Tutorial 11

Problem 1. Let us have a random vector $\mathbf{x} = (x_1, x_2)$, with a priori probability distribution density $p(x) = Ce^{-(x_1^2+x_2^2-x_1x_2)}$ ($C > 0$ is the normalizing constant). We measured (without any error/noise) $x_1 + 2x_2 = 10$.

Formulate a convex problem to estimate $\mathbf{x}$ using the maximum a posteriori probability method (MAP).

Problem 2. Formulate a convex problem that corresponds to the maximum likelihood estimate for the following situation:

Our goal is to estimate $\mathbf{x} \in \mathbb{R}^n$. We measured $A\mathbf{x} + \mathbf{v} \in \mathbb{R}^m$, where we know the matrix A and we know that the noise vector $\mathbf{v}$ has independent identically distributed components with probability density

$$p(z) = \begin{cases} 1/(2|a|) & \text{if } |z| \le a, \\ 0 & \text{else.} \end{cases}$$

We do not know the parameter $a > 0$, though, so $p_{\mathbf{x},a}(\mathbf{y})$ depends on $\mathbf{x}$ as well as on a. Estimate $\mathbf{x}$ by estimating a together with $\mathbf{x}$.

Problem 3. Assume we have $\mathbf{y} = A\mathbf{x} + \mathbf{v}$ where we know A and the entries of $\mathbf{v}$ are all independent identically distributed with mean 0 and variance 1.

We can propose any function $\hat{\mathbf{x}} = f(\mathbf{y})$ as an estimator of $\mathbf{x}$. One important property of an estimator is its bias. An estimator is *unbiased* if for any $\mathbf{x}_{true}$ the expected value of $\hat{\mathbf{x}}$ is $\mathbf{x}_{true}$ when we draw $\mathbf{y}$ from the distribution $p_{\mathbf{x}_{true}}$.

1. Show that the least squares estimator is unbiased as long as the mean of $\mathbf{v}$ is $\mathbf{0}$. You can assume that A^TA is regular.

2. Show that minimizing $\|\mathbf{y} - A\mathbf{x}\|_1$ can give a biased estimator.

   Hint: It is enough to find one A, one $\mathbf{x}_{true}$, and one distribution for $\mathbf{v}$ where the expected value of $\hat{\mathbf{x}}$ is not $\mathbf{x}_{true}$; choose something that is easy to analyze.

Problem 4 (Relevant for the future study of ellipsoids)**.** In this problem, we will show that for $X, Y \in S^n_{++}$ we have $X \succeq Y$ if and only if $X^{-1} \preceq Y^{-1}$.

1. Prove that $X \succeq Y$ if and only if $E \succeq X^{-1/2}YX^{-1/2}$ (E is the unit matrix).

2. Prove that if $Z \succ 0$ is an $n \times n$ matrix then $E \succeq Z$ if and only if $E \preceq Z^{-1}$.

3. Use the previous two points to prove that $X \succeq Y$ if and only if $E \preceq X^{1/2}Y^{-1}X^{1/2}$.

4. Use the previous point to show that $X \succeq Y$ if and only if $X^{-1} \preceq Y^{-1}$.

Recall from Linear algebra that for X symmetric matrix the matrix $X^{1/2}$ is the symmetrical matrix for which we have $X = X^{1/2}X^{1/2}$; $X^{-1/2}$ is the inverse matrix to $X^{1/2}$.