# Linear Datalog and $k$-permutability = symmetric Datalog

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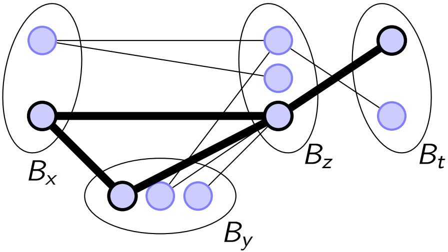
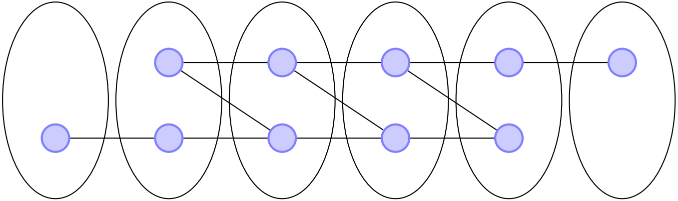
# Constraint Satisfaction Problem




- An instance of CSP($\mathbb{A}$). Unary and binary relations must be from the relational clone of $\mathbb{A}$.
- Is there a solution?

# Consistency checking

- Solving CSP is NP-complete in general.
- If a part of a CSP instance has no solution, then the whole instance has no solution.



- Good news: Consistency can be checked quickly.
- Bad news: An instance can have no solution, yet be locally consistent.

# Datalog

- A Datalog program adds tuples to predicates using local rules until it reaches its goal or can't apply any rule.
- Example of a Datalog program with unary predicates $Z, c_0, c_1$, binary predicate $E$, and the goal predicate $G$.

$$
\begin{aligned}
Z(x) &\Leftarrow c_0(x) \\
Z(y) &\Leftarrow Z(x) \wedge E(x,y) \\
G &\Leftarrow Z(y) \wedge c_1(x)
\end{aligned}
$$

- The above program checks if there is a directed path from something in $c_0$ to something in $c_1$.
- How to use Datalog to solve $\neg$ CSP: A Datalog program verifies local consistency. Goal: Prove inconsistency.

# Linear and Symmetric Datalog

- Predicates on left hand side of some rule: IDBs
- Linear Datalog: At most one IDB on the right hand side of any rule.
- Symmetric Datalog: At most one IDB on the right hand side of any rule. If there is an IDB on both sides of a rule, we can switch the IDBs.
- Example: Rule

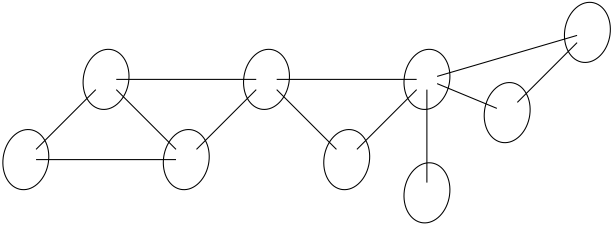
$$Z(y) \Leftarrow T(x) \wedge E(x, y)$$

gives

$$T(x) \Leftarrow Z(y) \wedge E(x, y).$$

# Bounded pathwidth duality

- There is a linear Datalog program solving $\neg\,\mathrm{CSP}(\mathbb{A})$ iff $\mathbb{A}$ has bounded pathwidth duality.
- Bounded pathwidth duality: Unsatisfiable instances of $\mathrm{CSP}(\mathbb{A})$ always have unsatisfiable "path-like" parts.

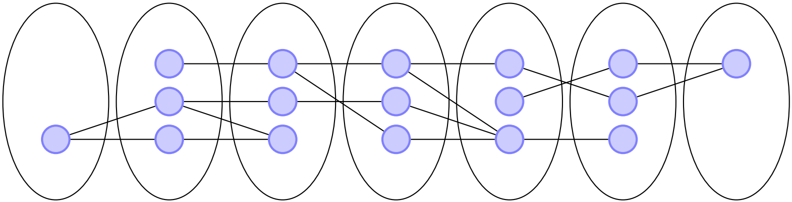


# Solving path instances is enough

**Theorem**

*If* $\neg\,\mathrm{CSP}(\mathbb{A})$ *is solvable by a linear Datalog program and* $\mathbb{A}$ *is n-permutable for some n, then* $\neg\,\mathrm{CSP}(\mathbb{A})$ *can be solved by a symmetric Datalog program.*

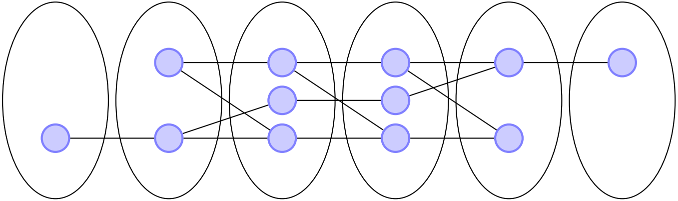
- $\mathbb{A}$ has bounded pathwidth duality.
- It is enough to look at path shaped unsatisfiable CSP instances.



- How to use symmetric Datalog to decide path CSP instances?

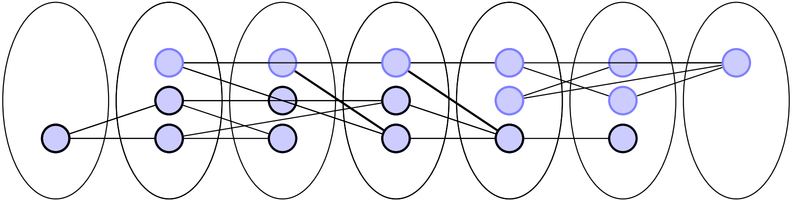
# Using $n$-permutability

- If $\mathbb{A}$ is 4-permutable, then the following can't be an unsatisfiable instance of CSP($\mathbb{A}$):



- Applying Hagemann-Mitschke terms gives us a solution.

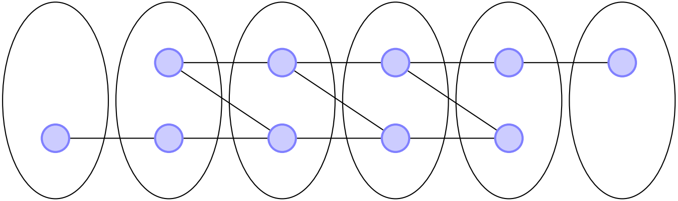
# Glued potatoes



- We label black the vertices that can be reached from the starting potato.
- We want lots of backwards edges: Edges from blue to black vertices.
- We use some trickery to glue together potatoes without blue to black edges.
- Nature of the trickery: We can run a smaller symmetric Datalog program inside our original program.

# Subdirect subinstance

- Additional trickery gives us a long part of the path instance where everything is subdirect.
- Then it is a matter of pigeonhole principle to find something like this:



# Summing up, part 1

**Theorem**

*Given n-permutatable relational structure $\mathbb{A}$ such that $\mathrm{CSP}(\mathbb{A})$ has bounded pathwidth duality, there is a symmetric Datalog program that decides $\mathrm{CSP}(\mathbb{A})$.*

- V. Dalmau and B. Larose previously proved this for 2-permutable structures.
- $n$-permutability is necessary by L. Egri, B. Larose, and P. Tesson.

## Summing up, part 2

Algebraic view (due to L. Egri, B. Larose, and P. Tesson):

- If $\neg\,\mathrm{CSP}(\mathbb{A})$ is solvable by linear Datalog, then the algebra of $\mathbb{A}$ must be (join) semidistributive.
- If $\neg\,\mathrm{CSP}(\mathbb{A})$ is solvable by symmetric Datalog, then the algebra of $\mathbb{A}$ must be (join) semidistributive and $n$-permutable.

**Conjecture**

*Semidistributive = linear Datalog = NL.*

**Conjecture**

*Semidistributive and n-permutable = symmetric Datalog = L.*

- Our result plus first conjecture gives the second conjecture.

Thank you for your attention.

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