NMMA 583 1. přednáška

$$-\operatorname{div} A(\nabla u) = f \quad (-\operatorname{div} f) \quad v\ \Omega \subset \mathbb{R}^n$$

$n \geq 2$

+ okrajové podmínky

Př: $-\Delta u = \operatorname{div} f\ \Omega$; $u = 0$ na $\partial\Omega$

- slabé řešení: $f \in L^2(\Omega) \Rightarrow u \in W_0^{1,2}(\Omega)$

? : $f \in L^p(\Omega), p > 2 \overset{?}{\Rightarrow} u \in W_0^{1,p}(\Omega)$

$f \in L^p(\Omega), p \in (1,2) \overset{?}{\Rightarrow} u \in W_0^{1,p}(\Omega)$

$f \in C^{0,\alpha}(\Omega), \alpha \in (0,1) \Rightarrow \nabla u \in C^{0,\alpha}(\Omega)$

$f \in X$ B.p. $\Rightarrow \nabla u \in X$

- pro systémy: • parcial regularity
  • regularita, je-li navíc struktura

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pro nás: regularita je lokální

Pokud f je v X v B_{2R}, pak $u(\nabla u) \in X$ v B_R

Mariano Giaquinta: Multiple integrals in calculus of variations, Chapter II

19tý Hilbertův problém: Jsou řešení problémů variačního počtu analytická?

Př: $$J(u) = \int_\Omega F(\nabla u)\,dx$$

E-L: $$-\partial_\alpha A_{i\alpha}(\nabla u) = 0$$

růstové podm.:

$$\exists C, L, \lambda > 0, \forall p, \xi \in \mathbb{R}^{n\times N}: \begin{cases} \cdot\ |A_{i\alpha}(p)| \leq C\cdot|p| \\ \cdot\ |\partial_{j\beta} A_{i\alpha}(p)| \leq L \\ \cdot\ \partial_{j\beta} A_{i\alpha}\xi_{i\alpha}\xi_{j\beta} \geq \lambda|\xi|^2 \end{cases}$$

Používáme sumační konvenci:

$$\partial_{j\beta} A_{i\alpha}\xi_{i\alpha}\xi_{j\beta} = \sum_{\substack{i,j\in N\\ \alpha,\beta\in M}} \partial_{j\beta} A_{i\alpha}\cdot\xi_{i\alpha}\xi_{j\beta}$$

$$u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$$

$$F : \mathbb{R}^{n\times N} \to \mathbb{R}, \quad A_{i\alpha} = \partial_{i\alpha} F$$

např. $F(\nabla u) = |\nabla u|^2$

ať $u \in W^{1,2}(\Omega)$ a platí

$$\int_\Omega A_{i\alpha}(\nabla u)\, \partial_\alpha \varphi_i = 0$$

$s \in \{1, \dots, n\}$, $h > 0$

$$\int_\Omega \Big[A_{i\alpha}(\nabla u(x+h e_s)) - A_{i\alpha}(\nabla u(x))\Big] \partial_\alpha \varphi_i = 0$$

$$\Big[A_{i\alpha}(\nabla u(x+h e_s)) - A_{i\alpha}(\nabla u(x))\Big] = \int_0^1 \frac{d}{dt} A_{i\alpha}\big(t \nabla u(x+h e_s) + (1-t)\nabla u(x)\big)\, dt = \underbrace{\int_0^1 \partial_{j\beta} A_{i\alpha}(\dots)\, dt}_{\tilde{A}^{\alpha\beta}_{ij(h)}(x)} \Big[\nabla_\beta u(x+h e_s) - \nabla_\beta u(x)\Big]_j$$

$$\int_\Omega \tilde{A}^{\alpha\beta}_{ij,h}(x)\, \partial_\beta \left(\frac{u(x+h e_s) - u(x)}{h}\right)_j \partial_\alpha \varphi_i = 0$$

$$\forall \xi \in \mathbb{R}^{nN} : \tilde{A}^{\alpha\beta}_{ij(h)}(x)\, \xi_{i\alpha} \xi_{j\beta} \geq \lambda |\xi|^2$$

$$|\tilde{A}^{\alpha\beta}_{ij,h}(x)| \leq L$$

Volme: $\varphi = \dfrac{u(x+h e_s) - u(x)}{h} \zeta^2$, ζ je cut-off

$$\int_\Omega \xi^2 \left|\frac{\nabla(u(x+he_s)-u(x))}{h}\right|^2 \leq C\int_\Omega \left|\frac{u(x+he_s)-u(x)}{h}\right|^2 |\xi'|^2$$

$$\leq C$$

$u \in W^{1,2}(\Omega)$ nezávisle na h

$\Rightarrow u \in W^{2,2}_{loc}(\Omega)$

Limitní přechod:

$$(2) \quad \int_\Omega \boxed{\partial_{j\beta} A_{i\alpha}(\nabla u)\,\partial_\beta(\partial_s u_j)}\,\partial_\alpha \varphi_i = 0$$

$\forall \varphi \in W^{1,2}_0(\Omega)$, $\operatorname{spt} \varphi \subset \Omega$

Lze na (2) použít stejnou metodu?

→ Pokud A bylo lineární, tak ano.

→ Obecně, ne! Formálně derivujme (2) $(\cdot\, \partial_r)$

$$\int_\Omega \partial_{j\beta} A_{i\alpha}(\nabla u)\,\partial_\beta(\partial_r\partial_s u_j)\,\partial_\alpha\varphi_i + \underbrace{\partial_{a\gamma}\partial_{j\beta} A_{i\alpha}(\nabla u)(\partial_r\partial_\gamma u_a)(\partial_s\partial_\beta u_j)\partial_\alpha\varphi_i}_{\text{problém}} = 0$$

Pokud $\nabla u \in C^{0,\alpha}$ pro $\alpha \in (0,1)$

(2) vidíme jako

$$\int_\Omega \bar{A}^{\alpha\beta}_{ij}(x)\, \partial_\beta \mathcal{U}_j\, \partial_\alpha \varphi_i = 0 \quad \forall \varphi \in \dots ,$$

kde $\mathcal{U} = \partial_s u$ a $\bar{A}^{\alpha\beta}_{ij} \in C^{0,\alpha}$ v Ω.

Obecná teorie říká, že $\underbrace{}_{\text{lineární}}$ $\mathcal{U} \in C^{1,\alpha} \Rightarrow \bar{A}_i \in C^{1,\alpha}$

$\Rightarrow \mathcal{U} \in C^{2,\alpha}_1$ atd....

– $\operatorname{div} A(\nabla u) = 0 \dots$ systém N rovnic.

$$u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$$

Historie:

$N = 1$: '58 de Giorgi, Nash – regularita

$n = 2$: regularita $\left(W^{2,2}(\Omega) \hookrightarrow C^{1,\alpha}(\Omega) \right)$

$N > 1$: '68 de Giorgi protipříklad na reg.

Př: $\Omega = \mathcal{U}(0,1)$ v $\mathbb{R}^n$, $n \geq 3$

$$J(v) := \int_\Omega \sum_{\alpha,i=1}^{n} |\partial_\alpha v_i|^2 + \left[\sum_{\alpha,i=1}^{n} \left((n-2)\delta_{i\alpha} + n \frac{x_i x_\alpha}{|x|^2} \right) \partial_\alpha v_i \right]^2$$

EL: $$\int_\Omega A_{ij}^{\alpha\beta}(x)\, D_\beta u^j D_\alpha \varphi_i = 0 \quad \forall \varphi \in W_0^{1,2}$$

$$A_{ij}^{\alpha,\beta} = \delta_{\alpha\beta}\delta_{ij} + \left[(n-2)\delta_{\alpha i} + n \frac{x_i x_\alpha}{|x|^2} \right] \left[(n-2)\delta_{\beta j} + n \frac{x_j x_\beta}{|x|^2} \right]$$

řešení je: $u(x) = x \cdot |x|^{-\gamma}$; $\gamma = \frac{n}{2}\left(1 - \left[(2n-2)^2 + 1\right]^{-1/2}\right)$

Protipříklad pro kvazilineární:

Giusti, Miranda '68

$$\int_\Omega A_{ij}^{\alpha\beta}(u)\, \partial_\alpha u_i\, \partial_\beta \varphi_j = 0 \quad \forall \varphi \in W_0^{1,2}(\Omega)$$

$$A_{ij}^{\alpha\beta}(u) := \delta_{ij}\delta_{\alpha\beta} + \left[\delta_{\beta j} + \frac{4}{n-2} \frac{u_j u_\beta}{1+|u|^2} \right] \left[\delta_{\alpha i} + \frac{4}{n-2} \frac{u_i u_\alpha}{1+|u|^2} \right]$$

řešení $u(x) = x \cdot |x|^{-1}$

→ Nečas, Stará, Šverák