4th problem set

Reminder: A relation $R \leq A^n$ is subdirect if projections of R to all coordinates are A.

Problem 1. We will show that Abelian groups don't have definable principal congruences. The signature of Abelian groups will be $(+, -, 0)$ where we take expressions like $(-3)x$ to be abbreviations for $(-x) + (-x) + (-x)$.

1. Verify that in Abelian groups we have $(x, y) \in Cg(a, b)$ if and only if there is $n \in \mathbb{Z}$ such that $x + n \cdot a = y + n \cdot b$.

2. Show that if Abelian groups had DPC, then there exists k such that for any G Abelian group and any $x, y, a, b \in G$ we have

$$(x, y) \in Cg(a, b) \Leftrightarrow \bigvee_{n \in \mathbb{Z}, |n| \leq k} x + n \cdot a = y + n \cdot b.$$

   Hint: Look at your lecture notes.

3. Show that formulas from the previous point are do not work for the Abelian group $\mathbb{Z}$.

Problem 2. Consider the CSP where we decide if a primitive positive sentence of the form

$$\exists x_1 \exists x_2 \dots \exists x_n E(x_{i_1}, x_{j_1}) \wedge E(x_{i_2}, x_{j_2}) \wedge \dots \wedge E(x_{i_k}, x_{j_k}),$$

(where $k, n \in \mathbb{N}$, and $i_1, \dots, i_k, j_1, \dots, j_k \in [n]$) is true. The possible values of the x_i's are in $\{1, 2, \dots, 5\}$ and the relation E is the "house" symmetric graph given by the 12 pairs

$$E = \{(5,1), (1,5), (1,2), (2,1), (2,3), (3,2), (2,4), (4,2), (3,4), (4,3), (4,5), (5,4)\}$$

(draw a picture). Show that we can reduce 3-coloring to this problem.

Problem 3. Find a subdirect binary relation $R \subset A^2$ and an unsatisfiable primitive positive sentence (=CSP instance) of the form

$$\exists x_1 \exists x_2 \dots \exists x_n R(x_{i_1}, x_{j_1}) \wedge R(x_{i_2}, x_{j_2}) \wedge \dots \wedge R(x_{i_k}, x_{j_k}).$$

Problem 4. Let R be a subdirect binary relation (on a finite set A) invariant under a semilattice operation. Show that then any primitive positive sentence of the form

$$\exists x_1 \exists x_2 \dots \exists x_n R(x_{i_1}, x_{j_1}) \wedge R(x_{i_2}, x_{j_2}) \wedge \dots \wedge R(x_{i_k}, x_{j_k}).$$

is true.

Problem 5. One variant of CSP is the counting CSP where the goal is to find the number of assignments that satisfy a primitive positive sentence. Let's say that our sentences use (multiple copies of) only one relation – the relation R which is

$$R = \{(x, y, z) \in \{0, 1\}^3 \colon x + y + z = 1 \pmod 2\}.$$

Use linear algebra to formulate a polynomial time algorithm that solves this counting CSP.