

Deciding the existence of a k -wnu operation in polynomial time

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Maltsev conditions

- A strong Maltsev condition is a system of equalities for some operations and variables, eg.

$$p(p(x, y), r(y)) \approx x.$$

- A variety V satisfies the Maltsev condition M if we can replace the operations in M by terms of V and get a system of equalities true in V .
- Linear Maltsev: No nested operations. Example:

$$m(x, x, y) \approx y$$

$$m(y, x, x) \approx y.$$

- Libor Barto: CSP complexity can be characterized by linear Maltsev conditions.

The problem

- Given finite \mathbf{A} , we want to know if $V(\mathbf{A})$ satisfies a given fixed Maltsev condition.
- Example: “Does \mathbf{A} have a Maltsev term?” (Complexity: Unknown.)
- Known to be EXPTIME-complete for numerous common Maltsev conditions (Freese, Valeriote, Horowitz).
- Restricting \mathbf{A} to be idempotent often allows us to find a polynomial time algorithm (Freese, Valeriote, Horowitz, K.).
- Big goal: Characterize, the complexity of all versions of the problem for all Maltsev conditions.
- Today’s goal: Characterize, the complexity of deciding k -ary weak near unanimity for idempotent algebras when k is fixed.

Weak near unanimity

- A k -ary weak near unanimity (k -WNU) is any idempotent operation that satisfies

$$t(y, x, \dots, x) \approx t(x, y, \dots, x) \approx \dots \approx t(x, x, \dots, y).$$

- A k -WNU is an example of a Taylor term.
- Given \mathbf{A} idempotent, we can test in polynomial time whether \mathbf{A} has a Taylor term (Bulatov).

- We say that \mathbf{A} has n -local k -WNUs if for every $\bar{r}, \bar{s} \in A^n$ there exists a term t such that

$$t(s_i, r_i, \dots, r_i) = t(r_i, s_i, r_i, \dots, r_i) = \dots = t(r_i, r_i, \dots, s_i).$$

for all i .

- Translating to relations:

$$\text{Sg}_{\mathbf{A}^{kn}} \left(\left(\begin{array}{c} \bar{s} \\ \bar{r} \\ \vdots \\ \bar{r} \end{array} \right), \left(\begin{array}{c} \bar{r} \\ \bar{s} \\ \vdots \\ \bar{r} \end{array} \right), \dots, \left(\begin{array}{c} \bar{r} \\ \bar{r} \\ \vdots \\ \bar{s} \end{array} \right) \right)$$

has a member $(\bar{a}, \bar{a}, \dots, \bar{a})^T$ for some $\bar{a} \in A^n$.

Observation

If \mathbf{A} has k -WNU then it has n -local k -WNUs for all n .

Lemma

If \mathbf{A} is idempotent, has a Taylor term and has n -local k -WNUs then \mathbf{A} also has $(n + 1)$ -local k -WNUs.

Corollary

We only need to check for the existence of a Taylor term and for all 1-local k -WNUs. This is polynomial for k fixed.

Proof for $k = 4$

$$R = \text{Sg} \begin{pmatrix} \bar{s} & \bar{r} & \bar{r} & \bar{r} \\ \bar{r} & \bar{s} & \bar{r} & \bar{r} \\ \bar{r} & \bar{r} & \bar{s} & \bar{r} \\ \bar{r} & \bar{r} & \bar{r} & \bar{s} \\ d & c & c & c \\ c & d & c & c \\ c & c & d & c \\ c & c & c & d \end{pmatrix}.$$

Applying an n -local 4-WNU, we have

$$(\bar{a}, \bar{a}, \bar{a}, \bar{a}, b_1, b_2, b_3, b_4)^T \in R$$

The last four entries can be permuted.

Lemma (Barto, Kozik, Niven)

If \mathbf{G} is a Taylor algebra, $E \leq_s \mathbf{G}^2$ and (G, E) has a component of algebraic length 1, then E contains a loop.

In our case, let

$$G = \{(x_1, x_2) : \exists y, z, (\bar{a}, \bar{a}, \bar{a}, \bar{a}, x_1, x_2, y, z)^T \in R\}$$

$$E = \{((x_1, x_2), (x_2, x_3)) : \exists z, (\bar{a}, \bar{a}, \bar{a}, \bar{a}, x_1, x_2, x_3, z)^T \in R\}.$$

\bar{G} has a Taylor term, E is subdirect, and (G, E) contains a 3-cycle and a 4-cycle in the same component (\Rightarrow algebraic length 1).

The loop lemma gives us e, f such that $(\bar{a}, \bar{a}, \bar{a}, \bar{a}, e, e, e, f) \in R$.

Final step

$$\begin{pmatrix} \bar{a} \\ \bar{a} \\ \bar{a} \\ \bar{a} \\ e \\ e \\ e \\ f \end{pmatrix}, \begin{pmatrix} \bar{a} \\ \bar{a} \\ \bar{a} \\ \bar{a} \\ e \\ e \\ f \\ e \end{pmatrix}, \begin{pmatrix} \bar{a} \\ \bar{a} \\ \bar{a} \\ \bar{a} \\ e \\ f \\ e \\ e \end{pmatrix}, \begin{pmatrix} \bar{a} \\ \bar{a} \\ \bar{a} \\ \bar{a} \\ f \\ e \\ e \\ e \end{pmatrix} \in R.$$

Apply 4-WNU for e, f and obtain a tuple $(\bar{a}, \bar{a}, \bar{a}, \bar{a}, b, b, b, b)^T$.

- The “local to global” idea works for many linear Maltsev conditions in the idempotent case. . .
- . . . but Dmitriy Zhuk has recently shown that “local to global” does **not** work for the minority operation:

$$t(x, x, y) \approx t(x, y, x) \approx t(y, x, x) \approx y.$$

- Question: How hard is it to decide if \mathbf{A} idempotent has a minority?
- Related problem: Given \mathbf{A} that satisfies a strong Maltsev condition M , produce operations of \mathbf{A} that witness M .

Thank you for your attention.