Name:

Convex optimization

Homework set 6

Date due: **November 20 2019, 17:21**

Explain your reasoning in all the problems.

Problem	Pts max	Pts
1	2	
2	3	
3	2	
4	3	
Σ	10	

Problem 1. Let (P) be a convex optimization problem and X its set of feasible solutions. Prove in detail that X is convex.

Problem 2. Let (P) be the SDP in the form from the definition

$$\begin{aligned} \text{minimize } & 5x_1 - x_2 \\ \text{s.t. } & F_1 x_1 + F_2 x_2 + G \preceq 0 \\ & x_1 + x_2 + x_3 = 1, \end{aligned}$$

where

$$F_1 = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}, F_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, G = \begin{pmatrix} 100 & 56 \\ 56 & 100 \end{pmatrix}.$$

Rewrite (P) to obtain a program (Q) for a variable matrix Y that is equivalent to (P) and has the form

$$\begin{aligned} \text{minimize } & \operatorname{Tr}(CY) \\ \text{s.t. } & Y \succeq 0 \\ & \operatorname{Tr}(A_i Y) = b_i \quad i = 1, \ldots, p, \end{aligned}$$

where C, p, A_i and B_i are parameters of your choice.

Hint: The matrix Y will be filled with many 0s.

Problem 3 (the GP with bacteria from the lecture)**.** Use CVXOPT/CVXPY to formulate and solve a GP for minimizing the maximal eigenvalue of the bacteria-growth matrix from the lecture (or pages 165-167 of the textbook) in this concrete situation: $s_1 = s_2 = s_3 = 0.7$ and we have two chemicals C and D that we can apply on the bacteria. The concentration of C will be c and the concentration of D will be d (we assume $c, d > 0$ because this is a geometric program).

We have the following relationships between c, d and bacteria multiplication rates:

$$\begin{aligned} b_1 &= \frac{0,5}{c} \\ b_2 &= \frac{0,7}{cd} \\ b_3 &= 0,5\frac{c}{d} \\ b_4 &= 0,4cd \end{aligned}$$

Formulate and solve a GP that minimizes the maximal eigenvalue of

$$\begin{pmatrix} b_1(c,d) & b_2(c,d) & b_3(c,d) & b_4(c,d) \\ s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & s_3 & 0 \end{pmatrix}.$$

a) Use CVXOPT/CVXPY to find an optimal solution. CVXOPT has a built-in function for solving GPs but you will have to do the substitution $x = \exp(z)$. Send your code and the c, d, λ you found to Jiří.

b) Suppose we wanted to maximize the rate of bacteria growth instead of minimizing it. The naive way to do that would be to replace the objective function in our GP with $1/\lambda$ and keep everything else the same. Why is this not going to work?

Note: If your solver gives you a mysterious error about ranks of matrices, it is because CVXOPT struggles with too many possible choices of the vector $\mathbf{u}$ (the GP is homogeneous in $\mathbf{u}$). To resolve this, add some artificial conditions on $\mathbf{u}$ such as $u_1 = 1$.

Problem 4 (Vector optimization)**.** Our portfolio might contain four types of stocks, call them 1, 2, 3 and 4. The profit from investing into these stocks is a random variable. If x_i is the amount invested into the i-th stock, we the expected value of our profit is $0.07x_1 + 0.08x_2 + 0.09x_3 + 0.1x_4$. Without loss of generality assume that $x_1 + x_2 + x_3 + x_4 = 1$ and that $x_i \geq 0$.

We know the covariance matrix

$$\Sigma = \begin{pmatrix} 1.1 & -1.3 & 2.5 & -0.9 \\ -1.3 & 6.5 & 0.7 & -1.5 \\ 2.5 & 0.7 & 11.1 & 0.7 \\ -0.9 & -1.5 & 0.7 & 16.1 \end{pmatrix},$$

that allows us to calculate the variance of our profit as $\mathbf{x}^T \Sigma \mathbf{x}$.

Our goal is to maximize expected value and minimize the variance of the profit. We do this by choosing our risk-aversion parameter γ and will divide a unit of money into the stocks so that we maximize

$$\text{expected value of profit} - \gamma \cdot \text{variance of profit}.$$

Formulate this problem as a convex optimization problem with a parameter γ and solve it for γ going from 0 to 2 in steps of length 0.1.

Formulate your convex problem here on paper, then solve the problem in CVXOPT/CVXPY, *send you code to Jiří* and summarize the results in the form "I would recommend X to a cautious investor and Y to a risk-insensitive investor." (You do not have to list all 21 optimal values, but do comment on how the stocks chosen change as we move γ).

You can consult with your friends when solving the homework, but you have to **write** your solutions (including Python code) **on your own** and **do not show your fininished solutions** to your peers before the due date.