Minority is the join of two varieties defined by linear equations

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February 10, 2017

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Maltsev conditions

- A strong Maltsev condition is a system of equalities for some operations and variables, eg.

$$p(p(x,y),r(y)) \approx x.$$

- An algebra **A** satisfies M if some terms of **A** do.
- Linear conditions: No nested operations.
- Example (the Maltsev operation):

$$m(x,x,y) \approx y$$
$$m(y,x,x) \approx y.$$

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Deciding strong Maltsev conditions (idempotent case)

- Let M be a fixed strong Maltsev condition.
- Input: Idempotent algebra $\mathbf{A}$ given by tables of basic operations.
- Output: Does $\mathbf{A}$ satisfy M?
- Lies in EXPTIME.
- Conjecture: For any M, this problem is in P.

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Local to global

- How to decide if **A** has

$$m(x, x, y) \approx y$$
$$m(y, x, x) \approx y?$$

- It is enough to verify that for every choice of $a, b, c, d \in A$ we have m_{abcd} such that

$$m_{abcd}(a, a, b) = b$$
$$m_{abcd}(c, d, d) = c.$$

- Works when equalities in M allow bootstrapping: Maltsev operation, Jónsson terms, Gumm terms, NU(k), k-wnu ...

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Minority report

- Minority operation:

$$n(x,x,y) \approx n(x,y,x) \approx n(y,x,x) \approx y.$$

- Dmitriy Zhuk: There are algebras that have n locally, but not globally.
- How hard is it to decide if a given $\mathbf{A}$ has minority?
- Can check that we have a Maltsev operation, then use Peter Mayr's algorithm for subpower membership $\Rightarrow$ the problem is in NP.
- Open problem: Improve this.

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Embracing failure

- The proof of "local to global" fails for reasons that are hard to generalize.
- Is there a natural reason for the failure?
- Conjecture: Minority is join-irreducible in the lattice of interpretability of linear Maltsev conditions.
- This is not true.

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Joining up to minority

- N_1

$$\begin{aligned} w(x,x,y) \approx w(x,y,x) \approx w(y,x,x) \approx m(x,y,x) \\ m(x,x,y) \approx y \\ m(y,x,x) \approx y. \end{aligned}$$

- N_2

$$\begin{aligned} t(y,x,x,z,y,y,z) \approx y \\ t(x,y,x,y,z,y,z) \approx y \\ t(x,x,y,y,y,z,z) \approx y. \end{aligned}$$

- To get t from a generic minority, let $t(x_1,\ldots,x_7) := n(n(x_1,x_2,x_3), n(x_4,x_5,x_6), x_7)$.

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N_1 does not have minority

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- This condition is satisfied in $(\mathbb{Z}_4, +)$ by $w(x,y,z) = -x-y-z$ and $m(x,y,z) = x-y+z$.
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- This condition is satisfied in $(\mathbb{Z}_6,+)$ by $t(x_1,\ldots,x_7) := 3x_1 + 3x_2 + 3x_3 + 2x_4 + 2x_5 + 2x_6 + 4x_7$.
- Again, no minority in $(\mathbb{Z}_6,+)$.

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$N_1 \vee N_2$ give minority (and nothing more)

- $n(x, y, z) := t(x, y, z, m(z, x, y), m(x, y, z), m(y, z, x), w(x, y, z))$
- An example minority equality:

$$n(x, x, y) \approx t(x, x, y, m(y, x, x), m(x, x, y), m(x, y, x), w(x, x, y))$$
$$\approx t(x, x, y, y, y, w(x, x, y), w(x, x, y)) \approx y$$

- We used $t(x, x, y, y, y, z, z) \approx y$.
- Since free algebra with just minority satisfies both N_1 and N_2, we don't get any additional operations in $N_1 \vee N_2$.

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Where to go from here?

- We now have one more troublesome Maltsev condition: N_2. This can help us understand what is really going on.
- Generalize Dmitriy's counterexamples?
- Work on Subpower Membership Problem: A polynomial time algorithm for Maltsev SMP would get minority into P.
- A. Bulatov, A. Szendrei, P. Mayr: SMP is in P if $\mathbf{A}$ has a cube term and lies in a residually small variety.

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