Algorithms that decide absorption

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What is absorption?

Definition (Libor Barto, Marcin Kozik)

Let $\mathbf{B} \leq \mathbf{A}$ be algebras. We say that $\mathbf{B}$ absorbs $\mathbf{A}$ if there exists a term t in $\mathbf{A}$ such that for any $b_1, \ldots, b_n \in B, a \in A$ we have:

$$\begin{aligned} t(a, a, a, \ldots, a) &= a \\ t(a, b_2, b_3, \ldots, b_{n-1}, b_n) &\in B \\ t(b_1, a, b_3, \ldots, b_{n-1}, b_n) &\in B \\ &\vdots \\ t(b_1, b_2, b_3, \ldots, b_{n-1}, a) &\in B \end{aligned}$$

Notation for absorption: $\mathbf{B} \trianglelefteq \mathbf{A}$.

Ok, but what *is* absorption?

- If 0 is the minimal element of a semilattice $(L, \wedge)$, then $\{0\}$ absorbs L; absorption term is $t(x_1, x_2) = x_1 \wedge x_2$.
- If $\mathbf{A}$ is an algebra with a majority term m, then every singleton is an absorbing subalgebra; absorption term is m.
- $\mathbf{A} \trianglelefteq \mathbf{A}$ always.
- $\mathbf{A}$ has an NU term iff $\{a\} \trianglelefteq \mathbf{A}$ for every $a \in A$.
- If $\mathbf{A}$ is an abelian group, then $\mathbf{A}$ has no proper absorbing subalgebra.

What is absorption good for?

$\mathbf{A}$ "connected" and $\mathbf{B} \trianglelefteq \mathbf{A} \Rightarrow \mathbf{B}$ "connected".

Major recent results using absorption:

- CSP($\mathbb{A}$) is solvable by local consistency checking iff $\mathbf{A}$ is $SD(\wedge)$ (Barto, Kozik).
- If $\mathbf{A}$ is finitely related and CD, then $\mathbf{A}$ has an NU operation. (Barto and Zhuk independently)
- If $\mathbf{A}$ is finitely related and CM, then $\mathbf{A}$ has a cube term. (Barto)
- If $\mathbf{A}$ has an NU operation, then CSP($\mathbb{A}$) has bounded pathwidth duality and lies in NL (Barto, Kozik, Willard).
- A new proof of: $\mathbf{A}$ finite + solvable + Taylor $\Rightarrow$ $\mathbf{A}$ has a Maltsev term (Stanovský).

Deciding absorption

- Problem: Given $\mathbf{B} \leq \mathbf{A}$, can we decide if $\mathbf{B} \trianglelefteq \mathbf{A}$?
- Libor Barto, Jakub Bulín: Yes, if $\mathbf{A}$ is given by a finite set of relations.
- What about if $\mathbf{A}$ is given by a finitely many operations instead?
- Miklós Maróti: We can decide whether a finite algebra $\mathbf{A} = (A, f_1, \ldots, f_n)$ has an NU term.

Jónsson absorption

- Weaker notion of absorption inspired by Kozik's terms for congruence distributivity.
- Let $\mathbf{B} \leq \mathbf{A}$. Then $\mathbf{B} \trianglelefteq_J \mathbf{A}$ if there exist idempotent terms $d_0, d_1, \ldots, d_n$ such that:

$$\begin{aligned} \forall i = 1, \ldots, n,\ d_i(B, A, B) &\subset B \\ d_1(x, x, y) &= x \\ d_i(x, y, y) &= d_{i+1}(x, x, y) \\ d_n(x, y, y) &= y. \end{aligned}$$

- If $\mathbf{A}$ is finitely related, then $\mathbf{B} \trianglelefteq_J \mathbf{A}$ implies $\mathbf{B} \trianglelefteq \mathbf{A}$ (Barto, Bulín).

Jónsson is not enough

- Look at $(\{0,1\}, \to)$.
- $\{0\} \trianglelefteq_{\mathrm{J}} \{0,1\}$ as witnessed by the Jónsson absorbing terms:

$$\begin{aligned} d_1(x,y,z) &= (y \to (z \to x)) \to x, \\ d_2(x,y,z) &= (x \to (y \to z)) \to z, \end{aligned}$$

- However, $\{0\}$ does not absorb $\{0,1\}$.
- How to see that: Look at relations $\{0,1\}^n \setminus \{0\}^n$.

Blockers

- We call (C, D) a **B**-blocker if
  - $\emptyset \neq C \leq D \leq \mathbf{A}$,
  - $D \cap B \neq \emptyset$,
  - $C \cap B = \emptyset$,
  - $\{(x_1, \ldots, x_n) \in D^n : \exists i,\ x_i \in C\} \leq \mathbf{A}^n$ for every $n \in \mathbb{N}$.
- If $\mathbf{B} \trianglelefteq \mathbf{A}$, then there is no **B**-blocker.

Good and bad news about blockers

- Given idempotent $\mathbf{A}$ with finitely many operations, we can test if there are no $\mathbf{B}$-blockers.
- However, we can have no blockers and no absorption: Consider $\mathbf{A} = (\mathbb{Z}_2, m)$, where $m(x, y, z) = x + y + z \pmod 2$.

Putting it all together

Theorem

Let $\mathbf{A}$ *be a finite idempotent algebra,* $\mathbf{B} \leq \mathbf{A}$*. Then* $\mathbf{B} \trianglelefteq \mathbf{A}$ *iff there is no* $\mathbf{B}$*-blocker and* $\mathbf{B} \trianglelefteq_J \mathbf{A}$*.*

Corollary

We can decide $\mathbf{B} \trianglelefteq \mathbf{A}$ *algorithmically.*

Warning

- Decision problems come in two basic flavors: $\mathbf{A}$ can be given by tables of its basic operations, or by a set of invariant relations.
- The complexity of deciding something for $\mathbf{A}$ and for $\mathbb{A}$ can be different!
- If we have basic operations, we can generate subalgebras quickly, but have trouble deciding if a given operation belongs into the clone of $\mathbf{A}$.
- If we have relations, it is the other way around.

Idempotent algorithms

- Given $\mathbf{A}$ by basic operations, we can decide if $\mathbf{B} \trianglelefteq_J \mathbf{A}$ in polynomial time.
- For deciding B-blockers, we have one algorithm running in time $O(|\mathbf{A}|3^{|A|})$ and another running in time $O(|\mathbf{A}| \prod_i s_i)$ where s_i is the arity of the i-th basic operation of $\mathbf{A}$.
- Given $\mathbf{B} \leq \mathbf{A}$, deciding existence of a $\mathbf{B}$-blocker is NP-complete. (reduction from 3-SAT).
- Good news: Deciding if a given $\mathbf{A}$ has some blocker for some $\mathbf{B}$ can be done in polynomial time.

Algorithm for relational structures

- If $\mathbf{A}$ is given by a relational structure, all we need is to decide $\mathbf{B} \trianglelefteq_J \mathbf{A}$.
- Best known general algorithm: Time roughly $|\mathbb{A}||A|^{|A|^3}$.
- The issue: How to get 3-generated subalgebras of $\mathbf{A}^3$ quickly.
- Special cases can be much easier:
- If we can solve CSP($\mathbb{A}$) in P then deciding $\trianglelefteq_J$ is in P, too.
- Note: Deciding NU for relational structures is in P.

Nonidempotent algebras

- If **A** is not idempotent, we would also like to decide absorption.
- Problem with taking the idempotent reduct: We might lose the generators of the clone of **A**.
- Imitating some of Dmitriy Zhuk's ideas gives us an algorithm anyway.
- Deciding $\mathbf{B} \trianglelefteq_J \mathbf{A}$ can be done in time $O(|\mathbf{A}|^{|A|+3})$, deciding existence of a B-blocker is (roughly) doubly exponential in this way.

Thank you for your attention.