Tutorial 5

Problem 1. Show that the monomial (in convex optimization sense) $f(x, y) = xy$ with domain $x, y > 0$ is not a quasiconvex function.

Problem 2 (transforming FLPs to LPs)**.** Consider an FLP P

$$\begin{aligned} \text{min. } & \frac{\mathbf{a}^T\mathbf{x} + d}{\mathbf{e}^T\mathbf{x} + f} \\ \text{s.t.} & G\mathbf{x} \preceq \mathbf{h}. \end{aligned}$$

with the implicit constraint$\mathbf{e}^T\mathbf{x} + f > 0$).

Consider the almost-LP Q:

$$\begin{aligned} \text{min. } & \mathbf{a}^T\mathbf{y} + dz \\ \text{s.t. } & G\mathbf{y} \preceq \mathbf{h}z \\ & \mathbf{e}^T\mathbf{y} + fz = 1 \\ & z > 0. \end{aligned}$$

a) Prove that P and Q are equivalent (=figure out how to nicely map feasible/optimal solutions of P to Q and vice versa.)

b) The sharp inequality in Q is a nuisance. Let Q' be the LP we get from Q by replacing "$z > 0$" by "$z \geq 0$".

   If Q' has an optimal solution with $z > 0$, all is good. Assume that Q' has an optimal solution $(y^\star, z^\star)$ with $z^\star = 0$ while P has at least one feasible solution x. Show how to construct for every $\epsilon > 0$ a feasible solution of P whose objective function value is ϵ-close to the optimal value of Q'.

Problem 3. Let C be a convex subset of $\mathbb{R}^n$. Prove that the distance function $d(\mathbf{x}) = \inf\{\|\mathbf{x} - \mathbf{y}\|_2 : \mathbf{y} \in C\}$ is convex. Find a non-convex C for which d is not a convex function.

Problem 4. Let us have k types of stocks that we can invest in. The i-th type of stock has price c_i today; the price of the i-th stock in a year is a a random variable for which we know its expectation h_i and variance σ_i^2. We will (naively) assume that stock prices are independent random variables.

How to invest a fixed amount of money (say, 1000 Kč) so that the expected value of our stocks in a year is at least 1 200 Kč and the variance of the value of our stocks in a year is minimal? State this problem as a QP.

Problem 5 (connected to GP; corrected)**.** Let A be a matrix *with nonnegative entries* and $\lambda \in \mathbb{R}_{++}$ its eigenvalue such that $\lambda \geq |\tau|$ for any eigenvalue τ of A and λ has an eigenvector in $\mathbb{R}^n_{++}$. Prove that then

$$\lambda = \min\{\tau \in \mathbb{R}_{++} : \exists v \in \mathbb{R}^n_{++},\ Av \preceq \tau v\}.$$