Protipříklady

$u(x) = |x|^{-\alpha} x$

Chci $u \in W^{1,2}_{loc}(\mathbb{R}^n)$; $u : \mathbb{R}^n \to \mathbb{R}^n$

$$\partial_i u_\varkappa(x) = |x|^{-\alpha} \delta_{i\varkappa} - \alpha |x|^{-\alpha-2} x_i x_\varkappa$$

- $|\nabla u(x)| \sim |x|^{-\alpha}$; $|\nabla u| \in L^2(B_1(0)) \iff -2\alpha > -n$
  $x \neq 0$
  $\iff \alpha < \frac{n}{2}$

- bodová derivace = slabá derivace

$\boxed{\text{bereme } \alpha \in (0, \frac{n}{2})}$

Pomocné výpočty:

- $\sum_{i,\varkappa=1}^{n} \delta_{i\varkappa} \partial_i u_\varkappa = \delta_{i\varkappa} \partial_i u_\varkappa = |x|^{-\alpha} n - \alpha |x|^{-\alpha} = (n-\alpha)|x|^{-\alpha}$

- $\sum_{i,\varkappa}^{n} x_i x_\varkappa \partial_i u_\varkappa = |x|^{-\alpha+2} - \alpha |x|^{-\alpha+2} = (1-\alpha)|x|^{-\alpha+2}$

- $\partial_i u_\varkappa \partial_i u_\varkappa = |x|^{-2\alpha} n - 2\alpha |x|^{-2\alpha} + \alpha^2 |x|^{-2\alpha} = (n - 2\alpha + \alpha^2)|x|^{-2\alpha}$

$$A_{ij}^{\varkappa\lambda} \partial_i u_\varkappa = |x|^{-\alpha} \delta_{j\lambda} - \alpha |x|^{-\alpha-2} x_j x_\lambda +$$

$$\left(b_1 \delta_{j\lambda} + b_2 \frac{x_j x_\lambda}{|x|^2}\right)\left(b_1 \delta_{i\varkappa} + b_2 \frac{x_i x_\varkappa}{|x|^2}\right)\left(|x|^{-\alpha}\right)\left(\delta_{i\varkappa} - \alpha \frac{x_i x_\varkappa}{|x|^2}\right)$$

$$A_{ij}^{\varkappa\lambda}\, \partial_i u_\varkappa = |x|^{-\alpha}\left(\delta_{j\lambda} - \alpha \frac{x_j x_\lambda}{|x|^2}\right) +$$

$$+ \left(b_1 \delta_{j\lambda} + b_2 \frac{x_j x_\lambda}{|x|^2}\right)\left(b_1 (n-\alpha)|x|^{-\alpha} + b_2 (1-\alpha)|x|^{-\alpha}\right)$$

$$= |x|^{-\alpha}\Big(\delta_{j\lambda}\left[1 + b_1^2 (n-\alpha) + b_1 b_2 (1-\alpha)\right] +$$

$$\frac{x_j x_\lambda}{|x|^2}\left[-\alpha + b_2 b_1 (n-\alpha) + b_2^2 (1-\alpha)\right]\Big)$$

- $$\partial_j\left(|x|^{-\alpha}\delta_{j\lambda}\right) = \delta_{j\lambda}\left(-\alpha |x|^{-\alpha-2} x_j\right) =$$

$$= -\alpha |x|^{-\alpha-2} x_\lambda$$

- $$\partial_j\left(x_j x_\lambda |x|^{-\alpha-2}\right) = \sum_{j=1}^{n}\Big(x_\lambda |x|^{-\alpha-2} + \delta_{j\lambda}|x|^{-\alpha-2} +$$

$$+ x_j x_\lambda (-\alpha-2)|x|^{-\alpha-4} x_j\Big) =$$

$$= |x|^{-\alpha-2} x_\lambda (n + 1 - \alpha - 2)$$

$$\Rightarrow \partial_j\left(A_{ij}^{\varkappa\lambda}\,\partial_i u_\varkappa\right) = |x|^{-\alpha-2} x_\lambda \Big(-\alpha\left[1 + b_1^2 (n-\alpha) + b_1 b_2 (1-\alpha)\right]$$

$$+ \left[n - \alpha - 1\right]\left[-\alpha + b_2 b_1 (n-\alpha) + b_2^2 (1-\alpha)\right]\Big)$$

Podm.:

$$\alpha\left[1+b_1^2(n-\alpha)+b_1b_2(1-\alpha)\right]=\left[n-\alpha-1\right]\left[-\alpha+b_2b_1(n-\alpha)+b_2^2(1-\alpha)\right]$$

$$\alpha^2\left(-b_1^2-2b_1b_2-1-b_2^2\right)+\alpha\left(1+nb_1^2+b_1b_2(n+1)+b_2^2+(1-n)\left(-1-b_1b_2-b_2^2\right)\right)+(1-n)\left(nb_1b_2+b_2^2\right)=0$$

$$\alpha^2\Big(\underbrace{(b_1+b_2)^2+1}_{a}\Big)-\alpha\underbrace{\left(n(b_1+b_2)^2+n\right)}_{-b}+\underbrace{(n-1)\left(nb_1b_2+b_2^2\right)}_{c}=0$$

$$\alpha_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=-\frac{b}{2a}\pm\sqrt{\frac{b^2}{4a^2}-\frac{c}{a}}\quad\Big|\ b=-na$$

$$=\frac{n}{2}\pm\sqrt{\frac{n^2}{4}-\underbrace{\frac{(n-1)b_2(nb_1+b_2)}{(b_1+b_2)^2+1}}_{g(b_1,b_2)}}$$

spojitě závisí na b_1, b_2 ~~...~~ na Def oboru

g je malé pro b_2 malé a b_1 velké ... α může být ≈ 0

~~g ~~velké pro b_2 velké~~ $=\frac{n^2}{4}$ pro vhodnou volbu b_1 a b_2~~

~~$b_2=1:\ \frac{n^2}{4}=\frac{(n-1)(1+nb_1)}{(b_1+1)^2+1}$~~

~~$n^2(b_1^2+2b_1+2)=4(n-1)nb_1+4(n-1)$~~

~~$n^2b_1^2+b_1(2n^2-4n^2+4n)+2n^2-4n+4=0\to DU$~~

Po přednášce:
označme

$$h(t,\alpha) = \frac{nt+1}{(t+1)^2+\alpha} \quad ; \text{ kde } \alpha = \frac{1}{b_2^2} \ ; \ t = \frac{b_1}{b_2} .$$

pak $h\left(\frac{b_1}{b_2}, \frac{1}{b_2^2}\right) = g(b_1, b_2)$.

α, t můžeme volit nezávisle v $(0,+\infty) \times (-\infty, +\infty)$.

Navíc $\forall \alpha > 0$; $h(t,\alpha) \leq \ell(t) = \frac{nt+1}{(t+1)^2+0} = \frac{n(t+1)-n+1}{(t+1)^2}$

$$= n - \frac{n-1}{(t+1)^2}$$

$\mathcal{D}(\ell) = \mathbb{R}\setminus\{-1\}$; $\mathcal{H}(\ell) = (-\infty, n)$

$\Rightarrow$ Pro pevné $H \in (-\infty, n)$ najdeme t, α tak, aby $h(t,\alpha) = H$.
(nejdříve t tak, aby $\ell(t) > H$ a pak α)

Protože $\frac{n^2}{4} - (n-1)n < 0$ $\left(\frac{n}{4} < n-1 ,\ 1 < \frac{3}{4} n,\ n > \frac{4}{3} \right)$

pro $n > \frac{4}{3}$,

můžeme pro každé $\alpha \in [1, \frac{n}{2})$ (singulární řešení) najít b_1, b_2 tak, aby $u(x) = |x|^{-\alpha} x$ bylo řešení rce

$$\partial_j \left(A_{ij}^{\varkappa\lambda}(b_1, b_2, x)\, \partial_i u_\varkappa\right) = 0 .$$

- matice A je silně eliptická

$$A_{ij}^{\varkappa\lambda} \xi_{i\varkappa} \xi_{j\lambda} = |\xi|^2 + \left(\eta_{i\varkappa} \xi_{i\varkappa}\right)^2$$

$$\geq |\xi|^2 \ ; \ \text{kde } \eta_{i\varkappa} = b_1 \delta_{i\varkappa} + b_2 \frac{x_i x_\varkappa}{|x|^2}$$

- A je omezená: $|A| \leq C(n)\left(1 + |b_1|^2 + |b_2|^2\right)$

Př: Spec. volba: $b_1 = n-2$; $b_2 = n$, $n > 2$.

$$\Rightarrow \alpha := \frac{n}{2}\left(1-\left((2n-2)^2+1\right)^{-1/2}\right)$$

$\Rightarrow$ De Giorgiho protipříklad na reg. řešení el. rovnic s L^∞ koeficienty.

Pozn: Se spec. volbou b_1, b_2 dostáváme:

$$\alpha = \frac{n}{2} - \sqrt{\frac{n^2}{4} - \frac{(n-1)\cancel{(n)}(n^2-2n+n)}{4(n-1)^2+1}}$$

$$= \frac{n}{2}\left(1-\sqrt{1-\frac{4(n-1)^2}{4(n-1)^2+1}}\right) = \frac{n}{2}\left(1-\frac{1}{\sqrt{4(n-1)^2+1}}\right)$$

Je $\alpha > 1$?

$$\alpha > \frac{n}{2}\left(1-\frac{1}{2(n-1)}\right) = \frac{n}{2}\cdot\frac{2n-3}{2(n-1)}$$

Chci $2n^2-3n > 4n-4$, $2n^2-7n+4 > 0$

tj. $n > \frac{7+\sqrt{49-32}}{4} = \frac{7+\sqrt{17}}{4}$

Protože $\frac{7+\sqrt{17}}{4} < 3$, je $\alpha > 1$ pro $n \geq 3$.