Name:

Convex optimization

Homework set 8

Date due: **December 4 2019, 17:21**

Explain your reasoning in all the problems.

Problem	Pts max	Pts
1	2	
2	2	
3	2	
4	2	
5	2	
Σ	10	

Problem 1. Let A be an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$. State the dual problem to the two versions of the least squares optimization problem below and compare the resulting duals (the duals will differ):

a) Unconstrained least squares

$$\text{minimize} \quad \|A\mathbf{x} - \mathbf{b}\|_2^2$$

b) Least squares as a problem of minimizing the square of the norm

$$\begin{array}{ll} \text{minimize} & \mathbf{y}^T\mathbf{y} \\ \text{subject to} & \mathbf{y} = A\mathbf{x} - \mathbf{b} \end{array}$$

Problem 2. Consider the LP

$$\begin{aligned} \text{minimize} \quad & -x_1 \\ \text{subject to} \quad & x_1 - 2x_2 \leq 12 \\ & x_1 + x_2 \leq 36 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The optimal solution of this program is $\mathbf{x}^\star = (28, 8)$. Say we change the objective function to $-x_1 - tx_2$, where t is a constant (not a new variable). For which values of t will $(28, 8)$ remain an optimal solution.

Problem 3. Let K be a cone that is a closed subset of $\mathbb{R}^n$. Your task will be to show that $(K^\star)^\star = K$ as follows:

1. Show that $K \subseteq (K^\star)^\star$.

2. Show that for any $\mathbf{u} \notin K$ there exists a $d > 0$ such that for any $\mathbf{w}$ we have $\|\mathbf{u} - \mathbf{w}\| \le d \Rightarrow \mathbf{w} \notin K$.

3. Let $\mathbf{u}$ and d be as in the previous point. Apply the supporting hyperplane theorem to the set $L = \{\mathbf{k} - t\mathbf{w} \colon \mathbf{k} \in K, t \ge 0, \|\mathbf{u} - \mathbf{w}\| \le d\}$ at $\mathbf{0}$ to get a $\lambda \in K^\star$ such that $\lambda^T \mathbf{u} < 0$. Make sure to verify that L is convex and that $\mathbf{0}$ lies on the boundary of L.

4. Use the λ from the previous point to show that $\mathbf{u} \notin K$ implies that $\mathbf{u} \notin (K^\star)^\star$ and therefore $K \supseteq (K^\star)^\star$.

Problem 4. Recall Problem 4 from the Homework Set 4 (the rocket slalom) where you had to minimize fuel use. State the problem as a two-criterion optimization problem where you are trying to minimize fuel use and maximize the minimal (vertical) distance from the barriers $\min\{|y_i - a_i|, |y_i - b_i| \colon i = 1, 2, \dots, 10\}$. Use scalarization to find at least three Pareto optimal solutions to this problem for the race track given in the file rockets.csv where on the i-th line you find a_i, b_i.

Explain what you are doing and discuss the merits and flaws of the three optimal solutions you found here. Send your three optimal solutions to Jiří.

Problem 5 (Inspired by the board game Aréna. Altar, 1997)**.** We will simulate a sword duel as follows: There are two players, P and Q. Each player chooses a number from $\{1, 2, 3\}$ that describe a kind of attack/defense. If P's number is equal to Q's then Q has managed to deflect P's attack and both players gain 0 points. If the numbers differ then P gains and Q losses the number of points equal to the number chosen by P.

Use a computer to determine a worst case optimal strategy for P and Q. Calculate the value of this game (that is, expected number of points gained by P assuming both players play their worst case optimal strategies). You do not have to send your program, but do explain what your LP was and how it connects to the game.

You can consult with your friends when solving the homework, but you have to **write** your solutions (including Python code) **on your own** and **do not show your fininished solutions** to your peers before the due date.