Möbius number systems

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Outline

- Our goal: To use sequences of Möbius transformations to represent points on $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ or the unit circle $\mathbb{T}$.
- A Möbius tranformation (MT) is any nonconstant function $M : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ of the form

$$M(z) = \frac{az + b}{cz + d}$$

- We will consider MTs that preserve the upper half-plane
- or the unit disc $\mathbb{D}$.

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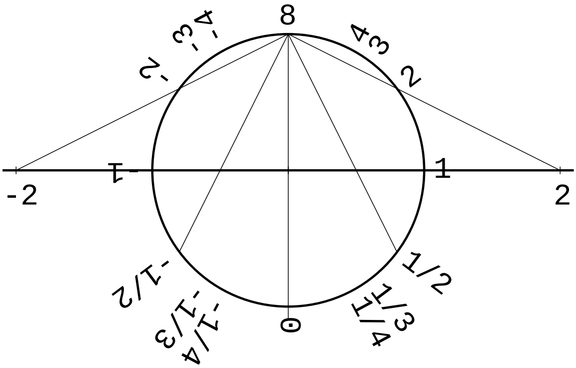
$\overline{\mathbb{R}}$ versus $\mathbb{T}$



- Using the stereometric projection, we have a one-to-one correspondence between the upper half-plane and unit disc.
- This projection is itself an MT.
- Therefore we can translate MTs that represent $\mathbb{T}$ to the ones that represent $\overline{\mathbb{R}}$.

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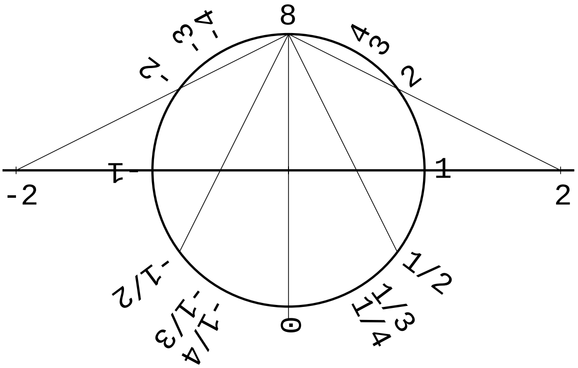

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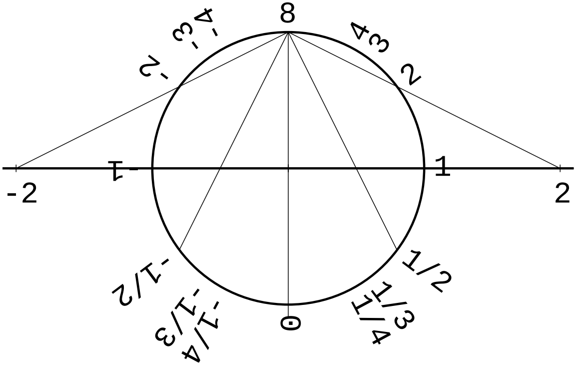

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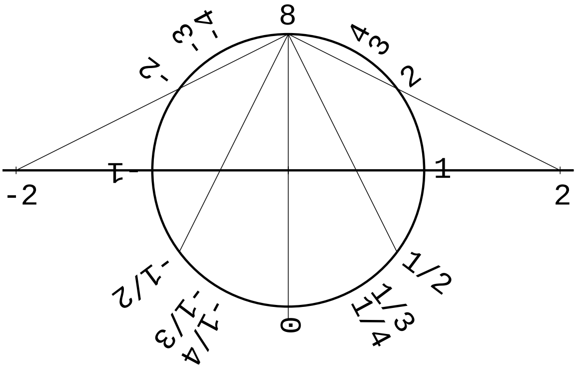

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- However, the example number systems represent $\overline{\mathbb{R}}$.
- How to tell them apart: half-plane-preserving MTs have a hat, disc-preserving MTs don't.

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Disc Möbius transformations $M : \mathbb{D} \to \mathbb{D}$

- A direct calculation shows that all MTs that preserve $\mathbb{D}$ must look like this:
- $$M(z) = \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}},$$
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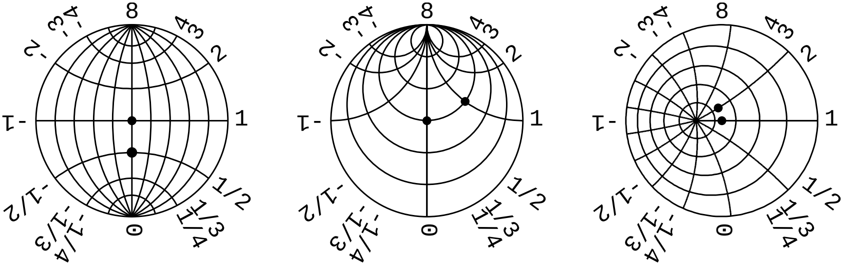

$M_0(z) = \frac{3z-i}{iz-3}$ $\qquad$ $M_1(z) = \frac{(2i+1)z+1}{2i-1}$ $\qquad$ $M_2(z) = \frac{(7+2i)z+i}{-iz+(7-2i)}$

$\hat{M}_0(x) = x/2$ $\qquad$ $\hat{M}_1(x) = x+1$ $\qquad$ $\hat{M}_2(x) = \frac{4x+1}{3-x}$

hyperbolic $\qquad$ parabolic $\qquad$ elliptic

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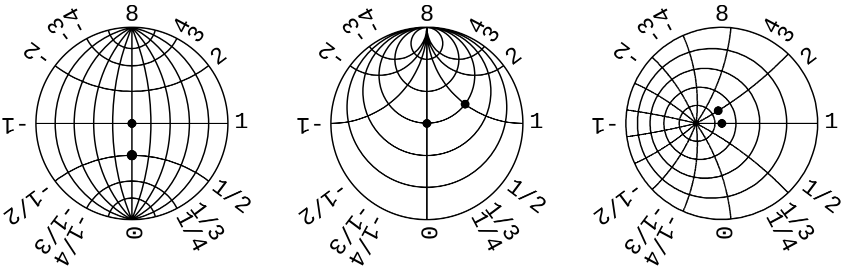

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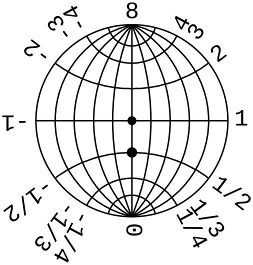

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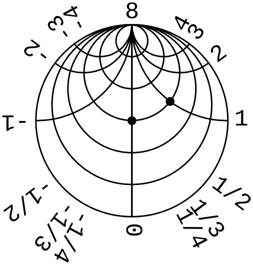

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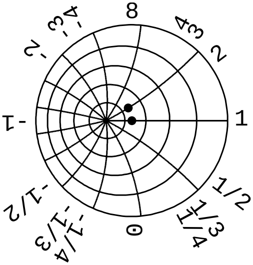

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Defining convergence

- A sequence $M_1, M_2, \ldots$ represents the number $x \in \mathbb{T}$ if $M_n(0) \to x$ for $n \to \infty$.
- Isn't it a bit arbitrary?
- No. This definition is quite natural.
- For example, if $M_1, M_2, \ldots$ represents x then $M_n(K) \to \{x\}$ for any $K \subset \mathbb{D}^o$ compact.

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Preliminaries from Symbolic dynamics

- Let A be finite alphabet. Then A^+ is the set of all finite nonempty words over A, A^ω the set of all one-sided infinite words.
- Recall that $\Sigma \subset A^\omega$ is a subshift if Σ can be defined by a set of forbidden (finite) words.
- For $v = v_0 v_1 \dots v_n$ a word, denote by F_v the transformation $F_{v_0} \circ F_{v_1} \circ \dots \circ F_{v_n}$.

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What is a Möbius number system?

Let us have a system of MTs $\{F_a : a \in A\}$. A subshift $\Sigma \subset A^\omega$ is a Möbius number system if:

- For every $w \in \Sigma$, the sequence $\{F_{w_0 w_1 \dots w_n}\}_{n=0}^{\infty}$ represents some point $\Phi(w) \in \mathbb{T}$.
- The function $\Phi : \Sigma \to \mathbb{T}$ is continuous and surjective.

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Getting the idea: Binary system

- Take transformations $\hat{F}_0(x) = x/2$ and $\hat{F}_1(x) = (x+1)/2$.
- Take the full shift $\Sigma = \{0,1\}^\omega$.
- The function Φ maps Σ to an interval on $\mathbb{T}$ corresponding to $[0,1]$.
- Essentialy, it is the ordinary binary system; $\Phi(w)$ corresponds to $0.w$.
- Note that this is not a Möbius number system yet, as it is not surjective...
- ...we will fix that soon.

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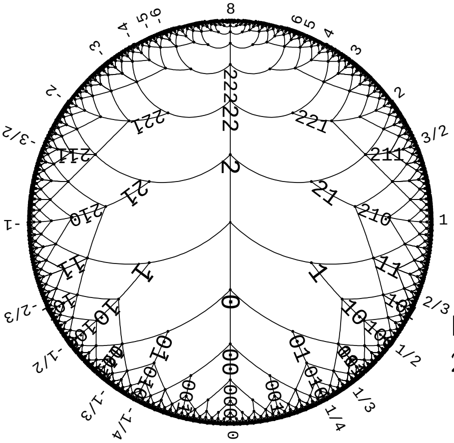
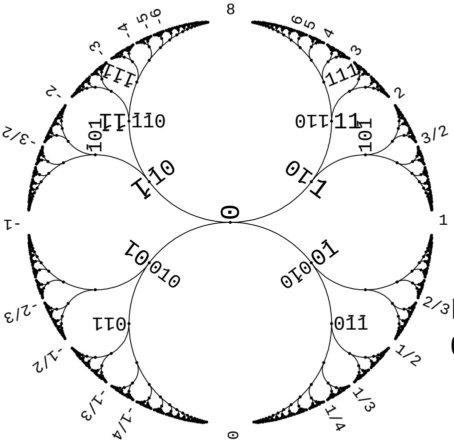
Binary signed system $A = \{\overline{1}, 0, 1, 2\}$



$$\begin{aligned}
\hat{F}_{\overline{1}}(x) &= (x-1)/2 \\
\hat{F}_0(x) &= x/2 \\
\hat{F}_1(x) &= (x+1)/2 \\
\hat{F}_2(x) &= 2x
\end{aligned}$$

Forbidden words:
$20, 02, 12, \overline{1}2, 1\overline{1}, \overline{1}1$

Regular continued fractions $A = \{\bar{1}, 0, 1\}$



$$\hat{F}_{\bar{1}}(x) = -1 + x$$
$$\hat{F}_0(x) = -1/x$$
$$\hat{F}_1(x) = 1 + x$$

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Existence problem

- The question: Given a system of MTs $\{F_a : a \in A\}$, does there exist a Möbius number system?
- The answer: It depends on whether $\{V_a : a \in A\}$ cover $\mathbb{T}$ in a certain way.

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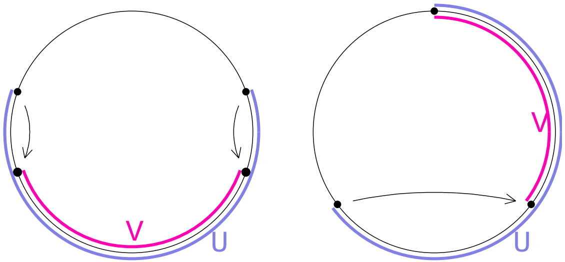
Intervals of contraction and expansion



$$
\begin{aligned}
\mathbf{U}_u &= \{z \in \mathbb{T} : \ |F_u'(z)| < 1\}, \\
\mathbf{V}_u &= \{z \in \mathbb{T} : \ |(F_u^{-1})'(z)| > 1\} \\
F_u(\mathbf{U}_u) &= \mathbf{V}_u, \ u \in A^+
\end{aligned}
$$

Theorem

Let $\{F_a : a \in A\}$ be MTs.

1. *If $\overline{\bigcup\{V_u : u \in A^+\}} \neq \mathbb{T}$, then there does not exist any Möbius number system.*
2. *If there exists a finite $B \subset A^+$ such that $\{\overline{V}_u : u \in B\}$ cover $\mathbb{T}$, then there exists a Möbius number system.*

Note that there can still be some situation in between (1) and (2).

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