How to decide absorption

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What is absorption?

Definition (Libor Barto, Marcin Kozik)

Let $B \leq A$ be algebras. We say that B absorbs A if there exists a term t in A such that for any $b_1, \ldots, b_n \in B, a \in A$ we have:

$$
\begin{aligned}
t(a, a, a, \ldots, a) &= a \\
t(a, b_2, b_3, \ldots, b_{n-1}, b_n) &\in B \\
t(b_1, a, b_3, \ldots, b_{n-1}, b_n) &\in B \\
&\vdots \\
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Ok, but what *is* absorption?

- If 0 is the minimal element of a finite semilattice $(L, \wedge)$ then $\{0\}$ absorbs L; absorption term is $t(x_1, x_2) = x_1 \wedge x_2$.
- If A is an algebra with a majority term m then every singleton is an absorbing subalgebra; absorption term is m.
- If A is an algebra then always $A \trianglelefteq A$.
- If A is an abelian group then A has no proper absorbing subalgebra.

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Deciding absorption

- Let A be idempotent finite algebra. Then A has an NU term iff every singleton $\{a\}$ absorbs A.
- Miklós Maróti: We can decide whether a finite algebra A has an NU term.
- Problem: Given $B \leq A$, can we decide if $B \trianglelefteq A$?
- Libor Barto, Jakub Bulín: Yes, if A is finitely related.
- Can we do more?

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Blockers

- Let $B \trianglelefteq A$ with absorption term t.
- If t is n-ary then for no C can we have $C \cap B \neq \emptyset$, $C \not\subset B$, and $C^n \setminus B^n \leq A^n$.
- We call (C, D) a blocker for B if
  - $\emptyset \neq D \subset C$,
  - $C \cap B \neq \emptyset$,
  - $D \cap B = \emptyset$,
  - $\{(x_1, \ldots, x_n) \in C^n : \exists i, x_i \in D\} \leq A^n$ for every $n \in \mathbb{N}$.
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No blockers $\Rightarrow$ absorption?

- Given idempotent A with finitely many operations, we can test if there are no blockers for B.
- However, we can have no blockers and no absorption: Consider $A = (\mathbb{Z}_2, m)$, where $m(x, y, z) = x + y + z \pmod 2$.

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Jónsson absorption

- Weaker notion of absorption inspired by terms for congruence distributivity.
- Let $B \leq A$. Then $B \trianglelefteq_J A$ if there exist idempotent terms $d_0, d_1, \ldots, d_n$ such that:

$$\begin{aligned} \forall i = 0, \ldots, n,\ d_i(B, A, B) &\subset B \\ d_0(x, y, z) &= x \\ d_i(x, y, y) &= d_{i+1}(x, y, y) \text{ for } i \text{ even} \\ d_i(x, x, y) &= d_{i+1}(x, x, y) \text{ for } i \text{ odd} \\ d_n(x, y, z) &= z. \end{aligned}$$

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Putting it all together

Conjecture

Let A be a finite algebra, $B \leq A$. Then $B \trianglelefteq A$ iff there is no blocker for B and $B \trianglelefteq_J A$.

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Thank you for your attention.