Convex optimization

Tutorial 9

Problem 1. Consider the LP (P)

$$\begin{array}{ll} \text{minimize} & -x_1 + x_2 \\ \text{subject to} & x_1 - 2 \leq 0 \\ & -x_1 - x_2 + 2 \leq 0 \\ & -x_1 + x_2 - 3 \leq 0. \end{array}$$

The optimal solution of the dual of (P) is $\lambda^\star = (2, 1, 0)$. Use complementary slackness to calculate the optimal solution of (P).

Problem 2. How does the solution of Problem 1 change if we consider the problem (P') from Quiz 8 instead? The dual optimal solution stays the same.

Problem 3. Solve the following problem (which you saw in Tutorial 1) using KKT conditions:

$$\begin{array}{ll} \text{minimize} & 4x + 5y + 3z \\ \text{subject to} & x^2 + 2y^2 + z^2 \leq 4. \end{array}$$

Compare what you did with the method of Lagrange multipliers from Analysis.

Problem 4. Consider the following game of rock-paper-partial scissors:

Player 1 can play rock, paper, or scissors, while player 2 can only play rock or scissors. Both players reveal their choices at the same time. Whoever wins (usual rules apply) gains 1 point, the loser loses 1 point (a tie is worth 0 points).

Find a worst case optimal strategy for both players for this game. What is the value of this game?

Problem 5. If $f\colon \mathbb{R}^n \to \mathbb{R}$ is a function, we define the *conjugate function to f* as a function $f^\star\colon \mathbb{R}^n \to \mathbb{R}$ with values

$$f^\star(\mathbf{y}) = \sup\{\mathbf{y}^T\mathbf{x} - f(\mathbf{x}) \mid \mathbf{x} \in \operatorname{dom} f\}.$$

Conjugate functions are important in convex analysis. Prove that the Lagrange dual function to the problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & A\mathbf{x} - \mathbf{b} \preceq \mathbf{0} \\ & C\mathbf{x} - \mathbf{d} = \mathbf{0} \end{array}$$

is $g(\lambda, \nu) = -f^\star(-A^T\lambda - C^T\nu) - \lambda^T\mathbf{b} - \nu^T\mathbf{d}$.