Algebraic Approach to Promise Constraint Satisfaction

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Noon Seminar

Disclaimer

- The results presented are not mine
- CS pioneers of algebraic PCSP: Per Austrin, Joshua Brakensiek, Venkatesan Guruswami, and Johan Håstad
- Coming soon: Jakub Bulín, Jakub Opršal. *Algebraic Approach to Promise Constraint Satisfaction*
- Any errors, typos etc. in the presentation belong to me

Promise Constraint Satisfaction

- $\mathbb{A}, \mathbb{B}$ are relational structures, $\mathbb{A} \to \mathbb{B}$ (wlog $\mathbb{A} \subseteq \mathbb{B}$)
- PCSP($\mathbb{A}, \mathbb{B}$): Input relational structure $\mathbb{C}$
  - Output "Yes" if $\mathbb{C} \to \mathbb{A}$
  - Output "No" if $\mathbb{C} \not\to \mathbb{B}$
- Example: PCSP($\mathbb{K}_3, \mathbb{K}_4$).
- PCSP($\mathbb{K}_3, \mathbb{K}_4$) is NP-hard because all of its polymorphisms are "almost projections"

Pol($\mathbb{A}, \mathbb{B}$)

- Pol($\mathbb{A}, \mathbb{B}$) are all polymorphisms from $\mathbb{A}$ to $\mathbb{B}$
- Polymorphism $f\colon \mathbb{A}^n \to \mathbb{B}$ sends $R^{\mathbb{A}}$ into $R^{\mathbb{B}}$
- Pol($\mathbb{A}, \mathbb{B}$) determines complexity of PCSP($\mathbb{A}, \mathbb{B}$) up to logspace reductions
- Can't compose, but can take minors:

$$f(x_1, x_2, x_3, x_4, x_5) \in \mathrm{Pol}(\mathbb{A}, \mathbb{B}) \Rightarrow f(x_{\mathbf{2}}, x_{\mathbf{2}}, x_{\mathbf{16}}, x_{\mathbf{4}}, x_{\mathbf{5}}) \in \mathrm{Pol}(\mathbb{A}, \mathbb{B})$$

- If $\mathbb{A} \subseteq \mathbb{B}$ then Pol($\mathbb{A}, \mathbb{B}$) contains all projections

$$\pi_i(x_1, \ldots, x_n) = x_i$$

Minions

- A ~~minor closed set~~ ~~clonoid~~ minion $\mathcal{C}$ on sets A, B is a nonempty family of operations from A to B closed under taking minors
- Taking minors: $\sigma\colon [n] \to [m]$ sends n-ary f to m-ary f^σ where

$$f^\sigma(x_1, \ldots, x_m) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

- Each $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$ is a minion

Minion homomorphisms

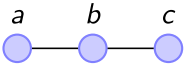
- $\phi\colon \mathcal{C} \to \mathcal{D}$ preserves arity and commutes with taking minors
- Another view: homomorphism sends identities of $\mathcal{C}$ to identities of $\mathcal{D}$
- Example:

$$f(x,x,y) \approx g(x,y,y,z) \Rightarrow \phi(f)(x,x,y) \approx \phi(g)(x,y,y,z)$$

- Jakub Bulín, Jakub Opršal: For $\mathbb{A}, \mathbb{B}, \mathbb{A}', \mathbb{B}'$ finite relational structures $\mathrm{Pol}(\mathbb{A}, \mathbb{B}) \to \mathrm{Pol}(\mathbb{A}', \mathbb{B}')$ gives a poly-time reduction from $\mathrm{PCSP}(\mathbb{A}', \mathbb{B}')$ to $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$

The reduction I

- Have: $\mathrm{Pol}(\mathbb{A},\mathbb{B}) \to \mathrm{Pol}(\mathbb{A}',\mathbb{B}')$
- Want: $\mathrm{PCSP}(\mathbb{A}',\mathbb{B}')$ reduces to $\mathrm{PCSP}(\mathbb{A},\mathbb{B})$
- $\mathrm{PCSP}(\mathbb{A},\mathbb{B})$ is equivalent to a different promise problem involving "functional equations" (Maltsev conditions).
- Example reduction: $\mathrm{PCSP}(\mathbb{K}_3,\mathbb{K}_4)$ and input graph



- Watch the blackboard!
- $\mathbb{G} \to \mathbb{K}_3 \Rightarrow$ solution by projections
- $\mathbb{G} \not\to \mathbb{K}_4 \Rightarrow$ no solution in $\mathrm{Pol}(\mathbb{K}_3,\mathbb{K}_4)$

The reduction II

- Have: $h\colon \mathrm{Pol}(\mathbb{A},\mathbb{B}) \to \mathrm{Pol}(\mathbb{A}',\mathbb{B}')$
- Want: $\mathrm{PCSP}(\mathbb{A}',\mathbb{B}')$ reduces to $\mathrm{PCSP}(\mathbb{A},\mathbb{B})$
- Input "functional equation" system

$$t(x_0,x_0,x_1,x_2) \approx s(x_0,x_3)$$
$$\vdots$$

- Given a system of functional equations, answer yes if the system has a solution by projections and no if it has no solution in $\mathrm{Pol}(\mathbb{A},\mathbb{B})$
- This problem is equivalent to $\mathrm{PCSP}(\mathbb{A},\mathbb{B})$ (takes work)
- Existence of $h \Rightarrow$ if no solution in $\mathrm{Pol}(\mathbb{A}',\mathbb{B}')$ then no solution in $\mathrm{Pol}(\mathbb{A},\mathbb{B})$

Applications

- Identities in minions determine PCSP complexity
- $\text{Pol}(\mathbb{K}_3, \mathbb{K}_4) \to \text{Pol}(\mathbb{K}_3, \mathbb{K}_3)$ (nontrivial) so $\text{PCSP}(\mathbb{K}_3, \mathbb{K}_4)$ is NP-hard.
- If $\text{Pol}(\mathbb{A}, \mathbb{B})$ maps to a minion of operations of bounded arity then $\text{PCSP}(\mathbb{A}, \mathbb{B})$ is NP-hard
- More on the way...

Going beyond homomorphisms

- $\mathrm{Pol}(\mathbb{A}, \mathbb{B}) \to$ bounded arity minion $\Rightarrow$ $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ is NP-hard
- Libor Barto, Jakub Bulín, Andrei Krokhin, Jakub Opršal: NP-hard PCSP whose polymorphisms don't map into a bounded arity minion
- Our best source of hardness: Reduction from GapLabelCover (variant of PCP) to PCSP.
- I weakened homomorphisms to ϵ-homomorphisms and tinkered with them, but it did not work out.
- TODO: Make sense of PCSP complexity.

Thank you for your attention.