Alexandr Kazda (with thanks to Petr Kůrka)

Möbius transformations

**[Convergence](#page-15-0)** 

Möbius number [systems](#page-19-0)

[Examples](#page-27-0)

Subshifts [admitting a](#page-39-0) number system

<span id="page-0-0"></span>[Conclusions](#page-47-0)

### Möbius number systems

Alexandr Kazda (with thanks to Petr Kůrka)

Charles University, Prague

NSAC Novi Sad August 17–21, 2009

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### **Outline**

**KOD KARD KED KED E YORA** 

Möbius number [systems](#page-0-0)

Alexandr Kazda (with thanks to Petr Kůrka)

Möbius transformations

**[Convergence](#page-15-0)** 

Möbius. number [systems](#page-19-0)

[Examples](#page-27-0)

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

### **1** Möbius transformations

2 [Convergence](#page-15-0)

**3** Möbius number systems

**4** [Examples](#page-27-0)

**5** [Subshifts admitting a number system](#page-39-0)

Alexandr Kazda (with thanks to Petr Kůrka)

Möbius transformations

**[Convergence](#page-15-0)** 

Möbius [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

<span id="page-2-0"></span>[Conclusions](#page-47-0)

- Our goal: To use sequences of Möbius transformations to represent points on  $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}.$
- A Möbius transformation  $(MT)$  is any nonconstant function  $M : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$  of the form

$$
M(z) = \frac{az+b}{cz+d}
$$

 $\mathbf{E} = \mathbf{A} \oplus \mathbf{A} + \mathbf{A$ 

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- We will consider MTs that preserve the upper half-plane.
- These are precisely the MTs with  $a, b, c, d$  real and  $ad - bc = 1$ .

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Möbius transformations

**[Convergence](#page-15-0)** 

Möbius [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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Möbius transformations

**[Convergence](#page-15-0)** 

Möbius [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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Möbius transformations

**[Convergence](#page-15-0)** 

Möbius [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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Möbius transformations

**[Convergence](#page-15-0)** 

Möbius number [systems](#page-19-0)

[Examples](#page-27-0)

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

### Classifying Möbius transformations

$$
M_0(x)=x/2
$$

• Hyperbolic, two fixed points.

$$
M_1(x) = x + 1
$$

• Parabolic, one fixed point.

$$
M_2(x) = -\frac{1}{x+1}
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 $\mathbf{A} \equiv \mathbf{A} + \math$ 

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Möbius transformations

**[Convergence](#page-15-0)** 

Möbius number [systems](#page-19-0)

[Examples](#page-27-0)

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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Möbius transformations

**[Convergence](#page-15-0)** 

Möbius number [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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Möbius transformations

**[Convergence](#page-15-0)** 

Möbius number [systems](#page-19-0)

[Examples](#page-27-0)

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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Möbius transformations

**[Convergence](#page-15-0)** 

Möbius number [systems](#page-19-0)

[Examples](#page-27-0)

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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Möbius transformations

**[Convergence](#page-15-0)** 

Möbius number [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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Möbius transformations

**[Convergence](#page-15-0)** 

Möbius number [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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Möbius transformations

**[Convergence](#page-15-0)** 

Möbius number [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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Möbius transformations

**[Convergence](#page-15-0)** 

Möbius number [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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Möbius trans-

#### **[Convergence](#page-15-0)**

Möbius [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

<span id="page-15-0"></span>[Conclusions](#page-47-0)

### • A sequence  $M_1, M_2, \ldots$  represents the number x if  $M_n(i) \to x$  for  $n \to \infty$ .

- Isn't it a bit arbitrary?
- No. This definition is quite natural.
- For example, if  $M_1, M_2, \ldots$  represents x then  $M_n(K) \to \{x\}$  for any K compact lying above the real line.

# Defining convergence

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Möbius trans-

**[Convergence](#page-15-0)** 

Möbius [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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**[Convergence](#page-15-0)** 

Möbius [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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**[Convergence](#page-15-0)** 

Möbius [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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Möbius transformations

**[Convergence](#page-15-0)** 

Möbius number [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

<span id="page-19-0"></span>[Conclusions](#page-47-0)

# Preliminaries from Symbolic dynamics

- Let  $A$  be finite alphabet. Let  $A^*$  denote the set of all finite words over A,  $A^{\omega}$  the set of all one-sided infinite words.
- $\bullet$   $A^*$  with the operation of concatenation is a monoid.
- $\bullet$  Let w: denote the *i*-th letter of the word w.
- A set  $\Sigma \subset A^\omega$  is a subshift if  $\Sigma$  can be defined by a set of forbidden (finite) factors.
- For  $v = v_1 \ldots v_n$  a word, denote by  $F_v$  the transformation  $F_{v_1} \circ \cdots \circ F_{v_n}$ .

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**[Convergence](#page-15-0)** 

Möbius number [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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**[Convergence](#page-15-0)** 

Möbius number [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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Möbius transformations

**[Convergence](#page-15-0)** 

Möbius number [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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Möbius trans-

**[Convergence](#page-15-0)** 

Möbius number [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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Möbius trans-

**[Convergence](#page-15-0)** 

Möbius number [systems](#page-19-0)

[Examples](#page-27-0)

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

# What is a Möbius number system?

Let us have a system of MTs  $\{F_{\bm a}: {\bm a}\in A\}$ . A subshift  ${\bm \Sigma}\subset A^\omega$ is a Möbius number system if:

- For every  $w \in \Sigma$ , the sequence  $\{F_{w_1...w_n}\}_{n=1}^{\infty}$  represents some point  $\Phi(w) \in \overline{\mathbb{R}}$ .
- The function  $\Phi : \Sigma \to \overline{\mathbb{R}}$  is continuous and surjective.

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**[Convergence](#page-15-0)** 

Möbius number [systems](#page-19-0)

[Examples](#page-27-0)

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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Möbius trans-

**[Convergence](#page-15-0)** 

Möbius number [systems](#page-19-0)

[Examples](#page-27-0)

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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Möbius transformations

**[Convergence](#page-15-0)** 

Möbius number [systems](#page-19-0)

### [Examples](#page-27-0)

Subshifts [admitting a](#page-39-0) number system

<span id="page-27-0"></span>[Conclusions](#page-47-0)

# Getting the idea: Binary system

• Take transformations  $F_0(x) = x/2$  and  $F_1(x) = (x + 1)/2$ .

• Take the full shift  $\Sigma = \{0,1\}^{\omega}$ .

- The function  $\Phi$  maps  $\Sigma$  to an interval on  $\mathbb R$  corresponding to [0, 1].
- Essentially, it is the ordinary binary system;  $\Phi(w)$ corresponds to 0.w.
- Note that this is not a Möbius number system yet, as it is not surjective. . .

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**[Convergence](#page-15-0)** 

Möbius number [systems](#page-19-0)

### [Examples](#page-27-0)

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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**[Convergence](#page-15-0)** 

Möbius number [systems](#page-19-0)

#### [Examples](#page-27-0)

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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**[Convergence](#page-15-0)** 

Möbius number [systems](#page-19-0)

#### [Examples](#page-27-0)

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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Möbius transformations

**[Convergence](#page-15-0)** 

Möbius [systems](#page-19-0)

#### [Examples](#page-27-0)

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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Möbius transformations

**[Convergence](#page-15-0)** 

Möbius [systems](#page-19-0)

#### [Examples](#page-27-0)

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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**[Convergence](#page-15-0)** 

Möbius number [systems](#page-19-0)

#### [Examples](#page-27-0)

Subshifts [admitting a](#page-39-0) number system



# Binary signed system  $A = \{\overline{1}, 0, 1, 2\}$

$$
F_{\overline{1}}(x) = (x - 1)/2
$$
  
\n
$$
F_0(x) = x/2
$$
  
\n
$$
F_1(x) = (x + 1)/2
$$
  
\n
$$
F_2(x) = 2x
$$

Forbidden words:  $20, 02, 12, \overline{1}2, 1\overline{1}, \overline{1}1$ 

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 $4$  ロ )  $4$   $\overline{r}$  )  $4$   $\overline{z}$  )  $4$   $\overline{z}$  )

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Möbius transformations

**[Convergence](#page-15-0)** 

Möbius [systems](#page-19-0)

#### [Examples](#page-27-0)

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

### • We forbid words to get rid of troublesome combinations.

Why forbid words?

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- In the binary signed system  $F_0$  and  $F_2$  are inverse to each other, so  $F_{02} = F_{20} = id$ .
- Forbidding 12 and  $\overline{1}2$  keeps twos at the beginning of every word.
- Finally,  $1\overline{1}$  and  $\overline{1}1$  are forbidden because  $\Phi((1\overline{1})^{\infty})$  and  $\Phi((11)^\infty)$  are not defined.
- We shall see that unregulated concatenation can break any Möbius number system.

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- Möbius transformations
- **[Convergence](#page-15-0)**
- Möbius number [systems](#page-19-0)

### [Examples](#page-27-0)

- Subshifts [admitting a](#page-39-0) number system
- [Conclusions](#page-47-0)

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- Möbius transformations
- **[Convergence](#page-15-0)**
- Möbius [systems](#page-19-0)

### [Examples](#page-27-0)

- Subshifts [admitting a](#page-39-0) number system
- [Conclusions](#page-47-0)

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- Möbius transformations
- **[Convergence](#page-15-0)**
- Möbius [systems](#page-19-0)

### [Examples](#page-27-0)

- Subshifts [admitting a](#page-39-0) number system
- [Conclusions](#page-47-0)

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Möbius trans-

**[Convergence](#page-15-0)** 

Möbius [systems](#page-19-0)

#### [Examples](#page-27-0)

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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Möbius transformations

**[Convergence](#page-15-0)** 

Möbius [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

<span id="page-39-0"></span>[Conclusions](#page-47-0)

# Forbidding words is necessary

### A non-erasing substitution is monoid homomorphism  $\rho: \mathsf{A}^{*} \to \mathsf{B}^{*}$  such that  $\rho(\mathsf{v})$  is the empty word only for  $\mathsf{v}$  empty.

If  $\Sigma$  is a Möbius number system then  $\Sigma \neq \rho(A^{\omega})$  for all alphabets A and all non-erasing substitutions  $\rho$ .

In particular, for  $\rho$  identity we obtain that  $\Sigma$  is never the full shift  $A^{\omega}$ .

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Möbius trans-

**[Convergence](#page-15-0)** 

Möbius [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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Möbius trans-

**[Convergence](#page-15-0)** 

Möbius [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

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Möbius number [systems](#page-0-0)

Möbius transformations

**[Convergence](#page-15-0)** 

Möbius [systems](#page-19-0)

[Examples](#page-27-0)

Subshifts [admitting a](#page-39-0) number system

[Conclusions](#page-47-0)

### • To simplify notation, we consider only the case  $\rho(v) = v$ .

- We first prove that for every  $w \in A^\omega$  and every  $x \in \overline{\mathbb{R}}$  it is true that  $\lim_{n\to\infty} F_{w_1w_2...w_n}(x) = \Phi(w)$ .
- This is highly suspicious...
- The only way we can obtain such pointwise convergence is when  $F_v$  is parabolic (like  $x \mapsto x + 1$ ) for every v nonempty finite word.
- But a simple case consideration shows that then all the  $F_v$ have the same fixed point and  $\Phi$  is a constant map.

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Möbius number [systems](#page-0-0)

Alexandr Kazda (with thanks to Petr Kůrka)

Möbius transformations

**[Convergence](#page-15-0)** 

Möbius [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

**KORK STRAIN A BAR SHOP** 

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Möbius number [systems](#page-0-0)

Alexandr Kazda (with thanks to Petr Kůrka)

Möbius transformations

**[Convergence](#page-15-0)** 

Möbius [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

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Möbius number [systems](#page-0-0)

Alexandr Kazda (with thanks to Petr Kůrka)

Möbius transformations

**[Convergence](#page-15-0)** 

Möbius [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

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Möbius number [systems](#page-0-0)

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Möbius transformations

**[Convergence](#page-15-0)** 

Möbius [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

 $\mathbf{E} = \mathbf{A} \oplus \mathbf{A} + \mathbf{A$ 

 $2Q$ 

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Möbius number [systems](#page-0-0)

- Möbius transformations
- **[Convergence](#page-15-0)**
- Möbius [systems](#page-19-0)
- [Examples](#page-27-0)
- Subshifts [admitting a](#page-39-0) number system
- <span id="page-47-0"></span>[Conclusions](#page-47-0)

### • Sequences of MTs can represent numbers.

- Möbius number systems can emulate more usual means of number representation.
- We can state (and sometimes prove) nontrivial existence conditions such as the one presented . . .
- ... however, there is a lot of room for improvements.

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Möbius number [systems](#page-0-0)

- Möbius transformations
- **[Convergence](#page-15-0)**
- Möbius [systems](#page-19-0)
- [Examples](#page-27-0)
- Subshifts [admitting a](#page-39-0) number system
- [Conclusions](#page-47-0)
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#### [systems](#page-0-0) Alexandr Kazda (with thanks to Petr Kůrka)

Möbius number

#### Möbius transformations

- **[Convergence](#page-15-0)**
- Möbius [systems](#page-19-0)
- **[Examples](#page-27-0)**
- Subshifts [admitting a](#page-39-0) number system
- [Conclusions](#page-47-0)
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Möbius number [systems](#page-0-0)

- Möbius transformations
- **[Convergence](#page-15-0)**
- Möbius [systems](#page-19-0)
- **[Examples](#page-27-0)**
- Subshifts [admitting a](#page-39-0) number system
- [Conclusions](#page-47-0)
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Möbius transformations

**[Convergence](#page-15-0)** 

Möbius number [systems](#page-19-0)

**[Examples](#page-27-0)** 

Subshifts [admitting a](#page-39-0) number system

<span id="page-51-0"></span>**[Conclusions](#page-47-0)** 

Thanks for your attention.

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