Jméno:

Konvexní optimalizace

Homework set 1

Due date: October 18th 2019, 10:41

Explain your reasoning in all the problems.

Problem	Max pts	Pts
1	2	
2	2	
3	2	
4	2	
5	2	
Σ	10	

Problem 1. Sketch the convex set given by the inequalities in variables x_1, x_2 (no need to prove anything, but make it clear how you got your sketch from the inequalities):

$$\begin{aligned} -x_1 &\leq 0 \\ x_1 + x_2 &\leq 4 \\ -x_1 - x_2 &\leq -4 \\ x_1 + 2x_2 &\leq 4 \end{aligned}$$

Problem 2. Prove that for every $\mathbf{a} \in \mathbb{R}^n$ the matrix $\mathbf{a}\mathbf{a}^T$ lies in S^n_+ (the position of the transposition is not a typo).

Problem 3. Are the following sets convex? If so, are they convex cones? If so, are they proper cones? Explain why.

a) $\{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 = 1\}$

b) $\{(x, y) \in \mathbb{R}^2 \colon \exists z \geq 0 \text{ such that } x > z, y > z\}$

c) $\{\mathbf{x} \in \mathbb{R}^3 \colon$ distance of $\mathbf{x}$ from the origin is less than or equal to the distance of the point $\mathbf{x}$ from $(2, -1, 7)\}$

d) $\left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \geq 1 \right\}$

Problem 4. Give a detailed proof that if $A, B \subset \mathbb{R}^n$ are convex sets then the following sets are also convex:

1. $\sqrt{2}A = \{c \in \mathbb{R}^n : \exists a \in A,\, c = \sqrt{2}a\}$
2. $A + B = \{c \in \mathbb{R}^n : \exists a \in A, \exists b \in B,\, a + b = c\}$

Problem 5. Model the following problem as a linear optimization problem (that is, into the form "minimize $f(x)$ subject to ..." with all functions linear/affine). You do *not* have to solve the problem, just state it in the linear programming form.

We are running a chemical factory that works with four types of chemicals: A, B, C, and D. Currently our storage of all of them is empty. We can buy chemicals: 1 kg of A costs 3 Kč, 1 kg of B costs 10 Kč and 1 kg of C costs 80 Kč. The chemical D is a hazardous waste that you can not buy and need to get rid of any excess of it for 1 Kč per kg.

You can run the following three reaction types in your factory (the numbers are ratios of weight of incoming and outgoing chemicals, i.e. $3A+2B \to C+4D$ means you can, say, take 3 kg of A, 2 kg of B and turn them into 1 kg of C and 4 kg of D).

$$\begin{aligned} 3A + 2B &\to C + 4D \\ 3B &\to C \\ A + D &\to B \end{aligned}$$

A customer has ordered 1 ton of A, 4 tons of B and 3 tons of C. How to fulfill this order with the least cost (including the disposal of any excess D)?

Your model does not have to be absolutely true to reality (as far as I know, you cannot describe this situation perfectly by a linear program, but you can make the distortion tiny), but it should be good enough to tell you what to do.

To get full points, explain (at least briefly) what your variables and constraints are and how they are connected with the factory situation.

You can consult with your friends when solving the homework, but you have to **write** your solutions (including Python code) **on your own** and **do not show your fininished solutions** to your peers before the due date.