Dichotomy for conservative digraphs

Alexandr Kazda

Department of Algebra
Charles University, Prague

June 9th, 2012

Where are we going

- A finite relational structure $\mathbb{A}$ is conservative if it contains all possible unary relations.
- Denote $\mathbf{A}$ the algebra of idempotent polymorphisms of $\mathbb{A}$.
- We show: If $\mathbf{A}$ contains a Taylor operation then $\mathbf{A}$ generates a congruence meet semidistributive variety.
- CSP translation: If CSP($\mathbb{A}$) is not obviously NP-complete, then local consistency checking solves CSP($\mathbb{A}$).

Where are we going

- A finite relational structure $\mathbb{A}$ is conservative if it contains all possible unary relations.
- Denote $\mathbf{A}$ the algebra of idempotent polymorphisms of $\mathbb{A}$.
- We show: If $\mathbf{A}$ contains a Taylor operation then $\mathbf{A}$ generates a congruence meet semidistributive variety.
- CSP translation: If CSP($\mathbb{A}$) is not obviously NP-complete, then local consistency checking solves CSP($\mathbb{A}$).

Where are we going

- A finite relational structure $\mathbb{A}$ is conservative if it contains all possible unary relations.
- Denote $\mathbf{A}$ the algebra of idempotent polymorphisms of $\mathbb{A}$.
- We show: If $\mathbf{A}$ contains a Taylor operation then $\mathbf{A}$ generates a congruence meet semidistributive variety.
- CSP translation: If CSP($\mathbb{A}$) is not obviously NP-complete, then local consistency checking solves CSP($\mathbb{A}$).

Where are we going

- A finite relational structure $\mathbb{A}$ is conservative if it contains all possible unary relations.
- Denote $\mathbf{A}$ the algebra of idempotent polymorphisms of $\mathbb{A}$.
- We show: If $\mathbf{A}$ contains a Taylor operation then $\mathbf{A}$ generates a congruence meet semidistributive variety.
- CSP translation: If CSP($\mathbb{A}$) is not obviously NP-complete, then local consistency checking solves CSP($\mathbb{A}$).

Where are we going

- A finite relational structure $\mathbb{A}$ is conservative if it contains all possible unary relations.
- Denote **A** the algebra of idempotent polymorphisms of $\mathbb{A}$.
- We show: If **A** contains a Taylor operation then **A** generates a congruence meet semidistributive variety.
- CSP translation: If CSP($\mathbb{A}$) is not obviously NP-complete, then local consistency checking solves CSP($\mathbb{A}$).

Shoulders of giants

- A. Bulatov: dichotomy for general conservative CSP
- L. Barto: proof of dichotomy using absorption
- P. Hell, A. Rafiey: combinatorial characterization of tractable conservative digraphs which implies our result

Shoulders of giants

- A. Bulatov: dichotomy for general conservative CSP
- L. Barto: proof of dichotomy using absorption
- P. Hell, A. Rafiey: combinatorial characterization of tractable conservative digraphs which implies our result

Shoulders of giants

- A. Bulatov: dichotomy for general conservative CSP
- L. Barto: proof of dichotomy using absorption
- P. Hell, A. Rafiey: combinatorial characterization of tractable conservative digraphs which implies our result

Shoulders of giants

- A. Bulatov: dichotomy for general conservative CSP
- L. Barto: proof of dichotomy using absorption
- P. Hell, A. Rafiey: combinatorial characterization of tractable conservative digraphs which implies our result

Polymorphisms on pairs

- If $\mathbb{A}$ is conservative and $a, b \in A$ then $\mathbf{A}$ contains some polymorphism f such that f is semilattice, majority or minority on a, b ...
- ...otherwise all operations on $\{a, b\}$ are projections...
- ...and so $\mathbf{A}$ has no Taylor operation.

Polymorphisms on pairs

- If $\mathbb{A}$ is conservative and $a, b \in A$ then $\mathbf{A}$ contains some polymorphism f such that f is semilattice, majority or minority on a, b ...
- ...otherwise all operations on $\{a, b\}$ are projections...
- ...and so $\mathbf{A}$ has no Taylor operation.

Polymorphisms on pairs

- If $\mathbb{A}$ is conservative and $a, b \in A$ then $\mathbf{A}$ contains some polymorphism f such that f is semilattice, majority or minority on a, b ...
- ...otherwise all operations on $\{a, b\}$ are projections...
- ...and so $\mathbf{A}$ has no Taylor operation.

Polymorphisms on pairs

- If $\mathbb{A}$ is conservative and $a, b \in A$ then $\mathbf{A}$ contains some polymorphism f such that f is semilattice, majority or minority on a, b ...
- ...otherwise all operations on $\{a, b\}$ are projections...
- ...and so $\mathbf{A}$ has no Taylor operation.

Colors

We color a pair $a, b \in A$:

- red if it admits a semilattice, else...
- ...yellow if it admits the majority operation, else...
- ...we color the pair blue if it admits a minority.

Colors

We color a pair $a, b \in A$:

- red if it admits a semilattice, else. . .
- . . . yellow if it admits the majority operation, else. . .
- . . . we color the pair blue if it admits a minority.

Colors

We color a pair $a, b \in A$:

- red if it admits a semilattice, else...
- ...yellow if it admits the majority operation, else...
- ...we color the pair blue if it admits a minority.

Colors

We color a pair $a, b \in A$:

- red if it admits a semilattice, else…
- …yellow if it admits the majority operation, else…
- …we color the pair blue if it admits a minority.

Colors

Theorem (Bulatov, shortened)

There are polymorphisms $f(x,y), g(x,y,z), h(x,y,z) \in \mathrm{Pol}(\mathbb{A})$ such that for every two-element subset $B \subset A$:

- *$f_{|B}$ is a semilattice operation whenever B is red, and $f_{|B}(x,y) = x$ otherwise,*
- *$g_{|B}$ is a majority operation if B is yellow and $g_{|B}(x,y,z) = x$ if B is blue*
- *$h_{|B}$ is a minority operation if B is blue.*

Colors

Theorem (Bulatov, shortened)

There are polymorphisms $f(x,y), g(x,y,z), h(x,y,z) \in \mathrm{Pol}(\mathbb{A})$ *such that for every two-element subset* $B \subset A$*:*

- $f_{|B}$ *is a semilattice operation whenever* B *is red, and* $f_{|B}(x,y) = x$ *otherwise,*
- $g_{|B}$ *is a majority operation if* B *is yellow and* $g_{|B}(x,y,z) = x$ *if* B *is blue*
- $h_{|B}$ *is a minority operation if* B *is blue.*

Colors

Theorem (Bulatov, shortened)

There are polymorphisms $f(x,y), g(x,y,z), h(x,y,z) \in \mathrm{Pol}(\mathbb{A})$ *such that for every two-element subset* $B \subset A$*:*

- $f_{|B}$ *is a semilattice operation whenever* B *is red, and* $f_{|B}(x,y) = x$ *otherwise,*
- $g_{|B}$ *is a majority operation if* B *is yellow and* $g_{|B}(x,y,z) = x$ *if* B *is blue*
- $h_{|B}$ *is a minority operation if* B *is blue.*

Colors

Theorem (Bulatov, shortened)

There are polymorphisms $f(x,y), g(x,y,z), h(x,y,z) \in \mathrm{Pol}(\mathbb{A})$ such that for every two-element subset $B \subset A$:

- *$f_{|B}$ is a semilattice operation whenever B is red, and $f_{|B}(x,y) = x$ otherwise,*
- *$g_{|B}$ is a majority operation if B is yellow and $g_{|B}(x,y,z) = x$ if B is blue*
- *$h_{|B}$ is a minority operation if B is blue.*

Blue is bad

- If we had no blue vertices, we could use the previous theorem to define 3ary and 4ary WNUs:

$$u(x,y,z) = g(f(f(x,y),z), f(f(y,z),x), f(f(z,x),y))$$
$$v(x,y,z,t) = g(f(f(f(x,y),z),t), f(f(f(y,z),x),t), f(f(f(z,x),y),t))$$

- Then **A** generates an SD($\wedge$) variety and CSP($\mathbb{A}$) is easy (see Barto, Kozik).

Blue is bad

- If we had no blue vertices, we could use the previous theorem to define 3ary and 4ary WNUs:

$$
\begin{aligned}
u(x,y,z) &= g(f(f(x,y),z), f(f(y,z),x), f(f(z,x),y)) \\
v(x,y,z,t) &= g(f(f(f(x,y),z),t), f(f(f(y,z),x),t)), \\
&\qquad f(f(f(z,x),y),t))
\end{aligned}
$$

- Then **A** generates an SD($\wedge$) variety and CSP($\mathbb{A}$) is easy (see Barto, Kozik).

Blue is bad

- If we had no blue vertices, we could use the previous theorem to define 3ary and 4ary WNUs:

$$
\begin{aligned}
u(x,y,z) &= g(f(f(x,y),z), f(f(y,z),x), f(f(z,x),y)) \\
v(x,y,z,t) &= g(f(f(f(x,y),z),t), f(f(f(y,z),x),t)), \\
&\qquad f(f(f(z,x),y),t))
\end{aligned}
$$

- Then **A** generates an SD($\wedge$) variety and CSP($\mathbb{A}$) is easy (see Barto, Kozik).

There is no blue pair

- Assume $\{a, b\}$ is a blue pair. We can now pp-define the relation

$$R = \{(a, a, b), (a, b, a), (b, a, a), (b, b, b)\}.$$

- This will lead us to a contradiction...

There is no blue pair

- Assume $\{a, b\}$ is a blue pair. We can now pp-define the relation

$$R = \{(a, a, b), (a, b, a), (b, a, a), (b, b, b)\}.$$

- This will lead us to a contradiction...

There is no blue pair

- Assume $\{a, b\}$ is a blue pair. We can now pp-define the relation

$$R = \{(a, a, b), (a, b, a), (b, a, a), (b, b, b)\}.$$

- This will lead us to a contradiction. . .

Combinatorics on potatoes

Assume $\mathbb{A}$ has a blue pair and I is the smallest constraint network for R. Then:

- Each potato contains two or three vertices.
- Each potato contains only blue pairs.
- There is no potato with three vertices.
- There are no interesting relations left and we win.

Combinatorics on potatoes

Assume $\mathbb{A}$ has a blue pair and I is the smallest constraint network for R. Then:

- Each potato contains two or three vertices.
- Each potato contains only blue pairs.
- There is no potato with three vertices.
- There are no interesting relations left and we win.

Combinatorics on potatoes

Assume $\mathbb{A}$ has a blue pair and I is the smallest constraint network for R. Then:

- Each potato contains two or three vertices.
- Each potato contains only blue pairs.
- There is no potato with three vertices.
- There are no interesting relations left and we win.

Combinatorics on potatoes

Assume $\mathbb{A}$ has a blue pair and I is the smallest constraint network for R. Then:

- Each potato contains two or three vertices.
- Each potato contains only blue pairs.
- There is no potato with three vertices.
- There are no interesting relations left and we win.

Combinatorics on potatoes

Assume $\mathbb{A}$ has a blue pair and I is the smallest constraint network for R. Then:

- Each potato contains two or three vertices.
- Each potato contains only blue pairs.
- There is no potato with three vertices.
- There are no interesting relations left and we win.

Thanks for your attention.