Tutorial 4

Problem 1. Show that the function $P(x,t) = x/t$ with domain $\{(x,t)\colon x \in \mathbb{R}, t \in \mathbb{R}_{++}\}$ is quasiconvex, but not convex.

Can you find for any $\alpha \in \mathbb{R}$ a nice convex function Ψ_α so that the α-sublevel set of P is $\{(x,t)\colon \Psi_\alpha(x,t) \leq 0\}$?

Problem 2. You have the problem (P)

$$\begin{gathered}\text{minimize } f_0(\mathbf{x})\\ \text{subject to } g(\mathbf{x}) \leq 0,\end{gathered}$$

where g is a convex function and f_0 is quasiconvex. Moreover assume that for every $\alpha \in \mathbb{R}$ you have efficient access to a convex function Ψ_α such that the α-sublevel set of f_0 is $\{(x)\colon \Psi_\alpha(x) \leq 0\}$. For simplicity assume that the optimum value of (P) is known to exist and to lie in the interval $[-10^{10}, 10^{10}]$.

Assume that you have an oracle (black box) that finds optimal solution to any convex optimization problem. Propose an algorithm that for any $\epsilon > 0$ finds an $\mathbf{x}$ that is a feasible solution of (P) and $f_0(\mathbf{x})$ is at most ϵ worse than the optimal value of (P).

Problem 3. Let C be a convex subset of $\mathbb{R}^n$. Prove that the distance function $d(\mathbf{x}) = \inf\{\|\mathbf{x} - \mathbf{y}\|_2 \colon \mathbf{y} \in C\}$ is convex. Find a non-convex C for which d is not a convex function.

Problem 4. Let us have k types of stocks that we can invest in. The i-th type of stock has price c_i today; the price of the i-th stock in a year is a a random variable for which we know its expectation h_i and variance σ_i^2. We will (naively) assume that stock prices are independent random variables.

How to invest a fixed amount of money (say, 1000 Kč) so that the expected value of our stocks in a year is at least 1 200 Kč and the variance of the value of our stocks in a year is minimal? State this problem as a (non-linear) convex optimization problem.

Problem 5. Find a convex non-decreasing function $f\colon \mathbb{R} \to \mathbb{R}$ and a convex function $g\colon \mathbb{R} \to \mathbb{R}$ such that $f \circ g$ is *not* convex.

Problem 6 (hard)**.** Let $n \in \mathbb{N}$. Prove that the function $f\colon S^n_{++} \to \mathbb{R}$ defined as $f(X) = \log\det X$ is *concave*.

To see how hard this is, try the case $n = 2$.