we denote the boundary of Σ. Notice that the condition on the rank of dF is always satisfied when $F(x) = (x, u(x))$ parametrizes a graph, which is the case we shall focus on.

Definition 11.19 (Minimal surface) *Let Σ be a Lipschitz n-dimensional submanifold of $\mathbb{R}^{n+m}$. We shall say that Σ is minimal if for every variation φ_t defined as in 11.15, we have*

$$\frac{d}{dt}\Big|_{t=0}\mathcal{A}(\Sigma_t) = 0.$$

Thanks to Propositions 11.16, 11.17 and 11.18, we have the following characterization of minimal surfaces.

Proposition 11.20 *Given a Lipschitz submanifold Σ of $\mathbb{R}^{n+m}$, the following facts are equivalent:*

1. *Σ is minimal;*
2. *for every vector field $X\in C^1_c(\mathbb{R}^{n+m},\mathbb{R}^{n+m})$ such that $X=0$ in a neighbourhood of $\partial\Sigma$*
$$\int_\Sigma \operatorname{div}^\Sigma X d\mathcal{H}^n = 0;$$
3. *for any local parametrization $F:\Omega\to\Sigma$ we have $\Delta_\Sigma F = 0$.*

Moreover, if Σ is of class C^2, the preceding statements are equivalent to $H=0$.

Proof. For every vector field $X \in C^1_c(\mathbb{R}^{n+m}, \mathbb{R}^{n+m})$ vanishing in a neighborhood of $\partial\Sigma$ we may find a family of diffeomorphisms φ_t as in Definition 11.15 satisfying $\frac{\partial \varphi_t}{\partial t}\big|_{t=0} = X$. For instance the solution φ of

$$\begin{cases} \dfrac{\partial \varphi(t,x)}{\partial t} = X(x) \\ \varphi(0,x) = 0, \end{cases}$$

which exists for small times thanks to ODE theory. Then the equivalence of 1, 2, and 3 follows from Propositions 11.16 and 11.17.

The last claim is an immediate consequence of Proposition 11.18. □

The minimal surface system

Consider a parametrization $F : \Omega \to \mathbb{R}^{n+m}$ of a Lipschitz submanifold $\Sigma \subset \mathbb{R}^{n+m}$. Thanks to Proposition 11.20, Σ is minimal if and only if F satisfies the following system, called *minimal surface system*:

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left(\sqrt{g} g^{ij} \frac{\partial F^\alpha}{\partial x^j}\right) = 0, \quad \alpha = 1, \dots, n+m, \tag{11.18}$$

where $g = \det(g_{ij})$, $g_{ij} = \frac{\partial F}{\partial x^i} \cdot \frac{\partial F}{\partial x^j}$ and $(g^{ij}) = (g_{ij})^{-1}$.

A non-parametric Lipschitz surface Σ is by definition the graph

$$\mathcal{G}_u := \{(x, u(x)) : x \in \Omega\}$$

of a Lipschitz function $u : \Omega \to \mathbb{R}^m$. Clearly $\mathcal{G}_u$ can be parametrized by

$$F : \Omega \to \mathbb{R}^{n+m}, \quad F(x) = (x, u(x)).$$

In this case, the minimal surface system becomes

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x^i}\left(\sqrt{g} g^{ij}\right) = 0 & j = 1, \dots, n \\ \displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left(\sqrt{g} g^{ij} \frac{\partial u^\alpha}{\partial x^j}\right) = 0 & \alpha = 1, \dots, m \\ \displaystyle g_{ij} := \delta_{ij} + \sum_{\alpha=1}^{m} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\alpha}{\partial x^j} & (g^{ij}) := (g_{ij})^{-1}. \end{cases} \tag{11.19}$$

If $u \in C^2(\Omega, \mathbb{R}^m)$, then the system (11.19) reduces to a quasilinear elliptic system in non-divergence form, as shown in the following proposition.

Proposition 11.21 *Let $u \in C^2(\Omega, \mathbb{R}^m)$. Then* (11.19) *is equivalent to*

$$\sum_{i,j=1}^{n} g^{ij} \frac{\partial^2 u^\alpha}{\partial x^i \partial x^j} = 0, \quad \alpha = 1, \dots, m, \tag{11.20}$$

where $g^{ij} = g^{ij}(Du)$ is as in (11.19).

Proof. Assume (11.20) and set $F(x) = (x, u(x))$. Then

$$\Delta_\Sigma F = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}g^{ij})\frac{\partial F}{\partial x^j} + g^{ij}\frac{\partial^2 F}{\partial x^i \partial x^j}.$$

The last term vanishes by (11.20) and because clearly $\frac{\partial^2 x^k}{\partial x^i \partial x^j} = 0$ for $k = 1, \dots, n$. Since $\Delta_\Sigma F \in N\Sigma$ and $\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}g^{ij})\frac{\partial F}{\partial x^j} \in T\Sigma$, they both vanish, whence $\Delta_\Sigma F = 0$.

Conversely, if (11.19) holds and u is C^2, then, thanks to Proposition 11.20, $H = 0$, and we use (11.7) to conclude. □

11.1.5 Area-decreasing maps

The area-decreasing condition will be useful when dealing with the regularity theory for minimal surfaces. Let us first recall

Proposition 11.22 (Singular-value decomposition)
Let $L \in M(m \times n)$ be any $m \times n$ matrix. Then there exist orthogonal matrices $U \in O(m)$ and $V \in O(n)$, such that $B := ULV \in M(m \times n)$ is a diagonal matrix, i.e. $B = \{\lambda_{\alpha i}\}_{\alpha=1,\dots,m}^{i=1,\dots,n}$, with $\lambda_{\alpha i} = 0$ whenever $\alpha \neq i$.

For the proof of this proposition, the reader can refer to e.g. [70].

Remark 11.23 We can always assume that $\lambda_{\alpha i} \geq 0$, as changing the sign of the basis vectors corresponds to an orthogonal transformation.

The numbers $\lambda_i := \lambda_{ii}$ in Proposition 11.22 are called *singular values* of L. They are the square roots of the eigenvalues of L^tL: indeed we can diagonalize L^tL as follows:

$$V^tL^tLV = V^tL^tU^tULV = B^tB = \mathrm{diag}\{\lambda_1^2, \dots, \lambda_n^2\},$$

where if $n > m$ we set $\lambda_i := 0$ for $n < i \leq m$.

Exercise 11.24 Let $u : \Omega \to \mathbb{R}^m$ be a Lipschitz map and let $\lambda_1(x), \dots, \lambda_n(x)$ be the singular values of $du(x) : \mathbb{R}^n \to \mathbb{R}^m$. Show that

$$\mathcal{A}(\mathcal{G}_u) = \int_\Omega \sqrt{\textstyle\prod_{i=1}^n (1 + \lambda_i^2(x))}dx. \tag{11.21}$$

[Hint: Use (11.10).]

Definition 11.25 (Area-decreasing map) *Let $u : \Omega \to \mathbb{R}^m$ be a Lipschitz map. Let $\{\lambda_i(x)\}_{i=1,\dots,n}$ be the singular values of $du(x)$. We shall say that u is area-decreasing if there exists $\varepsilon > 0$ such that for a.e. $x \in \Omega$ we have*

$$\lambda_i(x)\lambda_j(x) \leq 1 - \varepsilon, \quad 1 \leq i < j \leq n. \tag{11.22}$$

The name *area-decreasing* comes from the following geometric fact: consider $du(x)$ restricted to a 2 dimensional subspace V of $\mathbb{R}^n$. Then for each $A \subset V$ with $\mathcal{H}^2(A) < \infty$ we have $\mathcal{H}^2(du(x)(A)) < \mathcal{H}^2(A)$.

Remark 11.26 A scalar function $u : \Omega \to \mathbb{R}$ is always area-decreasing. This follows immediately from the definition because the nonzero singular values of $du(x)$ correspond to a basis of the image of $du(x)$, therefore there is at most one of them.

11.2 Minimal graphs in codimension 1

11.2.1 Convexity of the area; uniqueness and stability

The area functional in codimension 1 is

$$\mathcal{A}(u) = \int_\Omega \sqrt{1 + |Du|^2} dx, \tag{11.23}$$

compare Exercise 11.13 and (11.21).

Proposition 11.27 (Convexity) *The area functional* $\mathcal{A} : \mathrm{Lip}(\overline{\Omega}) \to \mathbb{R}$ *in codimension 1 is strictly convex, that is*

$$\mathcal{A}(\lambda u + (1 - \lambda)v) \le \lambda \mathcal{A}(u) + (1 - \lambda)\mathcal{A}(v), \tag{11.24}$$

for every $u, v \in \mathrm{Lip}(\overline{\Omega})$ *and* $\lambda \in (0, 1)$ *and equality holds if and only if* $u = v + c$ *for some* $c \in \mathbb{R}$.

Proof. The function $f(t) = \sqrt{1 + t^2}$, $t \in \mathbb{R}$, is strictly convex, as

$$f''(t) = \frac{1}{(1 + t^2)^{\frac{3}{2}}} > 0,$$

and it follows easily that the function

$$p \mapsto \sqrt{1 + |p|^2}, \quad p \in \mathbb{R}^n$$

is strictly convex. Then inequality in (11.24) follows at once, with identity if and only if $du = dv$ a.e., hence if and only if $u - v$ is constant. □

Remark 11.28 Convexity is a major property of the area functional in codimension 1. In fact we shall see that uniqueness and stability of minimal graphs in codimension 1 (which do not hold in higher codimension as shown by Lawson and Osserman [68]) come from convexity.

Uniqueness and minimizing properties

Theorem 11.29 *The graph of a Lipschitz solution $u : \overline{\Omega} \to \mathbb{R}$ of the minimal surface system* (11.19) *minimizes the area among the graphs of Lipschitz functions v such that*

$$v|_{\partial\Omega} = u|_{\partial\Omega} =: \psi.$$

Moreover u is the unique solution to the minimal surface equation

$$\sum_{i=1}^{n} D_i \frac{D_i u}{\sqrt{1+|Du|^2}} = 0, \tag{11.25}$$

with prescribed boundary data $u|_{\partial\Omega} = \psi$.

Proof. Step 1. The minimal surface system implies that the first variation of the area of the graph $\mathcal{G}_u$ vanishes. In particular, for a given function $\varphi \in C^1_c(\Omega)$ (or in fact $\varphi \in \mathrm{Lip}(\Omega)$ with $\varphi|_{\partial\Omega} = 0$) we have

$$\begin{aligned} 0 &= \frac{d}{dt}\bigg|_{t=0} \mathcal{A}(u+t\varphi) \\ &= \int_\Omega \frac{\partial}{\partial t}\sqrt{1+|Du+tD\varphi|^2}dx \\ &= -\int_\Omega \sum_{i=1}^{n} \frac{D_i u D_i \varphi}{\sqrt{1+|Du|^2}}dx, \end{aligned}$$

which is the minimal surface equation (11.25).

Step 2. Equation (11.25) means that u is a *critical point* for the area functional. On the other hand convexity implies

$$\mathcal{A}(v) \geq \mathcal{A}(u) + \frac{d}{dt}\bigg|_{t=0} \mathcal{A}(u+t(v-u)) = \mathcal{A}(u), \quad \forall v \in \mathrm{Lip}(\Omega),\ v\big|_{\partial\Omega} = u\big|_{\partial\Omega}.$$

Step 3. Uniqueness follows at once by the strict convexity of $\mathcal{A}$. $\square$

Remark 11.30 In fact one can also show that a Lipschitz solution u of (11.25) also solves (11.19).

Stability under parametric deformations

Theorem 11.31 *Let $u : \overline{\Omega} \to \mathbb{R}$ be a Lipschitz solution to the minimal surface equation* (11.25). *Then:*

1. *if Ω is homologically trivial (for instance, Ω convex, star-shaped or contractible; in fact we only need $H^n(\Omega) = 0$), then the graph of u minimizes the area among every Lipschitz submanifold $\Sigma \subset \overline{\Omega} \times \mathbb{R}$ having the same boundary;*

2. *if Ω is convex, then the graph of u minimizes the area among the Lipschitz submanifolds $\Sigma \subset \mathbb{R}^{n+1}$ having the same boundary.*

The proof is based on the existence of a *calibration*, that is an exact n-form ω of absolute value at most 1, whose restriction to $\mathcal{G}_u$ is the area form.

Proposition 11.32 (Calibration) *Let $\omega \in L^\infty(\Omega \times \mathbb{R}; \bigwedge^n \mathbb{R}^{n+1})$ be an exact n-form in $\Omega \times \mathbb{R}$, such that $|\omega| \leq 1$. Let $\Sigma_0 \subset \overline{\Omega} \times \mathbb{R}$ be a Lipschitz submanifold and assume that $\omega\big|_{\Sigma_0}$ is the volume form of Σ_0. Then Σ_0 has least area among all Lipschitz submanifolds $\Sigma \subset \overline{\Omega} \times \mathbb{R}$ such that $\partial\Sigma = \partial\Sigma_0$.*

Proof. Let $\eta \in W^{1,\infty}(\Omega \times \mathbb{R}; \bigwedge^{n-1} \mathbb{R}^{n+1})$ be an $n-1$ form such that $d\eta = \omega$. Let Σ be as in the statement of the proposition; then, by Stokes' theorem and since the two submanifolds have the same boundary,

$$\int_{\Sigma - \Sigma_0} \omega = \int_{\partial\Sigma - \partial\Sigma_0} \eta = 0.$$

On the other hand, since $|\omega| \leq 1$, and $\omega\big|_{\Sigma_0}$ is the volume form of Σ_0,

$$\mathcal{A}(\Sigma) \geq \int_\Sigma \omega = \int_{\Sigma_0} \omega = \mathcal{A}(\Sigma_0).$$

□

Proof of Theorem 11.31. Define in $\Omega \times \mathbb{R}$ the *calibration form*

$$\omega(x, y) := \frac{\left(\sum_{i=1}^n (-1)^{n+i-1} D_i u(x) \widehat{dx^i} \wedge dy\right) + dx^1 \wedge \cdots \wedge dx^n}{\sqrt{1 + |Du|^2}},$$

where

$$\widehat{dx^i} := dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n.$$

By the minimal surface equation (11.25) $d\omega = 0$; since $\overline{\Omega} \times \mathbb{R}$ is homologically trivial, ω is also exact. Moreover $|\omega| = 1$ and the restriction of ω to $\mathcal{G}_u$ is the volume form of $\mathcal{G}_u$, thus ω is a calibration for $\mathcal{G}_u$ and Proposition 11.32 applies.

The second claim follows from the first one: consider $\Sigma \subset \mathbb{R}^{n+1}$ with $\partial\Sigma = \partial\mathcal{G}_u$. The closest point projection $\pi : \mathbb{R}^{n+1} \to \Omega \times \mathbb{R}$ doesn't increase the area and fixes the boundary. By part 1

$$\mathcal{A}(\Sigma) \geq \mathcal{A}(\pi(\Sigma)) \geq \mathcal{A}(\mathcal{G}_u).$$

□

Remark 11.33 The topological hypothesis (or at least some kind of hypothesis) on Ω is necessary: R. Hardt, C. P. Lau and Fang-Hua Lin [55] proved the existence of a solution of the minimal surface equation whose graph doesn't minimize the area among the n-submanifolds of $\mathbb{R}^{n+1}$ having the same boundary.

11.2.2 The problem of Plateau: existence of minimal graphs with prescribed boundary

Finding a minimal graph with prescribed boundary is equivalent to solving the Dirichlet problem for the minimal surface system (11.20). In codimension one that is

$$\begin{cases} \displaystyle\sum_{i,j=1}^{n} g^{ij}\frac{\partial^2 u}{\partial x^i \partial x^j} = 0 & \text{in } \Omega \\ u = \psi & \text{on } \partial\Omega, \end{cases} \tag{11.26}$$

with, say, $\psi \in C^\infty(\partial\Omega)$, and

$$g^{ij} = g^{ij}(Du) = \delta_{ij} - \frac{D_i u D_j u}{1+|Du|^2}.$$

The counterexample of Bernstein seen in section 2.3 shows that finding minimal graphs on arbitrary domains and for any boundary value is not possible in general. In fact by Theorem 2.20 a necessary condition to have existence for any boundary value is that the mean curvature of $\partial\Omega$ be non-negative. The next theorem proves that this condition is also sufficient.

Theorem 11.34 (Jenkins-Serrin [62]) *Let Ω be a smooth, bounded, connected domain whose boundary has nonnegative mean curvature. Then for each $\psi \in C^{2,\alpha}(\overline{\Omega})$, there exists a unique function $u \in C^\infty(\Omega)\cap C^{2,\alpha}(\overline{\Omega})$ solving the Dirichlet problem for the minimal surface equation (11.26).*

Remark 11.35 Equation (11.26) is quasilinear and elliptic. On the other hand, it is not uniformly elliptic since the ellipticity constant $\lambda > 0$ in

$$g^{ij}(p)\xi_i\xi_j \geq \lambda|\xi|^2, \quad \forall \xi, p \in \mathbb{R}^n.$$

depends on $|p|$. In fact

$$\frac{1}{1+\eta}|\xi|^2 \leq g^{ij}(Du(x))\xi_i\xi_j \leq |\xi|^2, \quad \eta := \sup_{\Omega}|Du|^2, \tag{11.27}$$

for all $\xi \in \mathbb{R}^n$.

To prove Theorem 11.34, we shall use the fixed point theorem of Caccioppoli-Schauder, which we recall without proof, see e.g. [47].

Theorem 11.36 *Let $T : K \to K$ be a completely continuous operator*[1] *which sends a non-empty, convex, closed, bounded subset K of a Banach space B into itself. Then T has a fixed point, meaning that there exists $\overline{x} \in K$ such that $T(\overline{x}) = \overline{x}$.*

Proposition 11.37 *Consider a completely continuous Banach-space operator $T : B \to B$ and $M > 0$ such that for each pair $(\sigma, u) \in [0,1] \times B$ satisfying $u = \sigma T(u)$ we have $\|u\| < M$. Then T has a fixed point.*

Proof. Let $K = \{u \in B \,|\, \|u\| \le M\}$ and define the operator on B

$$\overline{T}(u) := \begin{cases} T(u) & \text{if } T(u) \in K \\ M\dfrac{T(u)}{\|T(u)\|} & \text{if } T(u) \in B \backslash K. \end{cases}$$

Then $\overline{T}$ sends K into itself, hence by Theorem 11.36, it has a fixed point $u \in K$. Were $\|T(u)\| \ge M$, we would have

$$u = \frac{M}{\|T(u)\|} T(u), \qquad \frac{M}{\|T(u)\|} \in [0,1], \tag{11.28}$$

thus $\|u\| < M$ by hypothesis, absurd because (11.28) implies that $\|u\| = M$. So $\|T(u)\| < M$ and $T(u) = \overline{T}(u) = u$. □

Proof of Theorem 11.34. Uniqueness follows from Theorem 11.29 (compare Lemma 11.40 below). We will prove the existence of a solution $u \in C^{2,\alpha}(\overline{\Omega})$, but then by Schauder estimates (Theorem 5.20) we will have $u \in C^\infty(\Omega) \cap C^{2,\alpha}(\overline{\Omega})$.

On the Banach space $B = C^{1,\alpha}(\overline{\Omega})$, consider the operator $\widetilde{T}$ which associates to a function $u \in C^{1,\alpha}(\overline{\Omega})$ the unique solution $v \in C^{2,\alpha}(\overline{\Omega})$ to the Dirichlet problem

$$\begin{cases} \displaystyle\sum_{i,j=1}^n g^{ij}(Du) \frac{\partial^2 v}{\partial x^i \partial x^j} = 0 & \text{in } \Omega \\ v = \psi & \text{on } \partial\Omega, \end{cases} \tag{11.29}$$

which exists thanks to Theorem 5.25 and is bounded, i.e. sets bounded sets into bounded sets, thanks to Theorem 5.23. The inclusion operator $\pi : C^{2,\alpha}(\overline{\Omega}) \hookrightarrow C^{1,\alpha}(\overline{\Omega})$ is compact thanks to Ascoli-Arzelà's theorem.[2]

[1] Continuous and sending bounded sets into relatively compact sets; we do not assume T to be linear.

[2] We are using

Therefore the operator

$$T := \pi \circ \widetilde{T} : C^{1,\alpha}(\overline{\Omega}) \to C^{1,\alpha}(\overline{\Omega})$$

sends bounded sets into relatively compact sets. We want to show that T is continuous. Consider the sequences

$$u^{(k)} \xrightarrow{C^{1,\alpha}} u, \quad v^{(k)} := \widetilde{T}u^{(k)}.$$

Given a subsequence $u^{(k')}$, thanks to the compactness of the immersion

$$C^{2,\alpha}(\overline{\Omega}) \hookrightarrow C^2(\overline{\Omega}),$$

there exists a sub-subsequence $u^{(k'')}$ such that

$$v^{(k'')} \xrightarrow{C^2} v$$

Also v solves (11.29), as the following diagram explains:

$$\begin{array}{ccc} g^{ij}(Du^{(k'')}) & D_{ij}v^{(k'')} & = 0 \\ {\scriptstyle C^{0,\alpha}}\big\downarrow & \big\downarrow{\scriptstyle C^0} & \\ g^{ij}(Du) & D_{ij}v & = 0 \end{array}$$

(the sum over i and j is understood) and by uniqueness we have $v = Tu$. The arbitrariness in the choice of the first subsequence implies

$$Tu^{(k)} \xrightarrow{C^{1,\alpha}} Tu,$$

whence the continuity.

The proof of the theorem will be completed if we can show that T has a fixed point. By Proposition 11.37, it only remains to prove that T satisfies the following *a priori* estimate: *there exists* $M > 0$ *such that* $\|u\|_{C_{1,\alpha}(\overline{\Omega})} < M$ *whenever* $u = \sigma T(u)$ *for some* $\sigma \in (0, 1)$. This is the content of the following section. $\square$

Corollary 11.38 *The immersion* $C^{r,\alpha}(\overline{\Omega}) \to C^r(\overline{\Omega})$, $0 < \alpha \leq 1$, $r \in \mathbb{N}$, *is compact.*

Proof. Let u_j be bounded in $C^{r,\alpha}(\overline{\Omega})$, that is $\|u_j\|_{r,\alpha} \leq M$ for some $M > 0$. Then the derivatives of highest order are equicontinuous thanks to the estimate

$$|D^r_j(x) - D^r_j(y)| \leq K|x - y|^\alpha, \quad \forall j \in \mathbb{N}, x, y \in \Omega.$$

Moreover the lower order derivatives are equicontinuous by boundedness of the highest order derivatives and we may apply the theorem of Ascoli and Arzelà to each derivative. $\square$

11.2.3 A priori estimates

This section is devoted to prove the following a priori estimate.

Theorem 11.39 *Let Ω be bounded, smooth, connected, and assume that the mean curvature of $\partial\Omega$ is everywhere non-negative. Let $\psi \in C^2(\overline{\Omega})$. Then there exists a constant $C = C(\psi, \Omega)$ such that any solution $u \in C^2(\overline{\Omega})$ of*

$$\begin{cases} \displaystyle\sum_{i,j=1}^{n} g^{ij}(Du)\frac{\partial^2 u}{\partial x^i \partial x^j} = 0 & \text{in } \Omega \\ u = \sigma\psi & \text{on } \partial\Omega. \end{cases} \tag{11.30}$$

satisfies $\|u\|_{C^{1,\alpha}(\overline{\Omega})} < C$.

This will be obtained in four steps: we shall estimate

1. $\sup_{\overline{\Omega}} |u|$
2. $\sup_{\partial\Omega} |Du|$
3. $\sup_{\overline{\Omega}} |Du|$
4. $\|u\|_{1,\alpha}$

The first step is a simple application of the maximum principle:

$$\sup_{\overline{\Omega}} |u| \leq \sup_{\partial\Omega} |\sigma\psi| \leq \sup_{\partial\Omega} |\psi|,$$

see e.g. Exercise 1.4.

Gradient estimates

To obtain an estimate of the gradient on the boundary we use barriers, already introduced in Chapter 2. Here is where the assumption on the mean curvature of $\partial\Omega$ plays a crucial role.

Lemma 11.40 *Let $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy*

$$\begin{cases} \displaystyle\sum_{i,j=1}^{n} g^{ij}(Du)\frac{\partial^2 u}{\partial x^i \partial x^j} = 0 & \text{in } \Omega, \\ \displaystyle\sum_{i,j=1}^{n} g^{ij}(Dv)\frac{\partial^2 v}{\partial x^i \partial x^j} \leq 0 & \text{in } \Omega, \\ u \leq v & \text{on } \partial\Omega. \end{cases} \tag{11.31}$$

Then $u \leq v$ on all of $\overline{\Omega}$.

Proof. By the mean value theorem of Lagrange there exists $\xi \in (0,1)$ such that

$$g^{ij}(Du) = g^{ij}(Dv) + \sum_{k=1}^{n} \frac{\partial g^{ij}}{\partial p^k}(\xi Dv + (1-\xi)Du)\Big(\frac{\partial u}{\partial x^k} - \frac{\partial v}{\partial x^k}\Big).$$

Subtracting in the previous system and setting $w := v - u$ we get

$$\begin{cases} \displaystyle\sum_{i,j=1}^{n} g^{ij}(Dv)\frac{\partial^2 w}{\partial x^i \partial x^j} + \sum_{k=1}^{n} b^k \frac{\partial w}{\partial x^k} \leq 0 & \text{in } \Omega \\ w \geq 0 & \text{on } \partial\Omega \end{cases}$$

to which the maximum principle, Exercise 1.4, applies. Hence $w \geq 0$ in $\overline{\Omega}$. □

Proposition 11.41 *Let Ω be such that $\partial\Omega$ has everywhere nonnegative mean curvature. Then there exists a constant $C = C(\Omega, \psi)$ such that every C^2-solution of* (11.30) *satisfies*

$$\sup_{\overline{\Omega}} |Du| \leq C.$$

Proof. First we prove that $\sup_{\partial\Omega} |Du| \leq C$ for a suitable $C = C(\psi, \Omega)$. Let $d : \Omega \to \mathbb{R}$ be the function *distance from the boundary*

$$d(x) := \inf_{y \in \partial\Omega} |x - y|,$$

and define

$$N_r := \big\{x \in \Omega \,\big|\, d(x) < r\big\}, \quad \Gamma_r := \{x \in \Omega \,|\, d(x) = r\};$$

these domains are smooth for r small enough. Consider on N_r the barrier v given by

$$v(x) = \sigma\psi(\pi(x)) + h(d(x)),$$

where $\pi : N_r \to \partial\Omega$ is the closest point projection and $h : [0, r] \to \mathbb{R}_+$ is a C^∞-function to be determined, which satisfies

$$h(0) = 0,\ h'(t) \geq 1,\ h''(t) \leq 0, \quad t \in [0, r]. \tag{11.32}$$

With these choices we get

$$(1 + |Dv|^2) \sum_{i,j=1}^{n} g^{ij}(Dv) D_{ij} v \leq h'' + Ch'^2 + h'^3 \Delta d.$$

The behaviour of Δd is determined by the mean curvature of $\partial\Omega$: if $\partial\Omega$ has nonnegative mean curvature, then $\Delta d \le 0$,[3] thus,

$$(1+|Dv|^2)\sum_{i,j=1}^{n} g^{ij}(Dv)D_{ij}v \le h'' + C(h')^2.$$

Now, setting $h(d) = k\log(1+\rho d)$, we may choose the constants k and ρ independent of σ in such a way that conditions (11.32) are satisfied, $h(r) \ge 2\sup_{\partial\Omega}|\psi| \ge 2\sup_{\partial\Omega}|\sigma\psi|$ and $h'' + C(h')^2 \le$. Then

$$\begin{cases} \displaystyle\sum_{i,j=1}^{n} g^{ij}(Dv)\frac{\partial^2 v}{\partial x^i \partial x^j} \le 0 & \text{in } N_r \\ v \ge u & \text{on } \partial N_r \end{cases}$$

Then, by Lemma 11.40, $u \le v$ in N_r. Since $u = v$ on $\partial\Omega$, we obtain

$$\frac{u(x)-u(y)}{|x-y|} \le \frac{v(x)-v(y)}{|x-y|}, \qquad x \in \Omega,\ y \in \partial\Omega. \tag{11.33}$$

The similar construction of a lower barrier, say w, yields the opposite inequality, hence

$$-k\rho = D_\nu w \le D_\nu u \le D_\nu v = k\rho,$$

where $Du = \left(D^{\partial\Omega}u, D_\nu u\right)$, and ν is the interior normal to $\partial\Omega$. Taking into account $u = \sigma\psi$ on $\partial\Omega$ we have $D^{\partial\Omega}u = \sigma D^{\partial\Omega}\psi$, which together with (11.33) gives

$$\sup_{\partial\Omega}|Du| \le \sqrt{\sup_{\partial\Omega}|D\psi|^2 + (k\rho)^2},$$

and this estimate extends to the interior points thanks to the method of Haar-Radò, Proposition 2.11. □

The $C^{1,\alpha}(\overline{\Omega})$ a priori estimates

To prove the a priori estimates in $C^{1,\alpha}(\overline{\Omega})$ we need a global (up to the boundary) version of De Giorgi's Theorem 8.13. That has been obtained by O. Ladyžhenskaya and N. Ural'tseva [67]:

Theorem 11.42 *Let $w \in W^{1,2}(\Omega)$ be a weak solution of*

$$\begin{cases} D_\alpha(A^{\alpha\beta}D_\beta w) = 0 & \text{in } \Omega, \\ w = \varphi & \text{on } \partial\Omega \end{cases} \tag{11.34}$$

[3] See [50] or [52]

where $\lambda|\xi|^2 \leq A^{\alpha\beta}\xi_\alpha\xi_\beta \leq \Lambda|\xi|^2$ *and* $\varphi \in \mathrm{Lip}(\Omega)$. *Then* $w \in C^{0,\alpha}(\overline{\Omega})$, $\alpha = \alpha(\Omega, \lambda, \Lambda)$, *and there is a constant* $C_1 = C_1(\Omega, \lambda, \Lambda)$ *such that*

$$\|w\|_{C^{0,\alpha}(\overline{\Omega})} \leq C_1 \|\varphi\|_{C^{0,\alpha}(\overline{\Omega})}. \tag{11.35}$$

We shall apply this theorem to the derivatives of the solution u of (11.30), according to the following proposition.

Proposition 11.43 *Set* $A(p) := \frac{p}{\sqrt{1+|p|^2}}$, *for* $p \in \mathbb{R}^n$ *and let* $u \in C^2(\Omega)$ *(or in fact just* $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$*) be a solution of the minimal surface equation*

$$\mathrm{div}\, A(Du(x)) = 0.$$

Then if we set $w := D_s u$ $(s = 1, \ldots, n)$, *we have*

$$D_\alpha(a^{\alpha\beta} D_\beta w) = 0,$$

where $a^{\alpha\beta}(x) := D_{p_\beta} A^\alpha(Du(x)) = \frac{1}{\sqrt{1+|Du|^2}}\left(\delta^{\alpha\beta} - \frac{D_\alpha u D_\beta u}{1+|Du|^2}\right)$.

Proof. Differentiate the minimal surface equation with respect to x^s. □

Corollary 11.44 *In the hypothesis of Proposition 11.41, there exists a constant* $M = M(\Omega, \psi)$ *such that*

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leq M.$$

Proof. By Proposition 11.41, $\sup_{\overline{\Omega}} |Du| \leq C$, with C depending on Ω and ψ; by Theorem 11.42 and Proposition 11.43,

$$\|Du\|_{C^{0,\alpha}(\overline{\Omega})} \leq C_1 \|D\psi\|_{C^{0,\alpha}(\overline{\Omega})}. \tag{11.36}$$

The constant C_1 depends on Ω, λ and Λ; by the ellipticity estimate (11.27), we may choose $\Lambda = 1$ and

$$\lambda = \frac{1}{1 + \sup |Du|^2} \geq \frac{1}{1+C},$$

depending only on ψ and Ω. Putting together (11.36) and Proposition 11.41 we conclude. □

11.2.4 Regularity of Lipschitz continuous minimal graphs

As a consequence of De Giorgi's theorem, every Lipschitz solution to the minimal surface equation is smooth.

Theorem 11.45 *Let $u \in \mathrm{Lip}(\Omega)$ be a Lipschitz solution to the minimal surface equation* (11.25)*. Then $u \in C^\infty(\Omega)$.*

Proof. Since $|Du| \le C$ the function

$$A(p) = \frac{p}{\sqrt{1+|p|^2}}$$

satisfies (8.5), at least for $|p| \le \sup_\Omega |Du| \le C$ (the behavior of $A(p)$ for $|p| > \sup_\Omega |Du|$ is of course irrelevant). Then by Proposition 8.6 we have $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$, and by Proposition 11.43 $D_s u$ satisfies an elliptic equation, whose ellipticity constant λ can be estimated as in Corollary 11.44. Then by Theorem 11.42 we have $Du \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$. The higher order regularity follows from Schauder estimates, Theorem 5.20. □

Let us remark that a minimal surface (defined in a suitable generalized sense) in codimension 1 which cannot be expressed locally as a Lipschitz graph (for instance a varifold, an area minimizing current, or the boundary of a Caccioppoli set of least area) need not in general be regular. Indeed Bombieri, De Giorgi and Giusti [12] have shown that Simons' cone

$$C := \left\{x = (x_1, \dots, x_8) \in \mathbb{R}^8 \;\middle|\; x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\right\},$$

which is singular at $\{0\}$, is an area-minimizing current (if given an orientation).

Moreover the product $C \times \mathbb{R}^p \subset \mathbb{R}^{8+p}$ is also minimal, and its singular set is $\{0\} \times \mathbb{R}^p$. This makes the conclusion of the following theorem, concerning area-minimizing integral currents, optimal, compare [97].

Theorem 11.46 *Let $T \subset \mathbb{R}^{n+1}$ be an n-dimensional area-minimizing integral current. Then*

(i) *If $n < 7$, T is regular.*

(ii) *If $n = 7$, T has only isolated singularities.*

(iii) *If $n > 7$, the Hausdorff dimension of the singular set of T is at most $n - 7$.*

11.2.5 The a priori gradient estimate of Bombieri, De Giorgi and Miranda

The *a priori estimate for the gradient* of solutions to the minimal surface equation was obtained by Bombieri, De Giorgi and Miranda in 1968 [13], and will be the key tool in the proof of the regularity of BV minimizers of the area problem, whose existence is granted by Theorem 2.34.

Theorem 11.47 (Bombieri-De Giorgi-Miranda [13]) *Let $u \in C^2(\Omega)$ be a solution of the minimal surface equation*

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=0.$$

Then

$$|Du(x_0)| \le C_1 \exp\left[C_2 \frac{\sup_\Omega u - u(x_0)}{d}\right], \qquad d := \operatorname{dist}(x_0, \partial\Omega), \tag{11.37}$$

where C_1 and C_2 are constants depending only on n.

The proof we shall present here is due to N. S. Trudinger [108].

Let $\Sigma := \mathcal{G}_u$ be the graph of u, and let $\nu = (\nu_1, \dots, \nu_{n+1}) \in \mathbb{R}^{n+1}$ be the normal unit vector to Σ pointing upward ($\nu^{n+1} > 0$). At $p = (x, u(x))$ this is given by

$$\nu_i = \frac{-D_i u}{\sqrt{1+|Du|^2}}, \text{ for } i = 1, \dots, n, \qquad \nu_{n+1} = \frac{1}{\sqrt{1+|Du|^2}},$$

where the derivatives are computed at x. Define also the following operators on Σ:

$$\begin{aligned} \delta_i &:= D_i - \nu_i \nu_j D_j, \qquad i = 1, \dots, n+1, \\ \delta &:= (\delta_1, \dots, \delta_{n+1}), \\ \Delta_\Sigma &:= \delta_i \delta_i, \end{aligned}$$

where the summation over repeated indices is understood, and D_i is the partial derivative in the i-th direction in $\mathbb{R}^n \times \mathbb{R}$. The operator Δ_Σ is the Laplace-Beltrami operator. The reader can verify that the scalar mean curvature of Σ (the length of the mean curvature vector) is

$$H(p) = \sum_{i=1}^n \delta_i \nu_i(p), \quad p \in \Sigma. \tag{11.38}$$

When Σ is minimal, i.e. $H = 0$, (11.38) yields

$$\Delta_\Sigma \omega \ge |\delta\omega|^2, \quad \text{on } \Sigma \tag{11.39}$$

where $\omega(x, u(x)) := \ln\sqrt{1+|Du(x)|^2} = -\ln \nu_{n+1}(x, u(x))$. This means that ω is *subharmonic* on Σ.

The following lemma can be considered a generalization of the mean value inequality for subharmonic functions on $\mathbb{R}^n$, compare Proposition 1.9.

Lemma 11.48 *Let ω be as before. Then, for*

$$x_0 \in \Omega, \quad 0 < R < \operatorname{dist}(x_0, \partial\Omega), \quad p := (x_0, u(x_0)),$$

we have

$$\omega(x_0) \le \frac{c}{R^n} \int_{\Sigma_R(p)} \omega \, d\mathcal{H}^n, \tag{11.40}$$

where $\Sigma_R(p) := \{q \in \Sigma : |p - q| < R\}$ and the constant c depends only on n.

Proof. We can assume that $p = 0$. Assume also $n > 2$; the case $n = 2$, being similar, will be omitted. For $0 < \varepsilon < R$ and $z \in \mathbb{R}^{n+1}$ we set

$$\varphi_\varepsilon(z) = \begin{cases} \dfrac{1}{2(n-2)}(\varepsilon^{2-n} - R^{2-n}) + \dfrac{1}{2n}(R^{-n} - \varepsilon^{-n})|z|^2 & \text{if } 0 \le |z| < \varepsilon \\ \dfrac{|z|^{2-n}}{n(n-2)} + \dfrac{1}{2n}|z|^2 R^{-n} - \dfrac{1}{2(n-2)} R^{-n} & \text{if } \varepsilon \le |z| \le R \\ 0 & \text{if } |z| > R. \end{cases}$$

Since $\varphi_\varepsilon \ge 0$ and both φ_ε and $D\varphi_\varepsilon$ vanish on $\partial\Omega$, we have by (11.39)

$$\int_\Sigma \omega \Delta_\Sigma \varphi_\varepsilon d\mathcal{H}^n = \int_\Sigma \varphi_\varepsilon \Delta_\Sigma \omega d\mathcal{H}^n \ge 0.$$

Since

$$\Delta_\Sigma |z|^\alpha = \alpha(\alpha - 2)|z|^{\alpha-2}\left(1 - \frac{z \cdot \nu}{|z|^2}\right) + \alpha n |z|^{\alpha-2},$$

we have

$$\Delta_\Sigma \varphi_\varepsilon(z) := \begin{cases} R^{-n} - \varepsilon^{-n} & \text{if } 0 \le |z| < \varepsilon \\ R^{-n} - |z|^{-2-n}(z \cdot \nu)^2 & \text{if } \varepsilon \le |z| \le R \\ 0 & \text{if } |z| > R. \end{cases}$$

Therefore

$$\begin{aligned} 0 &\le \int_{\Sigma_\varepsilon(0)} (R^{-n} - \varepsilon^{-n}) \omega d\mathcal{H}^n + \int_{\Sigma_R(0) \setminus \Sigma_\varepsilon(0)} (R^{-n} - |z|^{-2-n}(z \cdot \nu)^2) \omega d\mathcal{H}^n \\ &\le \frac{1}{R^n} \int_{\Sigma_R(0)} \omega d\mathcal{H}^n - \frac{1}{\varepsilon^n} \int_{\Sigma_\varepsilon(0)} \omega d\mathcal{H}^n. \end{aligned}$$

To complete the proof it is enough to observe that

$$\omega(0) = \lim_{\varepsilon \to 0} \frac{1}{\omega_n \varepsilon^n} \int_{\Sigma_\varepsilon(0)} \omega d\mathcal{H}^n.$$

□

Proof of Theorem 11.47. After a translation, we can assume that $0 \in \Omega$ and $u(0) = 0$. We may rewrite (11.40) as

$$\omega(0) \leq \frac{c}{R^n} \int_{|x|^2+|u|^2 \leq R^2} \omega\sqrt{1+|Du|^2}dx \leq \frac{c}{R^n} \int_{\substack{|x|\leq R\\ |u|\leq R}} \omega\sqrt{1+|Du|^2}dx. \tag{11.41}$$

Now take any $R < \frac{1}{3}\operatorname{dist}(0, \partial\Omega)$, and set

$$u_R := \begin{cases} 2R & \text{if } u \geq R \\ u + R & \text{if } |u| \leq R \\ 0 & \text{if } u \leq -R. \end{cases}$$

Take $\eta \in C^1_c(B_{2R}(0))$, with $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_R(0)$, $|D\eta| \leq \frac{2}{R}$. Inserting the test function $\varphi := \omega u_R \eta$ in the minimal surface equation

$$\int_\Omega \frac{D_i u D_i \varphi}{\sqrt{1+|Du|^2}} dx = 0, \qquad \varphi \in C^1_c(\Omega), \tag{11.42}$$

and observing that $|Du| \leq \sqrt{1+|Du|^2}$, we get

$$\int_{\substack{|u|\leq R\\ |x|\leq R}} \frac{\omega|Du|^2}{\sqrt{1+|Du|^2}} dx \leq 2R \int_{\substack{|x|\leq 2R\\ u>-R}} \big(\omega|D\eta| + \eta|D\omega|\big) dx. \tag{11.43}$$

Multiplying (11.39) by a test function $\phi^2 \in C^1_c(C_{2R}(0))$, integrating over Σ intersected with the cylinder $C_{2R}(0) := B_{2R}(0) \times \mathbb{R}$, and integrating by parts, we infer

$$\int_{\Sigma\cap C_{2R}(0)} \phi^2 |\delta\omega|^2 d\mathcal{H}^n \leq -2 \int_{\Sigma\cap C_{2R}(0)} \phi\delta_i\omega\delta_i\phi d\mathcal{H}^n,$$

and, since $ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon}$,

$$\int_{\Sigma\cap C_{2R}(0)} \phi^2 |\delta\omega|^2 d\mathcal{H}^n \leq c_2 \int_{\Sigma\cap C_{2R}(0)} |\delta\phi|^2 d\mathcal{H}^n.$$

This implies, by Hölder's inequality,

$$\int_{\Sigma\cap C_{2R}(0)} \phi|\delta\omega| d\mathcal{H}^n \leq c_2 \max_{\Sigma\cap C_{2R}(0)} |\delta\phi| \mathcal{H}^n(\Sigma \cap \operatorname{spt}\phi). \tag{11.44}$$

Now choose $\phi(x, y) := \eta(x)\tau(y)$, with $\tau \in C^1_c(-2R, R + \sup_{B_{2R}(0)} u)$,

$$0 \leq \tau \leq 1, \quad \tau \equiv 1 \text{ in } (-R, \sup_{B_{2R}(0)} u), \quad \left|\frac{d\tau}{dy}\right| \leq \frac{c}{R}.$$

Then, since $D_{n+1}\omega = 0$ implies $|D\omega|\nu_{n+1} \le |\delta\omega|$, using (11.44) we find

$$\begin{aligned}\int_{\substack{|x|\le 2R\\ u\ge -R}} \eta|D\omega|dx &= \int_{\substack{|x|\le 2R\\ u\ge -R}} \eta|D\omega|\nu_{n+1}d\mathcal{H}^n\\ &\le \int_{\substack{|x|\le 2R\\ u\ge -R}} \eta|\delta\omega|d\mathcal{H}^n\\ &\le \int_{\Sigma\cap C_{2R}(0)} \phi|\delta\omega|d\mathcal{H}^n\\ &\le \frac{c_3}{R}\mathcal{H}^n(\Sigma\cap \operatorname{spt}\phi)\\ &\le \frac{c_3}{R}\int_{\substack{|x|\le 2R\\ u\ge -2R}} \sqrt{1+|Du|^2}dx.\end{aligned}$$

Since $\omega \le \sqrt{1+|Du|^2}$, we also have

$$\int_{\substack{|x|\le 2R\\ u\ge -R}} \omega|D\eta|dx \le \frac{2}{R}\int_{\substack{|x|\le 2R\\ u\ge -R}} \sqrt{1+|Du|^2}dx.$$

Combining these last two estimates with (11.43), we find

$$\begin{aligned}\int_{\substack{|u|\le R\\ |x|\le R}} \omega\sqrt{1+|Du|^2}dx &\le \int_{\substack{|u|\le R\\ |x|\le R}} \frac{\omega}{\sqrt{1+|Du|^2}}dx\\ &\quad+\int_{\substack{|u|\le R\\ |x|\le R}} \frac{\omega|Du|^2}{\sqrt{1+|Du|^2}}dx\\ &\le c_4\left(R^n + \int_{\substack{u\ge -2R\\ |x|\le 2R}} \sqrt{1+|Du|^2}dx\right).\end{aligned} \tag{11.45}$$

To estimate the last integral, take $\varphi = \eta\max\{u+2R,0\}$ in (11.42), where

$$\eta \in C^1_c(B_{3R}(0)), \quad \eta\equiv 1 \text{ in } B_{2R}(0), \quad |D\eta| \le \frac{2}{R}.$$

We then obtain

$$\int_{\substack{|x|\le 2R\\ u\ge -2R}} \sqrt{1+|Du|^2}dx \le R^n\Big(C_1 + \frac{C_2}{R}\sup_{B_{3R}(0)} u\Big). \tag{11.46}$$

Putting together (11.41), (11.45) and (11.46) and exponentiating we finally obtain

$$|Du(0)| \le \sqrt{1+|Du|^2} \le C_1\exp\left(\frac{C_2}{R}\sup_{B_{3R}(0)} u\right),$$

from which (11.37) follows by translation. $\square$

11.2.6 Regularity of BV minimizers of the area functional

We shall now prove that a BV minimizer u of the area functional (as defined for BV-functions in Section 2.5) is smooth. Thanks to Theorem 11.45, it suffices to prove that u is locally Lipschitz continuous.

Recall that for $u \in BV(\Omega)$ we define the relaxed area of its graph as

$$\begin{aligned}\mathcal{A}(u) &= \int_\Omega \sqrt{1+|Du|^2} \\ &:= \sup\left\{\int_\Omega \left(u\sum_{i=1}^n D_i g_i + g_{n+1}\right)dx : g \in C_c^1(\Omega, \mathbb{R}^{n+1}), |g| \le 1\right\}.\end{aligned} \tag{11.47}$$

The absence of the term "dx" in the first integral of (11.47) comes from the fact that $\sqrt{1+|Du|^2}$ is a measure which is in general not absolutely continuous with respect to the Lebesgue measure.

Exercise 11.49 If $u \in W^{1,1}(\Omega)$, then (11.47) reduces to

$$\mathcal{A}(u) = \int_\Omega \sqrt{1+|Du|^2}dx,$$

where the last integral is intended is the classical sense, since Du is absolutely continuous with respect to the Lebesgue measure.
[Hint: Start with $u \in C^1(\Omega)$. The vector $(-Du(x), 1) \in \mathbb{R}^{n+1}$ has length $\sqrt{1+|Du|^2}$.]

Lemma 11.50 *Let Ω be connected, let $g \in L^1(\partial\Omega)$ and assume that there are two functions $u, v \in BV(\Omega)$, with v locally Lipschitz, both minimizing the area functional*

$$\mathcal{I}(u) := \int_\Omega \sqrt{1+|Du|^2}$$

over the set

$$\mathcal{S} := \{w \in BV(\Omega) : w|_{\partial\Omega} = g\}.$$

Then $u = v$.

Proof. With the same proof as in Proposition 11.27, it is easy to infer that the functional

$$\mathcal{A}(p) := \int_\Omega \sqrt{1+|p|^2}dx,$$

is strictly convex on $L^1(\Omega, \mathbb{R}^n)$. Now apply the Lebesgue decomposition to the *vector measure* Du with respect to the Lebesgue measure:

$$Du = Du^{(a)} + Du^{(s)},$$

where $Du^{(a)}$ is the absolutely continuous part, and $Du^{(s)}$ is the singular part. We can still denote by $Du^{(a)} \in L^1(\Omega, \mathbb{R}^n)$ the Radon-Nikodym representative of the absolutely continuous part. Since $Du^{(s)}$ is concentrated on a set of measure zero, we have

$$\mathcal{I}(u) = \int_\Omega \sqrt{1 + |Du^{(a)}|^2} dx + \int_\Omega |Du^{(s)}|,$$

where

$$\int_\Omega |Du^{(s)}| := |Du^{(s)}|(\Omega)$$

is the total variation of $Du^{(s)}$ on Ω. By the strict convexity of $\mathcal{A}(p)$ we have that, unless $Du^{(a)} = Dv$ a.e.,

$$\begin{aligned}
\mathcal{I}\Big(\frac{u+v}{2}\Big) &= \int_\Omega \sqrt{1 + \Big|\frac{Du^{(a)} + Dv}{2}\Big|^2} dx + \int_\Omega \Big|\frac{Du^{(s)}}{2}\Big| \\
&< \frac{1}{2}\Big(\int_\Omega \sqrt{1 + |Du^{(a)}|^2} dx + \int_\Omega \sqrt{1 + |Dv|^2} dx\Big) \\
&\quad + \frac{1}{2}\int_\Omega |Du^{(s)}| \\
&= \frac{1}{2}(\mathcal{I}(u) + \mathcal{I}(v)),
\end{aligned}$$

contradicting the minimality of u and v. Therefore $Du^{(s)} = Dv$ a.e., but then

$$\mathcal{I}(u) = \mathcal{I}(v) + \int_\Omega |Du^s|,$$

hence, again by minimality of u, $Du^{(s)} = 0$. Therefore $Du = Dv$, and since $u = v$ on $\partial\Omega$, we get $u = v$. This follows for instance by the Poincaré inequality applied to $u - v$, observing that $u, v \in W^{1,1}(\Omega)$ since their derivatives are absolutely continuous. □

Lemma 11.51 *Let $u \in BV(\Omega)$ be a local minimizer of the area functional $\mathcal{A}$, i.e.*

$$\mathcal{A}(u) \leq \mathcal{A}(v)$$

for every $v \in BV(\Omega)$ such that $\mathrm{spt}(u - v) \Subset \Omega$. *Then u is locally bounded.*

Proof. Assume that there exists a minimizer u such that $\sup_K |u| = \infty$, where $K \subset \Omega$ is compact, say $\mathrm{dist}(K, \partial\Omega) > \varepsilon$. Then we can find a sequence of points $p_j = (x_j, u(x_j)) \in \mathcal{G}_u$ such that $|p_j - p_k| \geq 2\varepsilon$ for every $j \neq k$. Moreover, by standard measure theory, such points can be chosen such that

$$\Theta(p_j) := \lim_{\rho \to 0^+} \mathcal{H}^n \frac{(\mathcal{G}_u \cap B_\rho(p_j))}{\omega_n \rho^n} = 1.$$

By the monotonicity formula, Proposition 11.94,[4] and since $B_\varepsilon(p_j) \cap B_\varepsilon(p_k) = \emptyset$ for $j \neq k$, it follows that

$$\mathcal{H}^n(\mathcal{G}_u) \geq \sum_{j=1}^{\infty} \mathcal{H}^n(\mathcal{G}_u \cap B_\varepsilon(p_j)) \geq \omega_n \sum_{j=1}^{\infty} \varepsilon^n \Theta(p_j) = \infty,$$

hence $\mathcal{J}(u) = \infty$, contradiction. □

Theorem 11.52 *Let $u \in BV_{\mathrm{loc}}(\Omega)$ be a local minimizer of the area functional*

$$\mathcal{A}(u, \Omega) := \int_\Omega \sqrt{1 + |Du|^2},$$

compare (11.47)*. Then u is locally Lipschitz continuous, hence smooth.*

The following proof is due to C. Gerhardt [36].

Proof. Set $u_\varepsilon := u * \rho_\varepsilon$, where ρ_ε is a family of smooth mollifiers, and let $x_0 \in \Omega$, $R > 0$ be such that $B_{3R}(x) \Subset \Omega$ (if ε is small enough u_ε is well-defined on $B_{3R}(x_0)$). According to Theorems 11.29 and 11.34, there exist a unique minimizer $v_\varepsilon \in C^\infty(\overline{B_R(x_0)})$ of the area functional

$$\mathcal{A}(w, B_R(x_0)) := \int_{B_R(x_0)} \sqrt{1 + |Dw|^2} dx,$$

in the class

$$B := \left\{ w \in \mathrm{Lip}(B_R(x_0)) \;:\; w\big|_{\partial B_R(x_0)} = u_\varepsilon\big|_{\partial B_R(x_0)} \right\}.$$

By the maximum principle, and assuming $\varepsilon \leq R$,

$$\sup_{B_R(x_0)} |v_\varepsilon| \leq \sup_{B_R(x_0)} |u_\varepsilon| \leq \sup_{B_{3R}(x_0)} |u| =: L,$$

where $L < \infty$ by the Lemma 11.51. By the a priori estimate (11.37), for every $\rho < R$ there is a constant M depending on ρ, R and L such that

$$\sup_{B_\rho(x_0)} |Dv_\varepsilon| \leq M.$$

By Ascoli-Arzelà's theorem a sequence v_{ε_k} converges uniformly on $B_\rho(x_0)$ to a Lipschitz function v, and by a diagonal procedure, we have locally

[4] In fact, in order to apply the monotonicity formula one should first prove that $(\partial(\mathcal{SG}_u)) \cap \Omega \times \mathbb{R}$ can be seen as a minimal variold, where

$$\mathcal{SG}_u := \{(x, y) : y < u(x)\}.$$

We skip the details.

uniform convergence of a subsequence (still denoted by v_{ε_k}) to a locally Lipschitz function v in $B_R(x_0)$. Set now

$$\overline{v}_{\varepsilon_k} := \begin{cases} v_{\varepsilon_k} & \text{in } B_R(x_0), \\ u_{\varepsilon_k} & \text{in } B_{2R}(x_0)\backslash B_R(x_0). \end{cases}$$

Then

$$\begin{aligned} \int_{B_{2R}(x_0)} |D\overline{v}_{\varepsilon_k}|dx &\le \int_{B_{2R}(x_0)} \sqrt{1+|D\overline{v}_{\varepsilon_k}|^2}dx \\ &\le \int_{B_{2R}(x_0)} \sqrt{1+|Du_{\varepsilon_k}|^2}dx \\ &\le \int_{B_{2R+\varepsilon}(x_0)} \sqrt{1+|Du|^2}, \end{aligned}$$

where the last inequality is justified by the convexity of the area and the fact that the convolution is an average. We then have that the $\overline{v}_{\varepsilon_k}$'s are equibounded in $BV(B_{2R}(x_0))$, and by Theorems 2.32 and 2.33, a subsequence, still denoted by v_{ε_k} converges in L^1 to

$$\overline{v} := \begin{cases} v & \text{in } B_R(x_0), \\ u & \text{in } B_{2R}(x_0)\backslash B_R(x_0), \end{cases}$$

and

$$\begin{aligned} \int_{B_{2R}(x_0)} \sqrt{1+|D\overline{v}|^2} &\le \liminf_{k\to\infty} \int_{B_{2R}(x_0)} \sqrt{1+|D\overline{v}_{\varepsilon_k}|^2} \\ &\le \liminf_{k\to\infty} \int_{B_{2R}(x_0)} \sqrt{1+|Du_{\varepsilon_k}|^2}dx \\ &\le \int_{B_{2R+\varepsilon}(x_0)} \sqrt{1+|Du|^2}, \end{aligned}$$

and letting $\varepsilon \to 0$ we conclude

$$\int_{B_{2R}(x_0)} \sqrt{1+|D\overline{v}|^2} \le \int_{B_{2R}(x_0)} \sqrt{1+|Du|^2}. \tag{11.48}$$

We now have

$$\overline{v}\big|_{\partial B_{2R}(x_0)} = u\big|_{\partial B_{2R}(x_0)} \tag{11.49}$$

and of course u minimizes

$$\mathcal{I}(w) := \int_{B_{2R}(x_0)} \sqrt{1+|Dw|^2},$$

in

$$\mathcal{S} := \{w \in BV(B_{2R}(x_0)) : w|_{\partial B_{2R}(x_0)} = u|_{\partial B_{2R}(x_0)}\}.$$

By (11.48) and (11.49) also $\overline{v}$ minimizes $\mathcal{I}$ over $\mathcal{S}$, hence, by Lemma 11.50, $u = \overline{v}$ in $B_{2R}(x_0)$, therefore $u = v$ is locally Lipschitz in $B_R(x_0)$, hence in all of Ω by the arbitrariness of the ball $B_R(x_0)$. From Theorem 11.45 we finally infer $u \in C^\infty(\Omega)$. □

11.3 Regularity in arbitrary codimension

To study the regularity of minimal graphs in arbitrary codimension we shall use a blow-up procedure, i.e. we rescale a minimal graph and analyse the limit. The blow-up of the graph of a smooth function converges to a plane. Allard's theorem says that, in the case of minimal graphs, the converse is also true: if the blow-up at a point p of a minimal graph converges to a plane, then the graph is smooth in a neighborhood of p. This reduces the regularity problem to the classification of the objects arising as blow-ups of minimal graphs. Since such objects are entire minimal graphs, the result we need is a Bernstein-type theorem: entire minimal graphs, under suitable assumptions, are planes. In fact we shall prove that any *area-decreasing* entire minimal graph (see Definition 11.25) is a plane (Theorem 11.59), and consequently that an *area decreasing* minimal graph is smooth (Theorem 11.69).

Throughout this section $V = \mathbf{v}(\Sigma, \theta) := \theta\mathcal{H}^n \llcorner \Sigma$ will denote a rectifiable varifold with support Σ and multiplicity θ, where Σ is rectifiable and $\theta \geq 0$ is locally $\mathcal{H}^n$-integrable. In fact we will make a very limited use of varifolds, and we refer the reader to Section 11.4 for a brief introduction to the subject.

11.3.1 Blow-ups, blow-downs and minimal cones

The following propositions are the basic tools in the blow-up argument.

Proposition 11.53 *Consider a sequence of equi-Lipschitz equibounded maps*

$$u_j : \Omega \to \mathbb{R}^m \quad \textit{with} \quad \sup_\Omega(|u_j| + |Du_j|) \leq M \textit{ for some } M.$$

Assume that each u_j satisfies the minimal surface system (11.19), *i.e. each $\mathcal{G}_{u_j}$ is a minimal Lipschitz submanifold. Then there exists a subsequence $u_{j'}$ uniformly converging to a Lipschitz function*

$$v : \Omega \to \mathbb{R}^m \quad \textit{with} \quad \sup_\Omega(|v| + |Dv|) \leq M,$$

which is a solution to the minimal surface system. Moreover

$$\mathbf{v}(\mathcal{G}_{u_{j'}}, 1) \rightharpoonup \mathbf{v}(\mathcal{G}_v, 1)$$

in the sense of varifolds.

Proof. This proof requires the notion of varifolds, discussed in Section 11.4, and can be skipped at a first reading.
Step 1. By Ascoli-Arzelà's theorem there exists a subsequence, still denoted by u_j, with $u_j \to v$ uniformly and

$$\sup_\Omega(|v| + |Dv|) \le M.$$

We shall prove that the convergence is also in the sense of varifolds.

By Proposition 11.93, the rectifiable varifolds $U_j := \mathbf{v}(\Sigma_j, 1)$, $\Sigma_j := \mathcal{G}_{u_j}$ are minimal, and this implies that

$$\|\delta U_i\| = 0 \quad \text{in } \Omega \times \mathbb{R}^m,$$

compare Definition 11.105 (here we identify a rectifiable varifold and the corresponding abstract varifold, see Remark 11.101). By Allard's compactness theorem, Theorem 11.108, up to extracting a further subsequence we have $U_j \to V$ in the sense of varifolds, where $V = \mathbf{v}(\Gamma, \theta)$ is a minimal integer multiplicity rectifiable varifold. We only need to prove that $V = \mathbf{v}(\mathcal{G}_v, 1)$, i.e. $\Gamma = \mathcal{G}_v$ and $\theta = 1$ $\mathcal{H}^n$-a.e on Γ.

Step 2. Let us show that $V = \mathbf{v}(\mathcal{G}_v, 1)$. Clearly $\operatorname{spt} V \subset \mathcal{G}_v$: indeed let $A \subset \mathbb{R}^{n+m}$ be open with $A \cap \mathcal{G}_v = \emptyset$. Since $\mathcal{G}_v$ is closed, for any $f \in C^0_c(A)$, we have $\operatorname{dist}(\operatorname{spt} f, \mathcal{G}_v) = \varepsilon > 0$. By uniform convergence we have $|u_j(x) - v(x)| < \varepsilon$ for j large enough, hence

$$U_j(f) = \int_{\Sigma_j} f(x) d\mathcal{H}^n(x) \to 0.$$

Hence $V(f) = 0$ and by the arbitrariness of A we have that $\Gamma = \operatorname{spt} V \subset \mathcal{G}_v$.

We now prove that for $\mathcal{H}^n$-a.e. $p \in \mathcal{G}_v$ we have $\theta(p) = 1$. Indeed the convergence $U_j \to V$ in the sense of varifolds implies that

$$\pi_\# U_j \to \pi_\# V \tag{11.50}$$

in the sense of varifolds, where $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ is the orthogonal projection onto $\mathbb{R}^n \times \{0\}$. Indeed by (11.80) we have

$$\begin{aligned}
\mathbf{v}(\Omega \times \{0\}, 1)(f) &= \pi_\# U_j(f) \\
&= \int_{G_n} f(\pi(x), d\pi_x S) J\pi(x, S) dU_j \\
&\to \int_{G_n} f(\pi(x), d\pi_x S) J\pi(x, S) dV \\
&= \pi_\# V(f) \\
&= \mathbf{v}(\Omega \times \{0\}, \tilde\theta)(f), \quad \tilde\theta(x) := \theta(x, v(x)),
\end{aligned}$$

for any $f \in C^0_c(G_n)$, where $G_n = G_n(\Omega \times \mathbb{R}^m)$ is the Grassmann bundle on $\Omega \times \mathbb{R}^m$, as defined in 11.99, and $\mathbf{v}(\Omega \times \{0\}, 1)$ is seen as an abstract varifold, compare Remark 11.101.

Therefore $\theta = 1$ $\mathcal{H}^n$-a.e on $\mathcal{G}_v$. $\square$

Proposition 11.54 (Blow-up) *Let $u : \Omega \to \mathbb{R}^m$ be a Lipschitz map solving the minimal surface system* (11.19). *Let $u_\lambda = u_{\lambda,x_0}$ be defined by*

$$u_\lambda(x) := \frac{1}{\lambda}(u(x_0 + \lambda x) - u(x_0)),$$

for a given $x_0 \in \Omega$. Then there exists a sequence $\lambda(i) \to 0$ such that $u_{\lambda(i)} \to v$ locally uniformly in $\mathbb{R}^n$, where v is a Lipschitz solution of the minimal surface system and the graph of $v : \mathbb{R}^n \to \mathbb{R}^m$ is a (minimal) cone with vertex at the origin[5].

Proof. The convergence of a sequence $u_{\lambda(i)}$ to a Lipschitz minimal graph is an immediate consequence of Proposition 11.53. From the convergence is the sense of varifolds, we have

$$\begin{aligned}
\frac{\mathcal{H}^n(B_\rho(0) \cap \mathcal{G}_v)}{\omega_n \rho^n} &= \lim_{i\to\infty} \frac{\mathcal{H}^n(B_\rho(0) \cap \mathcal{G}_{u_{\lambda(i)}})}{\omega_n \rho^n} \\
&= \lim_{i\to\infty} \frac{\mathcal{H}^n(B_{\lambda(i)\rho}(p_0) \cap \mathcal{G}_u)}{\omega_n (\lambda(i)\rho)^n} \\
&= \Theta^n(\mathcal{G}_u, p_0),
\end{aligned}$$

where $p_0 := (x_0, u(x_0))$ and the last limit exists thanks to the monotonicity formula (11.72). Then the ratio $\frac{\mathcal{H}^n(B_\rho(0)\cap\mathcal{G}_v)}{\omega_n\rho^n}$ does not depend on ρ. Letting $\rho \to \infty$ and $\sigma \to 0$ in the monotonicity formula (11.72) yields

$$\int_{\mathcal{G}_v} \frac{\left|(\nabla r)^\perp\right|^2}{r^n} d\mathcal{H}^n = 0,$$

where $r(p) := |p|$ for $p \in \mathbb{R}^{n+m}$, and $(\nabla r)^\perp$ is the projection of ∇r into the tangent bundle $T\mathcal{G}_v$. Therefore $\nabla r(p) \in T_p\mathcal{G}_v$ for a.e. every $p \in \mathcal{G}_v$, hence $\mathcal{G}_v$ is a cone, i.e. $v(\tau x) = \tau v(x)$ for every $\tau > 0$. $\square$

Proposition 11.55 (Blow-up of a cone) *Let $u : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map solving the minimal surface system* (11.19). *For $x_0 \in \mathbb{R}^n \setminus \{0\}$ set*

$$u_\lambda(x) := \frac{1}{\lambda}(u(x_0 + \lambda x) - u(x_0)).$$

[5] A cone $C \subset \mathbb{R}^{n+m}$ with vertex at the origin is a set such that for every $\lambda > 0$ we have $\lambda C = C$.

Then there exists a sequence $\lambda(i) \to 0$ such that $u_{\lambda(i)} \to v$, locally uniformly in $\mathbb{R}^n$ to a solution of the minimal surface system, and

$$\mathbf{v}(\mathcal{G}_{u_{\lambda(i)}}, 1) \rightharpoonup \mathbf{v}(\mathcal{G}_v, 1)$$

in the sense of varifolds and $\mathcal{G}_v$ is a (minimal) cone. Moreover $\mathcal{G}_v$ is a product of the form $C \times \mathbb{R}$ (up to a rotarion), where C is a minimal cone of dimension $n-1$ in $\mathbb{R}^{n+m-1}$ which is also a graph.

Let $\widetilde{x} = (x^1, \dots, x^{n-1})$. The last assertion means that there exists an orthonormal system of coordinates $\mathbb{R}^n$, a function $\widetilde{v} : \mathbb{R}^{n-1} \to \mathbb{R}^m$ and $\sigma \in \mathbb{R}^m$ such that

$$v(x^1, \dots, x^n) = \sigma x^n + \widetilde{v}(\widetilde{x}) \tag{11.51}$$

and $\mathcal{G}_{\widetilde{v}}$ is a minimal cone.

Proof. Considering Proposition 11.54, we have that $u_{\lambda(i)} \to v$ locally uniformly, where $v : \mathbb{R}^n \to \mathbb{R}^m$ solves the minimal surface system. Moreover the convergence of the graphs is in the sense of varifolds. That up to a rotation or $\mathbb{R}^n$ we can write v as in (11.51) is a simple exercise left for the reader. Hence $\mathcal{G}_v = C \times \mathbb{R}$ is the sense specified above. It remains to prove that $C = \mathcal{G}_{\tilde{v}}$, which is an $(n-1)$-dimensional cone in $\mathbb{R}^{n-1+m}$, is also minimal, i.e. also $\tilde{v}$ satisfies the minimal surface system. But this can be done easily using that v solves the minimal surface system and that Dv, hence $g_{ij} = g_{ij}(Dv)$, do not depend on x^n. □

Proposition 11.56 (Blow-down) *Let $u : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz solution to the minimal surface system 11.19. Let u_λ be defined by*

$$u_\lambda(x) = \frac{1}{\lambda}(u(\lambda x) - u(0)), \quad \lambda > 0.$$

Then there exists a sequence $\lambda(i) \to \infty$ such that $u_{\lambda(i)} \to v$ uniformly on compact sets, where v solves the minimal surface system. Moreover the convergence of the graphs $\mathcal{G}_{u_{\lambda(i)}}$ to $\mathcal{G}_v$ is in the sense of varifolds and $\mathcal{G}_v$ is a (minimal) cone.

Proof. The proof is identical to the proof of Proposition 11.54, with $\lambda \to \infty$ instead of $\lambda \to 0$, except that we shall use that the limit

$$\lim_{i\to\infty} \frac{\mathcal{H}^n(B_{\lambda(i)\rho}(p_0) \cap \mathcal{G}_u)}{\omega_n(\lambda(i)\rho)^n}, \quad p_0 := (0, u(0)),$$

exists thanks to the monotonicity formula (11.72) and is finite because

$$\sqrt{\det(I + (du)^* du)} \le C(M), \quad M := \sup_{\mathbb{R}^n} |du|.$$

□

11.3.2 Bernstein-type theorems

As we shall see, Bernstein-type theorems play a crucial role in the regularity theory of minimal graphs, particularly in codimension greater than 1.

A Bernstein-type theorem is a rigidity theorem which, under suitable hypothesis, implies that an entire minimal graph, i.e. the minimal graph of a function defined on all of $\mathbb{R}^n$, is an affine subspace. The following is the first such theorem, as formulated by Bernstein in a memoir published in 1927.

Theorem 11.57 (Bernstein [8]) *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function satisfying the minimal surface equation. Then u is affine, i.e. $u(x, y) = y_0 + \sigma_1 x + \sigma_2 y$, with $\sigma_1, \sigma_2 \in \mathbb{R}$.*

Several generalizations have been proved since then. In 1965 De Giorgi [26] proved a Bernstein-type theorem for 3 dimensional graphs in $\mathbb{R}^4$, while Simons [100] generalized Bernstein's theorem to $\mathbb{R}^{n+1}$ for $n \leq 7$. This result is sharp for what concerns the dimensions because Bombieri, De Giorgi and Giusti [12] showed that there exists a non-affine function $u : \mathbb{R}^8 \to \mathbb{R}$ whose graph is minimal. Some years before Moser [79] had proved that the minimal graph of a scalar function whose gradient is bounded is an affine subspace.

In higher codimension, Lawson and Osserman [68] have shown that the cone over Hopf's map (9.4) is minimal. Since it is the graph of a function with bounded gradient this shows that Moser's result does not extend to higher codimension.

The first Bernstein-type theorems in arbitrary codimension were proved by Hildebrandt, Jost and Widman [60] who studied the Gauss map of a minimal graph. With a similar approach, Jost and Y. L. Xin [64] improved the result of [60], obtaining the following theorem.

Theorem 11.58 *Let $u : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth function satisfying the minimal surface system* (11.19). *Let*

$$*\omega(x) = \frac{1}{\sqrt{\det(I + Du(x))^* Du(x))}}$$

and take $\beta_0 > 0$ such that

$$\beta_0 < \begin{cases} 2 & \text{if } m \geq 2 \\ \infty & \text{if } m = 1. \end{cases} \tag{11.52}$$

Then, if $*\omega \geq \frac{1}{\beta_0}$, u is affine.

Observe that

$$*\omega(x) \geq \frac{1}{\beta_0} \quad \text{implies} \quad |Du(x)| \leq \beta_0^2 - 1,$$

while in codimension 1, although Moser's theorem requires Du to be bounded, say $|Du| \leq M$, we do not have any restriction on M. The theorem we shall prove below, due to M-T. Wang [109], implies the result of Moser for codimension 1 and the result of Jost and Xin in arbitrary codimension. It is a natural extension of Moser's theorem because it only requires Du to be bounded and area-decreasing, Definition 11.25, and this latter assumption is always true in codimension 1, see Remark 11.26.

Theorem 11.59 (M-T. Want [109]) *Let* $u : \mathbb{R}^n \to \mathbb{R}^m$ *be a smooth* area-decreasing *map with bounded gradient and satisfying the minimal surface system* (11.19). *Then* u *is linear. The same is true if* $u \in C^\infty(\mathbb{R}^n \backslash \{0\}, \mathbb{R}^m)$, *as will be the case later.*

To prove the theorem we study the behaviour of the function

$$*\omega(x,y) = \frac{1}{\sqrt{\det(I + (Du(x))^* Du(x))}} = \frac{1}{\sqrt{\prod_{i=1}^n (1 + \lambda_i(x)^2)}},$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, (i.e. we now extend $*\omega$ to the product space $\mathbb{R}^{n+m}$) and the numbers $\lambda_i(x)$ are the singular values of $Du(x)$, i.e. the square roots of the eigenvalues of $(Du(x))^* Du(x)$.

Exercise 11.60 Verify that

$$*\omega > \frac{1}{\sqrt{2-\delta}} \Rightarrow |Du|^2 < 1 - \delta; \quad |Du| < \sqrt{(2-\delta)^{1/n} - 1} \Rightarrow *\omega > \frac{1}{\sqrt{2-\delta}}.$$

Let $\Sigma \subset \mathbb{R}^{n+m}$ be an n-dimensional submanifold of $\mathbb{R}^{n+m}$ and define ω to be the n-form on $\mathbb{R}^{n+m}$ given by

$$\omega(e_1, \dots, e_n) = 1$$

$$\omega(e_{i_1}, \dots, e_{i_n}) = 0 \quad \text{if } i_1 < \dots < i_n, \ i_n > n,$$

where $\{e_1, \dots, e_{n+m}\}$ is the standard basis of $\mathbb{R}^{n+m}$. Its covariant derivative and its Laplacian on Σ are, by definition,

$$\nabla_X^\Sigma \omega(Y_1, \dots, Y_n) := D_X(\omega(Y_1, \dots, Y_n)) - \sum_{i=1}^n \omega(Y_1, \dots, \nabla_X^\Sigma Y_i, \dots, Y_n).$$

$$\Delta_\Sigma \omega(p) := \sum_{k=1}^n \nabla_{\tau_k}^\Sigma \nabla_{\tau_k}^\Sigma \omega(p),$$

where $\{\tau_k\}$ is an orthonormal frame of $T\Sigma$ in a neighborhood of p.

Exercise 11.61 For $p \in \Sigma$, prove that

$$*\omega(p) = \omega(\tau_1, \dots, \tau_n)$$

for any orthonormal basis $\{\tau_1, \dots, \tau_n\}$ of $T_p\Sigma$.
[Hint: Choose $\{\tau_i\}$ according to the singular value decomposition of Du.]

Lemma 11.62 *Consider an orthonormal frame $\{\tau_1, \dots, \tau_n\}$ of $T\Sigma$ in a neighborhood of a point $p \in \Sigma$. Then*

$$(\Delta_\Sigma \omega)(\tau_1, \dots, \tau_n) = \Delta_\Sigma(\omega(\tau_1, \dots, \tau_n)) = \Delta_\Sigma(*\omega).$$

Proof. Set

$$\omega(\tau_1, \dots, \tau_n) = \omega_{1\cdots n}, \quad (\Delta_\Sigma \omega)(\tau_1, \dots, \tau_n) = (\Delta_\Sigma \omega)_{1\cdots n}.$$

Then

$$\begin{aligned}
(\Delta_\Sigma \omega)_{1\cdots n} &= D_{\tau_k}\left((\nabla^\Sigma_{\tau_k}\omega)(\tau_1, \dots, \tau_n)\right) - \sum_{i=1}^n \nabla^\Sigma_{\tau_k}\omega(\tau_1, \dots, \nabla^\Sigma_{\tau_k}\tau_i, \dots, \tau_n) \\
&= D_{\tau_k} D_{\tau_k}(\omega(\tau_1, \dots, \tau_n)) \\
&\quad - 2\sum_{i,k=1}^n D_{\tau_k}\left(\omega(\tau_1, \dots, \nabla^\Sigma_{\tau_k}\tau_i, \dots, \tau_n)\right) \\
&\quad + \sum_{\substack{i,j,k=1 \\ i \neq j}}^n \omega\left(\tau_1, \dots, \nabla^\Sigma_{\tau_k}\tau_i, \dots, \nabla^\Sigma_{\tau_k}\tau_j, \dots, \tau_n\right) \\
&\quad + \sum_{j,k=1}^n \omega(\tau_1, \dots, \nabla^\Sigma_{\tau_k}\nabla^\Sigma_{\tau_k}\tau_j, \dots, \tau_n) \\
&=: a + b + c + d.
\end{aligned}$$

Now

$$a = \Delta_\Sigma(\omega(\tau_1, \dots, \tau_n))$$

because $\{\tau_k\}$ is an orthonormal frame, $b = 0$ because

$$\tau_i \cdot (\nabla^\Sigma_{\tau_k}\tau_i) = \frac{1}{2}D_{\tau_k}(\tau_i \cdot \tau_i) = 0$$

and ω is alternating. Finally, also $c + d = 0$: using that for $i \neq j$

$$0 = D_{\tau_k}\langle \tau_i, \tau_j \rangle = \langle \nabla^\Sigma_{\tau_k}\tau_i, \tau_j \rangle + \langle \tau_i, \nabla^\Sigma_{\tau_k}\tau_j \rangle,$$

i.e.

$$\langle \nabla^\Sigma_{\tau_k}\tau_i, \tau_j \rangle = -\langle \tau_i, \nabla^\Sigma_{\tau_k}\tau_j \rangle,$$

and that

$$\nabla^{\Sigma}_{\tau_k}\tau_i = \sum_{\ell=1}^{n}\langle\nabla^{\Sigma}_{\tau_k}\tau_i,\tau_\ell\rangle\tau_\ell = \sum_{\substack{\ell=1\\ \ell\neq i}}^{n}\langle\nabla^{\Sigma}_{\tau_k}\tau_i,\tau_\ell\rangle\tau_\ell,$$

because

$$2\langle\nabla^{\Sigma}_{\tau_k}\tau_i,\tau_i\rangle = D_{\tau_k}\langle\tau_i,\tau_i\rangle = D_{\tau_k}1 = 0,$$

we compute

$$\begin{aligned} c+d &= -\sum_{\substack{i,j,k=1\\ i\neq j}}^{n}\langle\nabla^{\Sigma}_{\tau_k}\tau_i,\tau_j\rangle\langle\tau_i,\nabla^{\Sigma}_{\tau_k}\tau_j\rangle\omega_{1\cdots n} \\ &\quad -\sum_{j,k=1}^{n}\omega(\tau_1,\ldots,\langle\nabla^{\Sigma}_{\tau_k}\nabla^{\Sigma}_{\tau_k}\tau_j,\tau_j\rangle\tau_j,\ldots,\tau_n) \\ &= \left(\sum_{\substack{i,j,k=1\\ i\neq j}}^{n}\langle\nabla^{\Sigma}_{\tau_k}\tau_i,\tau_j\rangle^2 - \sum_{j,k=1}^{n}\langle\nabla^{\Sigma}_{\tau_k}\tau_j,\nabla^{\Sigma}_{\tau_k}\tau_j\rangle\right)\omega_{1\cdots n} \\ &= 0, \end{aligned}$$

where the last identity is justified by

$$\sum_{\substack{j=1\\ i\neq j}}^{n}\langle\nabla^{\Sigma}_{\tau_k}\tau_i,\tau_j\rangle^2 = \left|\nabla^{\Sigma}_{\tau_k}\tau_i\right|^2 = \langle\nabla^{\Sigma}_{\tau_k}\tau_i,\nabla^{\Sigma}_{\tau_k}\tau_i\rangle,$$

and we also used that

$$\langle\nabla^{\Sigma}_{\tau_k}\nabla^{\Sigma}_{\tau_k}\tau_j,\tau_j\rangle = D_{\tau_k}\langle\nabla^{\Sigma}_{\tau_k}\tau_j,\tau_j\rangle - \langle\nabla^{\Sigma}_{\tau_k}\tau_j,\nabla^{\Sigma}_{\tau_k}\tau_j\rangle = -\langle\nabla^{\Sigma}_{\tau_k}\tau_j,\nabla^{\Sigma}_{\tau_k}\tau_j\rangle.$$

□

We recall without proof an important identity from Riemannian geometry:

Lemma 11.63 (Codazzi's equation) *Let* $h^{\alpha}_{ij} := \big(\nabla_{\tau_i}\tau_j\big)\cdot\nu_\alpha$ *and* $h^\alpha := H\cdot\nu_\alpha$ *be the coefficients in local coordinates of the second fundamental form and of the mean curvature, respectively:*

$$h(X,Y) = h^{\alpha}_{ij}X^iY^j\nu_\alpha, \quad H = h^\alpha\nu_\alpha.$$

Then

$$h^{\alpha}_{ik;k} = h^{\alpha}_{;i} \tag{11.53}$$

where the semicolons denote the covariant derivatives.

Notation Assume given a local orthonormal frame $\{\tau_i\}_{i=1,\dots,n}$ of $T\Sigma$ and a local orthonormal frame

$$\{\nu_\alpha\}_{\alpha=1,\dots,m}$$

of the normal bundle $N\Sigma$ at a generic point $p \in \Sigma$. In what follows we shall write

$$\begin{aligned}\omega_{1\cdots\alpha^i\cdots\beta^j\cdots n} &:= \omega_{1\cdots(i-1)\alpha(i+1)\cdots(j-1)\beta(j+1)\cdots n}\\ &:= \omega(\tau_1,\dots,\tau_{i-1},\nu_\alpha,\tau_{i+1},\dots,\tau_{j-1},\nu_\beta,\tau_{j+1},\dots,\tau_n)\end{aligned}$$

to denote that ν_α occurs in the i-th place and ν_β in the j-th. With a little abuse of notation the Greek letters will always denotes components in the normal bundle. For instance in general

$$\omega(\nu_\alpha,\tau_2,\dots,\tau_n) =: \omega_{\alpha 2\dots n} \neq \omega_{i2\dots n} := \omega(\tau_i,\tau_2,\dots,\tau_n), \quad \text{even if } \alpha = i.$$

Proposition 11.64 *On a smooth (embedded) surface $\Sigma \subset \mathbb{R}^{n+m}$ which is minimal, i.e. which has $H \equiv 0$, ω satisfies*

$$-\Delta_\Sigma \omega_{1\cdots n} = \omega_{1\cdots n}|A|^2 - 2\sum_{\substack{i,j,k=1\\ i<j}}^{n}\sum_{\alpha,\beta=1}^{m} \omega_{1\cdots\alpha^i\cdots\beta^j\cdots n} h^\alpha_{ik} h^\beta_{jk}, \tag{11.54}$$

where

$$|A| := \sqrt{\sum_{i,k=1}^{n}\sum_{\alpha=1}^{m}(h^\alpha_{ik})^2}$$

is the norm of the second fundamental form. In fact the same holds if we just assume H to be parallel, i.e. $\nabla^\Sigma H \equiv 0$.

Proof. Since ω is constant in $\mathbb{R}^{n+m}$ we have $\nabla\omega = 0$. Thus

$$\left(\nabla^\Sigma_{\tau_k}\omega\right)_{1\cdots n} = \left((\nabla^\Sigma_{\tau_k} - \nabla_{\tau_k})\omega\right)_{1\cdots n} = \sum_{i=1}^{n}\omega(\tau_1,\dots,\nabla_{\tau_k}\tau_i - \nabla^\Sigma_{\tau_k}\tau_i,\dots,\tau_n).$$

Observing that

$$(\nabla_{\tau_k}\tau_i)^N = \sum_{\alpha=1}^{m} h^\alpha_{ik}\nu_\alpha$$

we get

$$\omega_{1\cdots n;k} := \nabla^\Sigma_{\tau_k}\omega(\tau_1,\dots,\tau_n) = \sum_{i=1}^{n}\sum_{\alpha=1}^{m}\omega_{1\cdots\alpha^i\cdots n}h^\alpha_{ik}. \tag{11.55}$$

Similarly

$$\omega_{1\cdots\alpha^i\cdots n;k} = -\sum_{\ell=1}^{n}\omega_{1\cdots\ell^i\cdots n}h^\alpha_{\ell k} + \sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{\beta=1}^{m}\omega_{1\cdots\beta^j\cdots\alpha^i\cdots n}h^\beta_{jk},$$

where we also defined
$$\nabla^{\Sigma}_{\tau_k}\nu^\alpha := (\nabla_{\tau_k}\nu^\alpha)^T$$
while originally $\nabla^{\Sigma}_{\tau_k}$ was defined only on tangent vector fields, and using that $\nu^\alpha \cdot \tau_i = 0$, we also computed

$$(\nabla_{\tau_k} - \nabla^{\Sigma}_{\tau_k})\nu^\alpha = -\sum_{\ell=1}^{n} h^\alpha_{k\ell}\tau_\ell.$$

Then

$$\omega_{1\cdots n;kk} = \sum_{\alpha=1}^{m}\sum_{i=1}^{n}\omega_{1\cdots\alpha^i\cdots n;k}h^\alpha_{ik} + \sum_{\alpha=1}^{m}\sum_{i=1}^{n}\omega_{1\cdots\alpha^i\cdots n}h^\alpha_{ik;k}. \tag{11.56}$$

By Codazzi's equation (11.53) and by (11.56) we get:

$$\begin{aligned}
\omega_{1\cdots n;kk} &= -\sum_{\alpha=1}^{m}\sum_{i,l,k=1}^{n}\omega_{1\cdots l^i\cdots n}h^\alpha_{lk}h^\alpha_{ik} + \sum_{\substack{i,j=1\\ i\neq j}}^{n}\omega_{1\cdots\beta^j\cdots\alpha^i\cdots n}h^\beta_{jk}h^\alpha_{ik} \\
&\quad + \sum_{\alpha=1}^{m}\sum_{i=1}^{n}\omega_{1\cdots\alpha^i\cdots n}h^\alpha_{;i} \\
&= -\omega_{1\cdots n}\sum_{i,k=1}^{n}\sum_{\alpha=1}^{m}(h^\alpha_{ik})^2 + 2\sum_{1\le i<j\le n}\omega_{1\cdots\beta^j\cdots\alpha^i\cdots n}h^\beta_{jk}h^\alpha_{ik} \\
&\quad + \sum_{\alpha=1}^{m}\sum_{i=1}^{n}\omega_{1\cdots\alpha^i\cdots n}h^\alpha_{;i}.
\end{aligned}$$

The last term vanishes because $H \equiv 0$ (in fact it sufficies $\nabla^\Sigma H \equiv 0$), and we conclude with Lemma 11.62. □

Let us come back to the case in which Σ is the graph of a smooth function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$. For any $x_0 \in \Omega$ by the singular value decomposition, Proposition 11.22, applied to the linear map $Du(x_0) : \mathbb{R}^n \to \mathbb{R}^m$, we can find orthonormal basis $\{v_i\}_{i=1,\ldots,n}$ and $\{w_\alpha\}_{\alpha=1,\ldots,m}$ of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively with respect to which Du is represented by a diagonal matrix, say $\lambda_{i\alpha}$, with $\lambda_{i\alpha} = 0$ if $i \neq \alpha$. To such basis we associate a basis of the tangent space and a basis of the normal space to $\Sigma := \mathcal{G}_u$ at $p := (x_0, u(x_0))$. Set $\lambda_i := \lambda_{ii}$ if $i \le \min\{m,n\}$, $\lambda_i = 0$ if $\min\{m,n\} < i \le n$, and

$$\left\{\tau_i := \frac{1}{\sqrt{1+\lambda_i^2}}\Big(v_i + \sum_{\beta=1}^{m}\lambda_{i\beta}w_\beta\Big)\right\}_{i=1,\ldots,n}$$
$$\left\{\nu_\alpha := \frac{1}{\sqrt{1+\lambda_\alpha^2}}(w_\alpha - \sum_{j=1}^{n}\lambda_{j\alpha}v_j)\right\}_{\alpha=1,\ldots,m}.$$

Since u is smooth, also the τ_i's and ν_α's can chosen to depend smoothly on x_0. Observe that if we define π to be the projection of $\mathbb{R}^{n+m}$ on the first n coordinates, we have

$$\pi(\nu_\alpha) = -\sum_{j=1}^{n} \lambda_{j\alpha}\pi(\tau_j). \tag{11.57}$$

Since $\omega(a_1, \dots, a_n) = \omega(\pi(a_1), \dots, \pi(a_n))$, we may use (11.57) to compute

$$\omega_{1\cdots\alpha^i\cdots\beta^j\cdots n} = \omega_{1\cdots n}(\lambda_{\beta j}\lambda_{\alpha i} - \lambda_{\beta i}\lambda_{\alpha j})$$

Now Proposition 11.64 can be written in terms of the singular values of Du:

$$\begin{aligned} -\Delta_\Sigma(*\omega) &= *\omega\Bigg(|A|^2 + 2\sum_{\substack{i,j,k=1\\ i<j}}^{n}\sum_{\alpha,\beta=1}^{m}(-\lambda_{\beta j}\lambda_{\alpha i} + \lambda_{\beta i}\lambda_{\alpha j})h^\alpha_{ik}h^\beta_{jk}\Bigg) \\ &= *\omega\Bigg(|A|^2 + 2\sum_{\substack{i,j,k=1\\ i<j}}^{n}(-\lambda_j\lambda_i h^i_{ik}h^j_{jk} + \lambda_j\lambda_i h^i_{jk}h^j_{ik})\Bigg). \end{aligned} \tag{11.58}$$

We are now ready to prove Theorem 11.59.

Proof of Theorem 11.59. Let $\varepsilon > 0$ be such that $\lambda_i(x)\lambda_j(x) \le 1 - \varepsilon$ for $i \neq j$ and for every $x \in \mathbb{R}^n$, where the $\lambda_i(x)$'s are the singulare values of $Du(x)$.

Step 1. Denote by Δ_Σ the Laplacian on $\Sigma := \mathcal{G}_u$. We have

$$\Delta_\Sigma(\ln *\omega) = \frac{*\omega\Delta_\Sigma(*\omega) - \left|\nabla^\Sigma(*\omega)\right|^2}{|*\omega|^2}. \tag{11.59}$$

The covariant derivative of $*\omega$ may be computed using the singular value decomposition of Du and equations (11.57) and (11.55):

$$(*\omega)_{;k} = -*\omega\left(\sum_{i=1}^{n}\sum_{\alpha=1}^{m}\lambda_{\alpha i}h^\alpha_{ik}\right) = -*\omega\left(\sum_{i=1}^{n}\lambda_i h^i_{ik}\right). \tag{11.60}$$

Inserting (11.58) and (11.60) into (11.59) yields

$$
\begin{aligned}
\Delta_\Sigma(-\ln *\omega) &= |A|^2 + 2\sum_{\substack{i,j,k=1\\ i<j}}^{n} \lambda_j\lambda_i h^i_{jk}h^j_{ik} - 2\sum_{\substack{i,j,k=1\\ i<j}}^{n} \lambda_j\lambda_i h^i_{ik}h^j_{jk} \\
&\quad + \sum_{k=1}^{n}\Big(\sum_{i=1}^{n}\lambda_i h^i_{ik}\Big)^2 \\
&= |A|^2 + 2\sum_{\substack{i,j,k=1\\ i<j}}^{n} \lambda_i\lambda_j h^i_{jk}h^j_{ik} + \sum_{i,k=1}^{n}\lambda_i^2 (h^i_{ik})^2 \\
&\geq |A|^2 + 2\sum_{\substack{i,j,k=1\\ i<j}}^{n} \lambda_i\lambda_j h^i_{jk}h^j_{ik} \\
&\geq |A|^2 + \sum_{i,j,k=1}^{n} \lambda_i\lambda_j (h^i_{jk})^2 \\
&\geq |A|^2 - (1-\varepsilon)|A|^2 \\
&= \varepsilon|A|^2,
\end{aligned}
\tag{11.61}
$$

where we also used the inequality $2h^i_{jk}h^j_{ik} \leq (h^i_{jk})^2 + (h^j_{ik})^2$.
Step 2. We perform a blow-down of the graph of u. By Proposition 11.56 there exists an equi-Lipschitz sequence

$$
u_{\lambda(j)}(x) = \frac{1}{\lambda(j)}u(\lambda(j)x), \qquad \lambda(j) \to \infty
$$

uniformly converging to an area-decreasing Lipschitz function $\overline{u}$. Moreover the convergence is also in the sense of varifolds and $\mathcal{G}_{\overline{u}}$ is a minimal cone with vertex in the origin. The differential of $\overline{u}$ is positively homogeneous, that is

$$
D\overline{u}(tx) = D\overline{u}(x), \quad t > 0,\ x \in \mathbb{R}^n\backslash\{0\}.
$$

We shall assume that $\overline{u}$ is smooth in $\mathbb{R}^n\backslash\{0\}$. The general case is studied in Step 3. The homogeneity of $D\overline{u}$ implies that on every annulus with center in the origin $*\omega$ attains an interior minimum; this, together with (11.61) and the maximum principle, implies $|A| = 0$ in every annulus and so in $\mathbb{R}^n\backslash\{0\}$. Therefore the cone is a linear subspace, i.e., $\overline{u}$ is linear. We now prove that $Du(x) = D\overline{u}(0)$ for every $x \in \mathbb{R}^n$, whence u is linear. Let δ and γ be as in Allard's Theorem 11.98, j_0 and ρ such that for every $j \geq j_0$

$$
\frac{\mathcal{H}^n(\mathcal{G}_{u_j} \cap B_1(0))}{\omega_n} \leq 1 + \delta,
$$

where this is possible because from the varifold convergence we get

$$\lim_{j\to\infty} \frac{\mathcal{H}^n(\mathcal{G}_{u_j}\cap B_1(0))}{\omega_n} = \frac{\mathcal{H}^n(\mathcal{G}_{\overline{u}}\cap B_1(0))}{\omega_n} = 1.$$

Then $u_j \in C^{1,\alpha}(\overline{B^n_\gamma(0)})$[6] with uniform bounds in $C^{1,\alpha}(\overline{B^n_\gamma(0)})$, thanks to (11.78). By Ascoli-Arzelà's theorem, a subsequence, still denoted by u_j, converges in $C^1(\overline{B^n_\gamma(0)})$ to the linear map $\overline{u}$. For every $x\in\mathbb{R}^n$ we have, for j large enough, $\frac{x}{\lambda(j)}\in B_\gamma(0)$ and

$$\left| Du_j\left(\frac{x}{\lambda(j)}\right) - D\overline{u}\left(\frac{x}{\lambda(j)}\right)\right| < \varepsilon.$$

As ε goes to 0, observing that $Du_j\left(\frac{x}{\lambda(j)}\right) = Du(x)$, we get that Du is constant hence u is linear.

Step 3. If the blow-up generates a cone which has at least a singularity in $x_0\neq 0$, we may perform a blow-up in x_0 and, by Proposition 11.55 we obtain a minimal cone C of dimension $n-1$ in $\mathbb{R}^{n+m-1}$. If such a cone is smooth except at the origin, we apply Step 2 to prove that C is actually a plane, which is a contradiction by Allard's theorem, since then

$$\lim_{r\to 0}\frac{\mathcal{H}^n(G_{\overline{u}}\cap B_r(p_0))}{\omega_n r^n} = 1, \quad p_0 := (x_0,\overline{u}(x_0)).$$

Otherwise we iterate the procedure until we get a cone of dimension 1, which cannot be singular and minimal at the same time, and obtain a contradition. □

Remark 11.65 Theorem 11.59 implies Theorem 11.58 because (11.52) yields, for $1\le i<j\le n$ and for some $\delta>0$

$$\begin{aligned} 4-\delta &\ge \frac{1}{(*\omega)^2} \\ &= \prod_{\ell=1}^n (1+\lambda_\ell^2) \\ &\ge 1+\lambda_i^2+\lambda_j^2+\lambda_i^2\lambda_j^2 \\ &\ge 1+2\lambda_i\lambda_j+\lambda_i^2\lambda_j^2 \\ &= (1+\lambda_i\lambda_j)^2, \end{aligned}$$

which yields

$$\lambda_i(x)\lambda_j(x) \le \sqrt{4-\delta}-1 = 1-\delta'$$

for some $\delta'>0$ and for every $x\in\mathbb{R}^n$.

[6] Here we let $B_r(x)$ denote the ball of radius r in $\mathbb{R}^{n+m}$ centered at $x\in\mathbb{R}^{n+m}$ and $B^n_r(x)$ denote the ball of radius r in $\mathbb{R}^n$ centered at $x\in\mathbb{R}^n$.

Remark 11.66 To prove that $-\ln *\omega$ is a subharmonic function, we only used the area-decreasing condition $|\lambda_i\lambda_j| \leq 1-\varepsilon$ and the minimal surface system (in fact a weaker version corresponding to $\nabla^\Sigma H \equiv 0$): thus we have shown that $-\ln *\omega$ is subharmonic on any area-decreasing smooth minimal graph (or any area-decreasing graph with parallel mean curvature).

Remarks on Bernstein's theorem: the Gauss map

Wang's proof of Theorem 11.59 is based on inequality (11.61) which says that $-\ln *\omega$ is a *subharmonic* function on Σ (with respect to the Riemannian metric of Σ). We shall sketch a geometric interpretation of that.

Definition 11.67 (Gauss map) *Given a smooth n-dimensional submanifold of $\Sigma \subset \mathbb{R}^{n+m}$, its Gauss map*

$$\gamma : \Sigma \to G(n,m)$$

is the map associating to each $x \in \Sigma$ the tangent space $T_x\Sigma$, seen as an element of the Grassmannian $G(n,m)$ of n-planes in $\mathbb{R}^{n+m}$.

The differentiable and Riemannian structures of $G(n,m)$ have been studied by Yung-Chow Wong [116] and Jost and Xin [64]. An important theorem concerning the Gauss map of a minimal surface is due to Ruh and Vilms:

Theorem 11.68 (Ruh-Vilms [91]) *The Gauss map γ of a submanifold $\Sigma \subset \mathbb{R}^{n+m}$ is harmonic if and only if the mean curvature H of Σ is parallel, i.e. if*

$$\nabla^\Sigma H \equiv 0.$$

In particular, if Σ is minimal, i.e. $H \equiv 0$, the Gauss map of Σ is harmonic on Σ. Jost and Xin observed that the condition $*\omega \geq \frac{1}{\beta_0}$, determines a region of the Grassmannian over which

$$f(L) := -\ln\sqrt{\det(I + L^*L)} \tag{11.62}$$

is convex[7] (in (11.62) we identify a plane with the linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ of which it is the graph; we shall only consider the region of the Grassmannian given by such planes). In [110], M-T. Wang proved that

[7] Convex here means that, given a geodesic $\sigma \to \Xi$, where $\Xi \subset G(n,m)$ is the subset of the Grassmannian containing the graphs of area-decreasing linear maps, we have that

$$\frac{d^2}{dt^2}\Big(-\ln\sqrt{\det(I+L^*L)} \circ \sigma\Big) \geq 0.$$

This notion of convexity is different from the one used in codimension 1 when we say that $\sqrt{1+|Du|^2}$ is a convex function: in the latter case, indeed, the $1\times n$-matrix space where Du lives is endowed with the flat metric, which is different from the Riemannian metric on the Grassmannian.

f is convex on a larger region of the Grassmannian: the graphs of area-decreasing linear maps. Thus $*\omega = f \circ \gamma$ is subharmonic because it is the composition of a harmonic map and a convex function.

11.3.3 Regularity of area-decreasing minimal graphs

As a consequence of Theorem 11.59, Allard's theorem (compare Theorem 11.98 below), and the dimension reduction argument of Federer, we obtain a regularity result for minimal graphs of Lipschitz maps which are area-decreasing (Theorem 11.69). As before, we remark that since the area-decreasing hypothesis is always met in codimension 1, this new result generalizes Theorem 11.45 to arbitrary codimension. We also remark that, due to the minimal cone of Lawson and Osserman (see (9.4)), an hypothesis on Du, other than its boundedness, is necessary.

Theorem 11.69 (M-T. Wang [111]) *Consider a Lipschitz map* $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ *satisfying the minimal surface system* (11.19) *and assume that there exists* $\varepsilon > 0$ *such that*

$$\lambda_i(x)\lambda_j(x) \le 1 - \varepsilon, \quad \textit{for } 1 \le i < j \le \min\{m, n\}, \quad x \in \Omega,$$

where the $\lambda_i(x)$*'s are the singular values of* $Du(x)$*. Then* $u \in C^\infty(\Omega, \mathbb{R}^m)$.

Proof.
Step 1. Let $x_0 \in \Omega$. Up to translation, we may assume $x_0 = 0$ and $u(0) = 0$. Performing a blow-up in 0, by Proposition 11.54 we get $u_i := u_{\lambda(i)} \to v$ uniformly and in the sense of varifolds for a sequence $\lambda(i) \to 0$ as $i \to \infty$, where $\mathcal{G}_v$ is a minimal cone. Moreover the uniform convergence preserves the area-decreasing.

If $v \in C^\infty(\mathbb{R}^n \backslash \{0\})$ then v is affine by Theorem 11.59. In particular

$$\mathcal{H}^n(\mathcal{G}_v \cap B_1(0)) = \omega_n.$$

From the varifold convergence we infer as $i \to \infty$

$$\frac{\mathcal{H}^n(\mathcal{G}_u \cap B_{\lambda(i)}(0))}{\omega_n \lambda(i)^n} = \frac{\mathcal{H}^n(\mathcal{G}_{u_{\lambda(i)}} \cap B_1(0))}{\omega_n} \to \frac{\mathcal{H}^n(\mathcal{G}_v \cap B_1(0))}{\omega_n} = 1.$$

In particular for every $\delta > 0$ we can find i large enough so that

$$\frac{\mathcal{H}^n(\mathcal{G}_u \cap B_{\lambda(i)}(0))}{\omega_n \lambda(i)^n} \le 1 + \delta.$$

By Allard's theorem (a suitably scaled version of Theorem 11.98) we get $u \in C^{1,\sigma}(\overline{B^n_{\gamma\lambda}(0)})$ for some $\sigma, \gamma > 0$.

Step 2. Now assume that v is not smooth in all of $\mathbb{R}^n\backslash\{0\}$. As in the proof of Theorem 11.59, assume that there exists a singularity in $x_0 \neq 0$. We may generate another cone in $(x_0, v(x_0))$ with another blow-up. Thanks to Proposition 11.55, such a cone factorizes and we obtain an $(n-1)$-dimensional area-decreasing cone which is minimal. If this cone is smooth except at most at the origin, then applying Step 1 we obtain that v is smooth is x_0, contradiction.

Then, by induction, we perform blow-ups and find cones with singularities until we find a minimal cone of dimension 1, union of two straight lines, which must be flat. Since x_0 was arbitrary, we obtain $u \in C^{1,\sigma}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ for some $\sigma > 0$.

Step 3. The existence of the higher order derivatives is consequence of Schauder estimates, Theorem 5.20. □

Remark 11.70 By a theorem of Allard's [4], the solutions u of the Dirichlet problem for the minimal surface system are smooth up to the boundary if Ω is strictly convex and the boundary data is smooth.

11.3.4 Regularity and Bernstein theorems for Lipschitz minimal graphs in dimension 2 and 3

The proof of Theorems 11.59 and 11.69 can be recast in dimensions 2 and 3 without the assumption that u be area decreasing.

Theorem 11.71 (Barbosa [7], Fisher-Colbrie [33]) *Assume that*

$$u \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$$

has bounded gradient and satisfies the minimal surface system (11.19). *Assume also that $n = 2$ or $n = 3$. Then u is linear. The same is true if $u \in C^\infty(\mathbb{R}^n\backslash\{0\}, \mathbb{R}^m)$.*

Proof. The only place in the proof of Theorem 11.59 where the area-decreasing assumption was used was (11.61). Let us assume $n = 2$. From the first lines of (11.61) we infer

$$\begin{aligned}\Delta_\Sigma(-\ln *\omega) &= |A|^2 + 2\sum_{\substack{i,j,k=1\\ i<j}}^{2} \lambda_i\lambda_j h^i_{jk}h^j_{ik} + \sum_{i,k=1}^{2}(\lambda_i h^i_{ik})^2 \\ &= |A|^2 + 2\lambda_1\lambda_2 h^1_{21}h^2_{11} + 2\lambda_1\lambda_2 h^1_{22}h^2_{12} \\ &\quad + (\lambda_1 h^1_{11})^2 + (\lambda_1 h^1_{12})^2 + (\lambda_2 h^2_{21})^2 + (\lambda_2 h^2_{22})^2 \\ &=: (I)\end{aligned} \tag{11.63}$$

Now using the assumption $H = 0$, i.e.

$$h^1_{11} + h^1_{22} = 0, \quad h^2_{11} + h^2_{22} = 0$$

and the symmetry $h^i_{jk} = h^i_{kj}$, we compute

$$(I) = |A|^2 + (\lambda_1 h^1_{12} + \lambda_2 h^2_{11})^2 + (\lambda_1 h^1_{11} - \lambda_2 h^2_{21})^2,$$

hence

$$\Delta_\Sigma(-\ln *\omega) \geq |A|^2, \tag{11.64}$$

which is the equivalent of (11.61), and the proof is complete for $n = 2$.

When $n = 3$, we first blow-down as in Step 2 of the proof of Theorem 11.59, obtaining a minimal cone which is the graph of a Lipschitz function v. Assuming that $v \in C^\infty(\mathbb{R}^3 \backslash \{0\}, \mathbb{R}^m)$, we can again prove (11.64), as for $n = 2$, this time using that the second fundamental form vanishes in one direction, hence reducing essentially to the one 2-dimensional case. If v has singularities away from 0, we apply the dimension reduction argument as in the proof of Theorem 11.59. The details are left for the reader. □

Remark 11.72 The above proof is due to M-T. Wang [109].

With the same proof of Theorem 11.69, replacing Theorem 11.59 with Theorem 11.71, we obtain:

Theorem 11.73 *Let $u \in \mathrm{Lip}(\Omega, \mathbb{R}^m)$ solve the minimal surface system* (11.19)*, with $\Omega \subset \mathbb{R}^2$ or $\Omega \subset \mathbb{R}^3$. Then u is smooth.*

Remark 11.74 Theorems 11.71 and 11.73 are sharp for what concerns the dimension ($n = 2$ or $n = 3$), since the cone of Lawson and Osserman (Section 9.1.3) is a 4-dimensional entire minimal graph which is Lipschitz continuous and singular at the origin.

11.4 Geometry of Varifolds

We recall a few facts about rectifiable and general varifolds. For more details see [97].

11.4.1 Rectifiable subsets of $\mathbb{R}^{n+m}$

Definition 11.75 *A Borel subset $M \subset \mathbb{R}^{n+m}$ is said to be* countably n-rectifiable *(or simply n-rectfiable) if*

$$M \subset N_0 \cup \Big(\bigcup_{j=1}^{\infty} N_j\Big), \tag{11.65}$$

where $\mathcal{H}^n(N_0) = 0$ and, for $j \geq 1$, N_j is a C^1-submanifold of $\mathbb{R}^{n+m}$ of dimension n.

The connection between rectifiable sets and Lipschitz functions is essentially a consequence of the theorems of Rademacher and Whitney, see [32], [49] and [97].

Theorem 11.76 (Rademacher) *Every Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is almost everywhere differentiable. In particular its gradient is a.e. well defined*

$$\nabla f := \Big(\frac{\partial f}{\partial x^1}, \cdots, \frac{\partial f}{\partial x^n}\Big)$$

and for a.e. $x_0 \in \mathbb{R}^n$ we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - \nabla f \cdot (x - x_0)}{|x - x_0|} = 0.$$

Remark 11.77 The gradient ∇f is the a.e. limit of measurable functions (the difference quotients) and is thus measurable. Moreover, if f is Lipschitz with Lipschitz constant K, it is clear that $|\nabla f| \le K$, so that $\nabla f \in L^\infty(\mathbb{R}^n)$.

Theorem 11.78 (Whitney) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function. Then for every $\varepsilon > 0$ there exists a function $h : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that*

$$\mathcal{L}^n\big(\{x \in \mathbb{R}^n : f(x) \neq h(x)\} \cup \{x \in \mathbb{R}^n : \nabla f(x) \neq \nabla h(x)\}\big) < \varepsilon,$$

where $\mathcal{L}^n$ is the Lebesgue measure on $\mathbb{R}^n$.

Observe that, thanks to Rademacher's theorem, the right term in the union is well defined up to $\mathcal{L}^n$-null sets.

Proposition 11.79 (Characterization of rectifiable sets) *A subset $M \subset \mathbb{R}^{n+m}$ is countably n-rectifiable if and only if there exists a sequence of Lipschitz maps $F_j : \mathbb{R}^n \to \mathbb{R}^{n+m}$ and a set M_0 with $\mathcal{H}^n(M_0) = 0$ such that*

$$M = M_0 \cup \Big(\bigcup_{j=1}^{n} F_j(A_j)\Big), \tag{11.66}$$

where $A_j \subset \mathbb{R}^n$ is measurable for every j.

Proof. ($\Rightarrow$) Every C^1-submanifold N_j in $\mathbb{R}^{n+m}$ is locally the image of C^1-maps which we denote by $h_{ij} : B_1(0) \subset \mathbb{R}^n \to \mathbb{R}^{n+m}$. Therefore

$$N_j \subset E_j \cup \Big(\bigcup_{i=1}^{\infty} h_{ij}(B_1(0))\Big), \quad \mathcal{H}^n(E_j) = 0. \tag{11.67}$$

If (11.65) holds, choose h_{ij} as in (11.67). Let

$$A_{ij} := h_{ij}^{-1}(M), \quad N_0 := \bigcup_{j=1}^{n} E_j \cap M.$$

Then

$$M = N_0 \cup \Big(\bigcup_{i,j=1}^{n} h_{ij}(A_{ij}) \Big).$$

Since A_{ij} is Borel (because inverse image of a Borel set) and since we may assume h_{ij} to be Lipschitz, we get (11.66).

($\Leftarrow$) Let F_j be as in (11.66). By Whitney's theorem we may find a family $h_{ij} : \mathbb{R}^n \to \mathbb{R}^{n+m}$ of C^1-maps and a measurable set E_j with $\mathcal{L}^n(E_j) = 0$ such that

$$F_j(A_j) \subset E_j \cup \Big(\bigcup_{i=1}^{\infty} h_{ij}(\mathbb{R}^n) \Big), \quad \forall j \geq 1. \tag{11.68}$$

Indeed we may choose h_{ij} as in the statement of Whitney's theorem with

$$D_{ij} := \{x \in \mathbb{R}^n : F_j(x) \neq h_{ij}(x)\} \cup \{x \in \mathbb{R}^n : \nabla F_j(x) \neq \nabla h_{ij}(x)\},$$

and

$$D_{1j} \supset D_{2j} \supset \cdots \supset D_{ij} \supset D_{(i+1)j} \supset \dots, \quad \mathcal{L}^n(D_{ij}) \leq \frac{1}{i}.$$

Setting $D_j := \cap_i D_{ij}$ we have $\mathcal{L}^n(D_j) = 0$ and by the area formula

$$\mathcal{H}^n(F_j(D_j)) = 0.$$

Then set $E_j := F_j(D_j)$ and we have (11.68).

Set $C_{ij} := \{x \in \mathbb{R}^n : \operatorname{rank} h_{ij}(x) < n\}$. Then $\mathcal{H}^n(h_{ij}(C_{ij})) = 0$ by Sard's lemma. Set

$$N_0 := \Big(\bigcup_{j=1}^{\infty} E_j \Big) \cup \Big(\bigcup_{i,j=1}^{\infty} C_{ij} \Big).$$

Then $\mathcal{H}^n(N_0) = 0$ and

$$M \subset N_0 \cup \Big(N_{ij} \Big),$$

with $N_{ij} := h_{ij}(\mathbb{R}^n \backslash C_{ij})$ countable union of C^1-submanifold thanks to the rank-max theorem (N_{ij} is a C^1-submanifold if h_{ij} is injective, otherwise we use the local injectivity of h_{ij} to write N_{ij} as countable union of C^1-submanifolds and a null set). $\square$

Corollary 11.80 *The image of a Lipschitz map*

$$F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+m}$$

is a countably n-rectifiable set. In particular the graph of a Lipschitz function $u : \Omega \to \mathbb{R}^m$ is n-rectifiable.

Since the only rectifiable sets Σ we shall use are the graphs of Lipschitz function, we shall assume that $\mathcal{H}^n \llcorner \Sigma$ is locally finite, that is, for every compact set $K \subset \mathbb{R}^{n+m}$, $\mathcal{H}^n(\Sigma \cap K) < \infty$.

Definition 11.81 (Tangent plane) *Given a countably n-rectifiable set Σ in $\mathbb{R}^{n+m}$ we define the* tangent plane to Σ at p, *if it exists, to be the only n-dimensional subspace P in $\mathbb{R}^{n+m}$ such that*

$$\lim_{\lambda\to 0} \int_{\eta_{p,\lambda}(\Sigma)} f(y) d\mathcal{H}^n(y) = \int_P f(y) d\mathcal{H}^n(y), \quad \forall f \in C_c^0(\mathbb{R}^{n+m}),$$

where $\eta_{p,\lambda(y)} := \lambda^{-1}(y-p)$ for every $y \in \mathbb{R}^{n+m}$. Such plane P will be denoted by $T_p\Sigma$.

Given an n-rectifiable $\Sigma \subset \mathbb{R}^{n+m}$, for instance a Lipschitz submanifold, its tangent plane at p is well defined for $\mathcal{H}^n$-a.e. $p \in \Sigma$. It is clear that if Σ is of class C^1, then the tangent plane just defined is the same as the tangent plane defined for smooth submanifolds as the set of tangent vectors. Given Σ n-rectifiable, thanks to Proposition 11.79, for $\mathcal{H}^n$-a.e. $p \in \Sigma$ there exists $N_{j(p)}$ C^1-submanifold such that $p \in N_{j(p)}$. It may be seen that $T_pM = T_pN_{j(p)}$ for $\mathcal{H}^n$-a.e. $p \in \Sigma$; in particular $T_pN_{j(p)}$ doesn't depend on the choice of the manifolds N_j covering Σ, nor on the choice of $j(p)$.

For these reasons, given $U \subset \mathbb{R}^{n+m}$ open and given $f \in \mathrm{Lip}(U)$, it is $\mathcal{H}^n$-a.e. well defined in $\Sigma \cap U$ the gradient $\nabla^\Sigma f := \nabla^{N_j} f$. The latter is $\mathcal{H}^n \llcorner N_j$-a.e. well defined thanks to Rademacher's theorem.

11.4.2 Rectifiable varifolds

Definition 11.82 *A rectifiable n-dimensional varifold in $\mathbb{R}^{n+m}$ with support Σ and multiplicity θ, $V = \mathbf{v}(\Sigma, \theta)$, where $\Sigma \subset \mathbb{R}^{n+m}$ is n-rectifiable and θ is positive and locally integrable on Σ, is the Radon measure (i.e. a Borel measure which is finite of compact sets)*

$$V := \theta \mathcal{H}^n \llcorner \Sigma,$$

i.e.

$$V(A) := \int_{A\cap\Sigma} \theta(y) d\mathcal{H}^n(y), \quad \forall A \subset \mathbb{R}^{n+m} \text{ Borel}.$$

Remark 11.83 Equivalently we may see a rectifiable varifold as an equivalence class of couples (Σ, θ) as above under the equivalence relation

$$(\Sigma_1, \theta_1) \sim (\Sigma_2, \theta_2) \text{ if } \mathcal{H}^n(\Sigma_1 \backslash \Sigma_2 \cup \Sigma_2 \backslash \Sigma_1) = 0 \text{ and } \theta_1 = \theta_2,\ \mathcal{H}^n - a.e. \tag{11.69}$$

The simple proof of the equivalence of the two definitions is left for the reader.

Remark 11.84 An n-rectifiable subset $\Sigma \subset \mathbb{R}^{n+m}$ such that $\mathcal{H}^n \llcorner \Sigma$ is locally finite can be seen as an n-rectifiable varifold with multiplicity $\theta = 1$ (in this case we identify, without further comments Σ and $V = \mathcal{H}^n \llcorner \Sigma$). For instance the graph of a Lipschitz function can be seen as a varifold.

Definition 11.85 *Given a rectifiable varifold $V = \mathbf{v}(\Sigma, \theta)$, the* tangent plane *of V at $p \in \Sigma$ is defined as*

$$T_pV := T_p\Sigma,$$

compare Definition 11.81. The definition is well posed $\mathcal{H}^n$-a.e. and does not depend on Σ, except for an $\mathcal{H}^n$-null set (recall that V is a measure which determines Σ only up to sets of null $\mathcal{H}^n$-measure).

The mass *of a varifold V is its mass in the sense of measures and is denoted by $\mathbf{M}(V)$:*

$$\mathbf{M}(V) := V(\mathbb{R}^{n+m}) = \int_\Sigma \theta d\mathcal{H}^n.$$

The convergence we define on the space of rectifiable varifolds, different from the convergence in the sense of varifolds which we shall define for abstract varifolds, is the weak-$*$ convergence induced by the duality between Radon measures and compactly supported continuous functions:

Definition 11.86 (Weak convergence) *We say that a sequence of varifolds V_j converges weakly to V (and we write $V_j \rightharpoonup V$) if*

$$\lim_{j\to\infty} \int_{\mathbb{R}^{n+m}} f dV_j = \int_{\mathbb{R}^{n+m}} f dV,$$

for every $f \in C^0_c(\mathbb{R}^{n+m})$.

Proposition 11.87 *The mass is continuous with respect to the weak convergence in a compact set $K \subset \mathbb{R}^{n+m}$, i.e. if $V_j \rightharpoonup V$, $\operatorname{spt} V_j \subset K$ for every $j \geq 0$, then $\mathbf{M}(V_j) \to \mathbf{M}(V)$.*

Proof. Set $R > 0$ such that $K \subset B_R(0)$ and $\varphi \in C^0_c(\mathbb{R}^{n+m})$ such that $\varphi = 1$ on $B_R(0)$. Then

$$\mathbf{M}(V_j) = \int_{\mathbb{R}^{n+m}} \varphi dV_j \to \int_{\mathbb{R}^{n+m}} 1 dV = \mathbf{M}(V),$$

since also $\operatorname{spt} V \subset K$. $\square$

11.4.3 First variation of a rectifiable varifold

The concept of first variation, which we defined for Lipschitz submanifolds of $\mathbb{R}^{n+m}$, compare Definition 11.15, may be easily extended to a rectifiable varifold $V = \mathbf{v}(\Sigma, \theta)$.

Definition 11.88 (Image varifold) *Given $f : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ Lipschitz and proper*[8] *and an n-rectifiable varifold $V = \mathbf{v}(\Sigma, \theta)$, the image varifold of V under f is defined by*

$$f_{\#}V := \mathbf{v}(f(\Sigma), \widetilde{\theta}),$$

where

$$\widetilde{\theta}(y) = \sum_{x \in \Sigma \cap f^{-1}(y)} \theta(x).$$

Thanks to Proposition 11.79, $f(\Sigma)$ is rectifiable and, since f is proper, we have that $\widetilde{\theta}\mathcal{H}^n \llcorner f(\Sigma)$ is locally finite: indeed, given a compact set K, by the area formula we get

$$f_{\#}V(K) = \int_{K \cap f(\Sigma)} \widetilde{\theta} d\mathcal{H}^n = \int_{f^{-1}(K) \cap \Sigma} (Jf)\theta d\mathcal{H}^n,$$

where

$$Jf := \sqrt{\det(df^* df)}.$$

The last integral is finite because Jf is bounded, $f^{-1}(K)$ is compact and $\theta\mathcal{H}^n \llcorner \Sigma$ is locally finite.

Definition 11.89 (First variation) *Let $\varphi : \mathbb{R}^{n+m} \times (-1, 1)$ be of class C^2 and such that*

1. *there exists a compact set $K \subset \mathbb{R}^{n+m}$ such that $\varphi_t(x) := \varphi(x, t) = x$ for every $x \notin K$;*
2. *$\varphi_0(x) = x$ for every $x \in \mathbb{R}^{n+m}$.*

Then the first variation of a varifold V with respect to φ is

$$\left.\frac{d}{dt}\right|_{t=0} \mathbf{M}(V_t), \quad V_t := (\varphi_t)_{\#}V.$$

With the same proof of Proposition 11.17 we get

Proposition 11.90 *Consider a family of diffeomorphisms φ_t as in Definition 11.89 and an n-rectifiable varifold $V = \mathbf{v}(\Sigma, \theta)$. Let*

$$X(x) := \left.\frac{d}{dt}\right|_{t=0} \varphi_t(x)$$

[8] For every compact $K \subset \mathbb{R}^{n+m}$ $f^{-1}(K)$ is compact.

be the first variation field of φ*. Then*

$$\frac{d}{dt}\bigg|_{t=0} \mathbf{M}((\varphi_t)_\# V) = \int_\Sigma \operatorname{div}^\Sigma X \theta d\mathcal{H}^n. \tag{11.70}$$

Definition 11.91 (Minimal varifold) *We say that an* n*-rectifiable varifold* $V = \mathbf{v}(\Sigma, \theta)$ *is minimal in an open set* $U \subset \mathbb{R}^{n+m}$ *if its first variation is zero for every choice of* φ *in 11.89 with* $K \Subset U$*; equivalently,* V *is minimal in* U *if for every vector field* $X \in C^1_c(U, \mathbb{R}^{n+m})$ *we have*

$$\int_\Sigma \operatorname{div}^\Sigma X \theta d\mathcal{H}^n = 0. \tag{11.71}$$

Definition 11.92 (Minimal graph) *We say that the graph* $\mathcal{G}_u$ *of a Lipschitz function* $u : \Omega \to \mathbb{R}^m$ *is* minimal in the sense of varifolds *if the associated varifold* $V := \mathbf{v}(\mathcal{G}_u, 1)$ *is minimal in* $\Omega \times \mathbb{R}^m$*.*

This means that the mass of $V(\mathcal{G}_u, \theta)$ is stationary with respect to variations compactly contained in $\Omega \times \mathbb{R}^m$, thus fixing the boundary of the graph.

Proposition 11.93 *Let* $u : \Omega \to \mathbb{R}^m$*,* $\Omega \subset \mathbb{R}^n$*, be a Lipschitz map. Then* u *satisfies the minimal surface system* (11.19) *if and only if the associated varifold* $\mathbf{v}(\mathcal{G}_u, 1)$ *is minimal in* $\Omega \times \mathbb{R}^m$*.*

Proof. This follows at once by Proposition 11.20 and (11.71), since choosing $F(x) = (x, u(x))$ in Proposition 11.20 the equation $\Delta_\Sigma F = 0$ (with $\Sigma = \mathcal{G}_u$) reduces to (11.19). $\square$

11.4.4 The monotonicity formula

Proposition 11.94 *Consider an* n*-dimensional rectifiable varifold* $V = \mathbf{v}(\Sigma, \theta)$ *in* $\mathbb{R}^{n+m}$ *which is minimal in* $U \subset \mathbb{R}^{n+m}$*. Then we have*

$$\frac{V(B_\rho(x_0))}{\rho^n} - \frac{V(B_\sigma(x_0))}{\sigma^n} = \int_{B_\rho(x_0)\setminus B_\sigma(x_0)} \frac{\left|(\nabla r)^\perp\right|^2}{r^n} dV, \tag{11.72}$$

for every $x_0 \in \mathbb{R}^{n+m}$*,* $0 < \sigma < \rho < \operatorname{dist}(x_0, \partial U)$*, where* $r(x) := |x - x_0|$ *and* $(\nabla r)^\perp$ *is the component of* ∇r *orthogonal to* Σ*. Therefore the function*

$$\rho \mapsto \frac{V(B_\rho(x_0))}{\rho^n}, \quad 0 < \rho < d(x_0, \partial U)$$

is monotone increasing. In particular the density at x_0 *is well defined as*

$$\Theta^n(V, x_0) := \lim_{\rho \to 0^+} \frac{V(B_\rho(x_0))}{\omega_n \rho^n}.$$

Proof. Fix $\rho > 0$ and define a function $\gamma \in C^1(\mathbb{R})$ such that

1. $\dot\gamma(t) \leq 0$ for every $t \geq 0$;
2. $\gamma(t) = 1$ for every $t \leq \frac{\rho}{2}$;
3. $\gamma(t) = 0$ for every $t \geq \rho$.

Consider the vector field

$$X(x) := \gamma(r)(x - x_0), \quad r := |x - x_0|.$$

Let $x \in \Sigma$ be such that $T_x\Sigma$ exists; then

$$\begin{aligned} \operatorname{div}^\Sigma X(x) &= \sum_{j=1}^{n+m} e_j \cdot (\nabla^\Sigma X^j) \\ &= \gamma(r) \sum_{j=1}^{n+m} e^{jj} + r\dot\gamma(r) \sum_{j,l=1}^{n+m} \frac{x^j - x_0^j}{r} \frac{x^l - x_0^l}{r} e^{jl} \end{aligned}$$

where e^{jl} is the $(n+m) \times (n+m)$-matrix projecting $\mathbb{R}^{n+m}$ onto $T_x\Sigma$. The trace of the projection is $\sum e^{jj} = n$; moreover the quantity

$$\sum_{j,l=1}^{n+m} \frac{x^j - x_0^j}{r} \frac{x^l - x_0^l}{r} e^{jl} = |(\nabla r)^T|^2 = 1 - |(\nabla r)^\perp|,$$

is equal to the scalar product between the projection of Dr onto $T_x\Sigma$ and $Dr = \frac{x - x_0}{r}$ itself. This implies

$$\operatorname{div}^\Sigma X(x) = n\gamma(r) + r\dot\gamma(r)(1 - |(\nabla r)^\perp|^2).$$

Apply (11.71) to X and get

$$n \int_\Sigma \gamma(r) dV + \int_\Sigma r\dot\gamma(r) dV = \int_\Sigma r\dot\gamma(r)|(\nabla r)^\perp|^2 dV. \tag{11.73}$$

Now consider a family of functions γ arising from the rescaling of a function $\Phi \in C^1(\mathbb{R})$ satisfying

1. $\dot\Phi(t) \leq 0$ for every $t \geq 0$;
2. $\Phi(t) = 1$ for every $t \leq \frac{1}{2}$;
3. $\Phi(t) = 0$ for every $t \geq 1$.

More precisely, let $\gamma(r) := \Phi\big(\frac{r}{\rho}\big)$ for a fixed $\rho > 0$. It is clear that

$$r\dot\gamma(r) = \frac{r}{\rho}\dot\Phi\Big(\frac{r}{\rho}\Big) = -\rho\frac{d}{d\rho}\Big(\Phi\Big(\frac{r}{\rho}\Big)\Big).$$

It follows that, defining

$$I(\rho) := \int_\Sigma \Phi\Big(\frac{r}{\rho}\Big) dV, \quad J(\rho) = \int_\Sigma \Phi\Big(\frac{r}{\rho}\Big) \left|(\nabla r)^\perp\right|^2 dV,$$

we obtain

$$nI(\rho) - \rho\dot{I}(\rho) = -\rho\dot{J}(\rho),$$

which may be rewritten multipling by ρ^{-n-1} as

$$\frac{d}{d\rho}\left(\frac{I(\rho)}{\rho^n}\right) = \frac{\dot{J}(\rho)}{\rho^n}. \tag{11.74}$$

If we let Φ converge from below to the characteristic function of $(-\infty, 1]$, we obtain

$$I(\rho) \to V(B_\rho(x_0)), \quad J(\rho) \to \int_{B_\rho(x_0)} \left|(\nabla r)^\perp\right|^2 dV,$$

thus, in the sense of distributions, (11.74) becomes

$$\frac{d}{d\rho}\left(\frac{V(B_\rho(x_0))}{\rho^n}\right) = \frac{d}{d\rho}\int_{B_\rho(x_0)} \frac{\left|(\nabla r)^\perp\right|^2}{r^n} dV.$$

The claim of the theorem follows integrating with respect to ρ. □

11.4.5 The regularity theorem of Allard

The theorem of Allard is a basic tool in the regularity theory for minimal surfaces. Generalized by Allard, the theorem was first proved by De Giorgi [25] in codimension 1. De Giorgi had the fundamental idea of approximating a minimal surface with harmonic functions, introducing the *excess* to estimate the error.

Definition 11.95 (Excess) *Given a varifold $V = \mathbf{v}(\Sigma, \theta)$, $x_0 \in \Sigma$, an n-plane T and $R > 0$, define the excess*

$$E(x_0, R, T) := \frac{1}{R^n}\int_{B_R(x_0)} \|p_{T_xM} - p_T\|^2 dV(x), \tag{11.75}$$

where p_T and p_{T_xM} are the projections onto T and T_xM respectively, and for an $(n+m)\times(n+m)$ matrix $A = (A_{ij})$ we set

$$\|A\| := \sqrt{\sum_{i,j=1}^{n+m} A_{ij}^2}.$$

Theorem 11.96 (Allard) *Let $V = \mathbf{v}(\Sigma, \theta)$ be a* rectifiable n-varifold *in $\mathbb{R}^{n+m}$ which is minimal in the open set $U \subset \mathbb{R}^{n+m}$, $x_0 \in \operatorname{spt} V$, $B_R(x_0) \Subset U$. Then for every $\sigma, \alpha \in (0,1)$ there are constants ε, γ depending on n, m, σ, α such that if*

1. $1 \leq \theta \leq 1 + \varepsilon$, *$V$-a.e. in U,*
2. $\frac{V(B_R(x_0))}{\omega_n R^n} \leq 2 - \alpha$,
3. $E(x_0, R, T) \leq \varepsilon$ *for some n-plane $T \subset \mathbb{R}^{n+m}$.*

Then

$$V \llcorner B_{\gamma R}(x_0) = \mathbf{v}(\mathcal{G}_u \cap B_{\gamma R}(x_0), 1),$$

where $u : T \cap B_{\gamma R}(x_0) \to T^\perp$ is a smooth function. This means, that up to a rotation of $\mathbb{R}^{n+m}$ sending T onto $\mathbb{R}^n \times \{0\}$ and $T^\perp$ onto $\{0\} \times \mathbb{R}^m$ and a translation sending x_0 to 0, we can take $u \in C^\infty(B^n_{\gamma R}(0), \mathbb{R}^m)$, where $B^n_{\gamma R}(0) \subset \mathbb{R}^n$.

Second version of Allard's theorem

The following version of Allard's theorem can be deduced from the previous one and it is the one we shall actually use.

Theorem 11.97 *Let $V = \mathbf{v}(\Sigma, \theta)$ be a* rectifiable n-varifold *in $\mathbb{R}^{n+m}$ which is minimal in the open set $U \subset \mathbb{R}^{n+m}$, $0 \in \operatorname{spt} V$, $B_1(0) \Subset U$. Then for every $\sigma \in (0,1)$ there exist positive numbers δ, γ and c depending on m, n and σ such that if*

$$\begin{cases} \theta \geq 1, \quad V - a.e., \\ \dfrac{V(B_1(0))}{\omega_n} \leq 1 + \delta, \end{cases} \tag{11.76}$$

then, up to a rotation of $\mathbb{R}^{n+m}$ there exists $u \in C^{1,\sigma}(\overline{B^n_\gamma(0)})$ with $u(0) = 0$ such that

$$V \llcorner B_\gamma(0) = \mathbf{v}\big(\mathcal{G}_u \cap B_\gamma(0), \theta\big).$$

Moreover

$$\|u\|_{C^{1,\sigma}(\overline{B_\gamma(0)})} \leq c\delta^{\frac{1}{4n}}. \tag{11.77}$$

In the case of Lipschitz minimal graphs Proposition 11.93 and Theorem 11.97 give:

Theorem 11.98 *Let $u \in \operatorname{Lip}(\Omega, \mathbb{R}^m)$, $\Omega \subset \mathbb{R}^n$, be a solution to the minimal surface system (11.19). Assume (up to a translation and a dilation) that $B_1(0) \Subset \Omega$ and $u(0) = 0$. There for $\sigma \in (0,1)$ there exist positive numbers δ, γ and c depending on m, n and σ such that if*

$$\mathcal{H}^n(\mathcal{G}_u \cap B_1(0)) \leq (1 + \delta)\omega_n,$$

then $u \in C^{1,\sigma}(\overline{B^n_\gamma(0)})$ *and*

$$\|u\|_{C^{1,\sigma}(\overline{B^n_\gamma(0)})} \le c\delta^{\frac{1}{4n}}. \tag{11.78}$$

11.4.6 Abstract varifolds

Rectifiable varifolds are Radon measures in $\mathbb{R}^{n+m}$. A compactness theorem for measures assures that a sequence of varifolds with equibounded masses admits a subsequence converging in the sense of measures. The limit, however, is a Radon measure whose support in general might not be rectifiable. This motivates the introduction of a stronger convergence and of a larger class of objects.

Definition 11.99 *Given an open set $U \subset \mathbb{R}^{n+m}$, the Grassmannian fiber bundle of n-planes on U is*

$$G_n(U) := U \times G(n,m), \qquad \pi : G_n(U) \to U$$

where

$$G(n,m) \cong \frac{O(n+m)}{O(n) \times O(m)}$$

is the Grassmannian of n-planes in $\mathbb{R}^{n+m}$ and $\pi(x,S) = x$ for every $x \in U$ and every n-plane S. We endow $G_n(U)$ with the product topology.

Definition 11.100 *An n-varifold in $U \subset \mathbb{R}^{n+m}$ is a Radon measure V on the Grassmannian fiber bundle $G_n(U)$. Associated to V there is a measure μ_V on U defined by*

$$\mu_V(A) := V(\pi^{-1}(A)), \quad \forall A \subset U \text{ measurable.}$$

Finally we define the mass of V,

$$\mathbf{M}(V) := \mu_V(U).$$

Remark 11.101 The class of abstract varifolds contains the class of rectifiable varifolds: to a rectifiable n-varifold $\mathbf{v}(\Sigma,\theta)$ we can associate the abstract varifold V defined by

$$V(B) = \mathbf{v}(\Sigma,\theta)(\pi(B \cap T\Sigma)), \quad B \subset G_n(U) \text{ measurable,}$$

being

$$T\Sigma := \{(x, T_x\Sigma) : x \in \Sigma_*\} \subset G_n(U)$$

the tangent bundle of Σ (Σ_* is the set of points of Σ where the approximate tangent plane is defined). Clearly in this case $\mu_V = \mathbf{v}(\Sigma,\theta)$ because

$$\mu_V(A) = V(\pi^{-1}(A)) = \mathbf{v}(\Sigma,\theta)(\pi(\pi^{-1}(A) \cap T\Sigma)) = \mathbf{v}(\Sigma,\theta)(A).$$

We give the space of n-dimensional varifolds in U the weak-$*$ topology of Radon measures, so that $V_n \rightharpoonup V$ if and only if for every $f \in C^0_c(G_n(U))$ we have

$$\int_{G_n(U)} f(x,S)dV_n(x,S) \to \int_{G_n(U)} f(x,S)dV(x,S).$$

Remark 11.102 The convergence just defined, which we call *convergence in the sense of varifolds*, is stronger than the convergence defined for rectifiable varifolds. For instance, if $V_k = \mathbf{v}(\Sigma_k, \theta_k) \rightharpoonup V$ is a sequence of rectifiable varifolds converging in the sense of varifolds, then in a certain sense, both the supports (with multiplicity) and the tangent planes of the varifolds V_k converge. As we shall see, this does not yet guarantee (without further assumptions) that V is also rectifiable.

11.4.7 Image and first variation of an abstract varifold

Definition 11.103 *Given a proper Lipschitz map $\varphi : U \subset \mathbb{R}^{n+m} \to U$ and an n-dimensional varifold V, define the image varifold*

$$\varphi_\# V(A) := \int_{F^{-1}(A)} J\varphi(x,S)dV(x,S), \qquad \forall A \subset G_n(U), \tag{11.79}$$

where $F : G_n(U) \to G_n(U)$ is given by

$$F(x,S) := (\varphi(x), d\varphi_x S), \quad x \in U,\ S \in G(n,m),$$

while

$$J\varphi(x,S) := \sqrt{\det\left((d\varphi_x\big|_S)^* d\varphi_x\big|_S\right)}.$$

Remark 11.104 The image varifold $\varphi_\# V$ can also be defined using the duality with continuous functions on $G_n(U)$:

$$\varphi_\# V(f) = \int_{G_n(U)} f d\varphi_\# V = \int_{G_n(U)} f(\varphi(x), d\varphi_x S) J\varphi(x,S)dV(x,S). \tag{11.80}$$

It is possible to pass from (11.79) to (11.80) using the characteristic functions of subsets $A \subset G_n(U)$ and then approximating.

We define the first variation of an abstract varifold in a way similar to that used for rectifiable varifolds: let φ_t be as in Definition 11.89. Then the first variation of a varifold V with respect to φ_t is

$$\delta V(X) := \frac{d}{dt}\bigg|_{t=0} \mathbf{M}(\varphi_{t\#} V), \tag{11.81}$$

with
$$X(x) := \frac{\partial \varphi_t(x)}{\partial t}(x,0).$$

The definition is well posed, since $\delta V(X)$ only depends on the vector field X. In fact with the same computation as in Propositions 11.16 and 11.17 and one can see that

$$\delta V(X) = \int_{G_n(U)} \operatorname{div}^S X(x) dV(x,S), \tag{11.82}$$

being
$$\operatorname{div}^S X(x) := \sum_{i=1}^n \tau_i \cdot (\nabla_{\tau_i} X),$$

for a choice of an orthonormal basis $\{\tau_1, \ldots, \tau_n\}$ of S.

Definition 11.105 *Given a varifold V in $U \subset \mathbb{R}^{n+m}$, its (total) first variation in $W \subset U$ is*

$$\|\delta V\| := \sup_{\substack{X \in C^1_c(U,\mathbb{R}^{n+m}) \\ \sup|X|\le 1,\ \operatorname{spt} X \subset W}} |\delta V(X)|, \tag{11.83}$$

where $\delta V(X)$ is defined in (11.82).

Remark 11.106 If V is the abstract varifold induced by a rectifiable varifold $\mathbf{v}(\Sigma,\theta)$ (compare Remark 11.101), then $\varphi_\# V$ is the abstract varifold corresponding to $\varphi_\# \mathbf{v}(\Sigma,\theta)$ (compare Definition 11.88). For this reason the first variation of a rectifiable varifold is the same as the first variation of the corresponding abstract varifold.

11.4.8 Allard's compactness theorem

Allard's compactness theorem gives a natural condition under which a sequence of rectifiable integer multiplicity varifolds admits a subsequence converging in the sense of varifolds (i.e. on the Grassmannian) to an *integer multiplicity rectifiable* varifold.

Example 11.107 Consider the sequence of functions $u_n : [0,1] \to \mathbb{R}$ defined by
$$u_n(x) = \frac{\{nx\}}{n}, \tag{11.84}$$

where $\{x\}$ denotes x minus its integral part.[9] The graph of u_n is an integer multiplicity rectifiable 1-varifold in $\mathbb{R}^2$ (see Figure 11.1), and as $n \to +\infty$,

[9] For instance $\{\pi\} = 0,14159265\ldots$

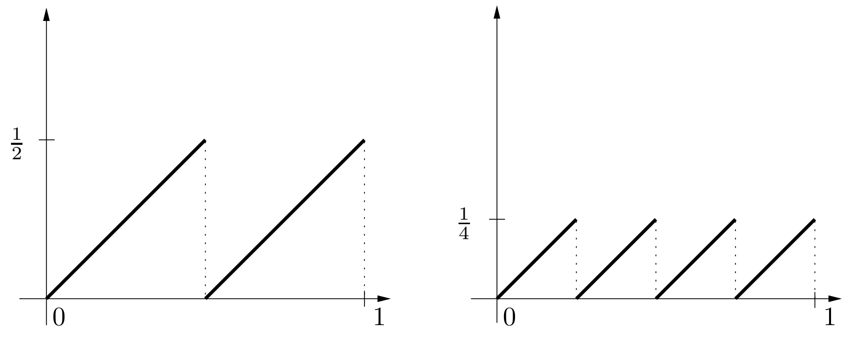

Figure 11.1: The functions u_2 and u_4 in (11.84).

the weak limit of $\mathbf{v}(G_{u_n}, 1)$ as rectifiable varifolds is $\sqrt{2}\mathcal{H}^1 \llcorner([0,1] \times \{0\})$, whose corresponding abstract varifold is

$$\sqrt{2}\mathcal{H}^1 \llcorner([0,1] \times \{0\}) \times \delta_0,$$

identifying a line in $\mathbb{R}^2$ (an element of $G_1(\mathbb{R}^2)$) with the angle it spans with the x-axis, in this case 0. On the other hand, the weak limit in the sense of varifolds is

$$\sqrt{2}\mathcal{H}^1 \llcorner([0,1] \times \{0\}) \times \delta_{\frac{\pi}{4}},$$

which is not rectifiable.

It is not hard to prove that in the preceding example $\|\delta\mathcal{G}_{u_n}\| \to +\infty$ and the following theorem of Allard, for the proof of which we refer to [97], does not apply.

Theorem 11.108 (Compactness) *Consider a sequence of integer multiplicity rectifiable varifolds V_j in a bounded open set U whose masses and first variations are locally equibounded, that is such that for every $W \Subset U$*

$$\sup_{j\geq 1} \big(\mathbf{M}(V_j|_W) + \|\delta V_j\|(W)\big) < +\infty.$$

Then there exists a subsequence $V_{j'}$ converging in the sense of varifolds to an integer multiplicity rectifiable varifold V, and

$$\|\delta V\|(W) \leq \liminf_{j\to+\infty} \|\delta V_j\|, \quad \forall W \Subset U.$$

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