Tutorial 12

Problem 1 (hypothesis testing for y continuous)**.** We want to guess unknown $x \in \{0, 1\}$. Let Y be a normally distributed random variable with variance 1 and mean x.

Find some nontrivial Pareto optimal detector for x when you observe Y. What kind of detector do we get from the maximum likelihood method?

Note: The density of normal distribution with variance 1 and mean x is

$$\rho(z) = \frac{1}{\sqrt{2\pi}} e^{-(z-x)^2/2}.$$

Problem 2. Find all Pareto optimal detectors for the Santa detection problem from the quiz.

Problem 3. Is the ellipsoid $\{\mathbf{x} \in \mathbb{R}^2 \colon x_1^2 + x_2^2 + x_1 x_2 \leq 1\}$ contained in the ellipsoid $\{\mathbf{x} \in \mathbb{R}^2 \colon 3x_1^2 + 2x_2^2 \leq 1\}$? How about the other way around?

Problem 4. Let $h \colon \mathbb{R}^n \to \mathbb{R}^n$ be a bijective affine mapping and let $\mathcal{E} \subset \mathbb{R}^n$ be an ellipsoid. Prove that $h(\mathcal{E})$ is also an ellipsoid.

Problem 5. Let $\mathcal{E} = \{\mathbf{x} \in \mathbb{R}^n \colon \mathbf{x}^T Q x \leq 1\}$ be an ellipsoid for some $Q \in S^n_{++}$. Prove that there is a constant c_n that depends only on n (not on Q) such that the volume of $\mathcal{E}$ is $c_n \sqrt{\det Q^{-1}}$.

Problem 6. Let $K \subset \mathbb{R}^m$ be a proper convex cone and let $f \colon \mathbb{R}^n \to \mathbb{R}^m$ be K-convex. We take an $n \times k$ matrix A and some $\mathbf{b} \in \mathbb{R}^n$. Show that the function $x \mapsto f(A\mathbf{x} + \mathbf{b})$ is also K-convex.