Absorption and reflexive digraphs

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- M. Maróti and L. Zádori: CM $\Rightarrow$ NU for reflexive digraphs.
- We show an alternative proof using absorption.
- All our graphs will be reflexive.

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CM $\Rightarrow$ MZ1 + 2

Let G be a CM reflexive digraph. Then for any K reflexive digraph:

MZ1 If H is a connected component of G, $R \leq G^K$ and $R \subset H^K$ then R is connected.

MZ2 If H is a strongly connected component of G, $R \leq G^K$ and $R \subset H^K$ then R is extremely connected.

Maróti and Zádori have given a nice proof that MZ1 + 2 implies NU.

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G^K

- The digraph G^K has as vertices all the homomorphisms $K \to G$.
- We have $f \to g$ if whenever $u \to v$ in K then $f(u) \to g(v)$ in G.
- In particular G^K is itself a reflexive digraph...
- ...that contains a copy of G on the "diagonal"...
- ...and if G was CM then so is G^K.

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Absorption

Let (V, E) be reflexive, $U \subset V$. Assume we have Gumm terms and $U \trianglelefteq_{\mathrm{g}} V$. Then:

- If (V, E) is connected then so is (U, E).
- If (V, E) is strongly connected then so is (U, E).

Note: Maróti and Zádori actually prove both claims in their paper (without mentioning absorption).

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Proving MZ1

Goal: If H is a connected component of G, $R \leq G^K$ and $R \subset H^K$ then R is connected.

- We show by induction that R is connected if it contains the diagonal.
- In the general case, we have some pp definition D of R. If we remove all constant constraints in D we get a pp definition of some $S \supset R$.
- Now S contains the diagonal and $R \trianglelefteq_{\mathrm{g}} S$.
- Therefore, R must be connected.

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- The previous argument can be easily modified to prove that if H is strongly connected then $R \subset H^K$ is strongly connected.
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Proving MZ2

Goal: If H is a strongly connected component of G, $R \leq G^K$ and $R \subset H^K$ then R is extremely connected.

- By the previous argument we know that R is strongly connected.
- All we need is CM + strongly connected $\Rightarrow$ all subalgebras extremely connected.

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Strongly connected ⇒ extremely connected

- Take the smallest counterexample G: CM, strongly connected, some subalgebra not extremely connected.
- MZ1.5: Any subalgebra of G must be strongly connected.
- By minimality, any proper subalgebra of G must be extremely connected and G is not extremely connected.
- Singletons are subalgebras $\Rightarrow$ G is extremely connected.

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Thanks for your attention.