Name:

Convex optimization

Homework set 3

Date due: **October 30 2019, 17:21**

Explain your reasoning in all the problems.

Problem	Pts max	Pts
1	2	
2	2	
3	2	
4	2	
5	2	
Σ	10	

Problem 1. Let $p \geq 1$.

a) Prove that the function $f(x) = x^p$ with domain $\mathbb{R}_{++}$ is convex by calculating second derivatives.

b) There are technical problems with taking the derivative of x^p at $x = 0$. These problems could be resolved, but we ask you instead to use the definition of convexity and the convexity of f from part a) to show that the function $g(x) = x^p$ for $x \geq 0$ is convex. Note: The only difference between f and g is at the point $x = 0$.

Problem 2. Prove in detail that a function $f \colon \mathbb{R}^n \to \mathbb{R}$ is convex if and only if its epigraph is a convex subset of $\mathbb{R}^{n+1}$.

Problem 3. Prove that the function $f_2(x_1, x_2) = \ln(\exp(x_1) + \exp(x_2))$ is convex.

Problem 4. 21 dwarves are mining gold in a mine. Each dwarf has a different productivity and we would like to be able to estimate the different dwarves' productivity. However, we only have data on how much total gold was mined in a given day and which dwarves were working on that day.

Formulate the problem as a least squares problem and solve it using CVX-OPT or CVXPY.

The data is in the file mining.csv; the columns are days, lines denote the attendance of a given dwarf and the bottom line are the kilograms of gold mined. (Python has the module "csv" for reading .csv files, but it is OK if you hard-code the productivity table into your program.)

Explain here on paper what your model is and *send* your code and the vector of estimated dwarf productivities by the due date to Jiří to
`pavluji@artax.karlin.mff.cuni.cz`

Problem 5. Let X be a system consisting of many simple pieces (molecules, people, ...). Each of these pieces can be in one of n states and switch states in some random manner. Let p_i be the fraction of the pieces that are in state i. To avoid some technicalities, assume that $p_i > 0$ for all i (notice that $\sum_{i=1}^{n} p_i = 1$).

For $(p_1, \dots, p_n) \in \mathbb{R}^n_{++}$ such that $\sum_{i=1} p_i = 1$, we define the *entropy* of the system by

$$S(p_1, \dots, p_n) = -\sum_{i=1}^{n} p_i \ln p_i.$$

We have some macroscopic data about the system. Under reasonable assumptions, systems tend to be in states that maximize entropy. These are the "most disorganized" states compatible with the macroscopic observations.

a) Prove that S is a *concave* function (do not forget to discuss its domain).

b) Assume that our system consists of Czech population and a person can be in one of 99 states, where the i-th state means "this person earns i thousand Kč per month." In addition, we know that the average Czech monthly salary is 31 000 Kč. Use CVXOPT/CVXPY to find the numbers p_i that maximize the entropy in the above situation.

I recommend using CVXPY here. The solution in CVXOPT is more work-intensive; you will need to use the function `cvxopt.solvers.cp` and to define the function S and its gradient and Hessian on your own.

You only have to write your solution of part a) here on paper. For part b), *send* your program and the output 99-tuple $(p_i)_{i=1}^{99}$ to Jiří to `pavluji@artax.karlin.mff.cuni.cz`

You can consult with your friends when solving the homework, but you have to **write** your solutions (including Python code) **on your own** and **do not show your fininished solutions** to your peers before the due date.