Zápočťák II

$$f(x) = \left(\sqrt{x^2+1} + x\right)^{1/x}, \quad x \in \mathbb{R}\setminus\{0\}$$

i) dokážte sudost

ii) dodefinovat spojitě v 0

Ad i) $-\mathbb{R}\setminus\{0\} = \mathbb{R}\setminus\{0\}$

$$f(-x) = \left(\sqrt{x^2+1} - x\right)^{-1/x} =$$

$$= \left(\frac{1}{\sqrt{x^2+1} - x}\right)^{1/x} = \left(\frac{1}{\sqrt{x^2+1} - x} \cdot \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} + x}\right)^{1/x} =$$

$x^2+1-x^2 = 1$

$$= \left(\frac{\sqrt{x^2+1} + x}{1} \right)^{1/x} = f(x)$$

Adiii)
$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \exp\left(\frac{\ln\left(\sqrt{x^2+1} + x\right)}{x} \right)$$

$$\lim_{x \to 0} \frac{\ln\left(\sqrt{x^2+1}+x\right)}{x} = \underbrace{\lim_{x \to 0} \frac{\ln\left(\sqrt{x^2+1}+x\right)}{\sqrt{x^2+1}+x-1}}_{1} \cdot$$

(VOLSF (P))

$$\cdot \lim_{x \to 0} \frac{\sqrt{x^2+1}+x-1}{x} = 1 + \lim_{x \to 0} \frac{\sqrt{x^2+1}-1}{x} =$$

$$= 1 + \lim_{x \to 0} \frac{x^2+1-1}{x\left(\sqrt{x^2+1}+1\right)} = 1 + \lim_{x \to 0} \frac{x}{\sqrt{x^2+1}+1} = 1$$

VOLSF(S)

$$\Rightarrow \quad \lim_{x\to 0} f(x) = \exp(1) = e$$

Potřebuji dodefinovat $f(0) = e$.

Ad VOLSF

$$\lim_{x\to 0} \frac{\ln\left(\sqrt{x^2+1}+x\right)}{\sqrt{x^2+1}+x-1} = 1$$

plyne z limity $\quad \lim_{y\to 1} \frac{\ln y}{y-1} = 1$

$$y = \sqrt{x^2+1}+x \;\begin{matrix} > 1, & \text{pro } x > 0 \\ < 1, & \text{pro } x < 0 \end{matrix}$$

$\Rightarrow \quad y = y(x)$ splňuje pod. (P).

Limity funkcí podruhé

$$\lim_{x\to\infty} x\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right) =$$

$$= \lim_{x\to\infty} \frac{x\left(\overbrace{(x^2+1)-(x^2-1)}^{2}\right)}{\sqrt{x^2+1}+\sqrt{x^2-1}} =$$

$$= \lim_{x\to\infty} \frac{2}{\sqrt{1+x^{-2}}+\sqrt{1-x^{-2}}} =$$

AL+VOLSF(S)

$$= \frac{2}{\sqrt{1+\lim\limits_{x\to\infty} x^{-2}}+\sqrt{1-\lim\limits_{x\to\infty} x^{-2}}} = \frac{2}{1+1}$$

$$= \underline{1}$$

L'Hospitalovo pravidlo

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Předpoklady: 1) PS má smysl

2) $g(x) \to 0$, $f(x) \to 0$ pro $x \to a$

(případ "$\frac{0}{0}$")

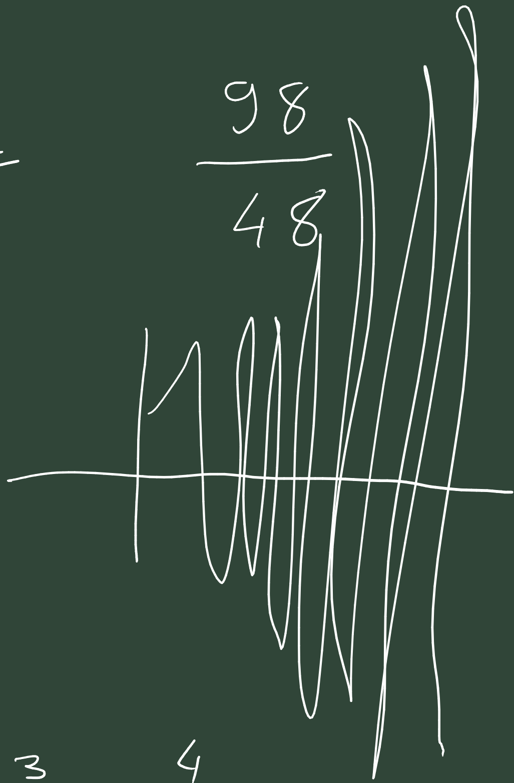
NEBO

$|g(x)| \to \infty$, $x \to a$ (případ "$\frac{\text{NĚCO}}{\infty}$")

Příklad

$$\lim_{x\to 1} \frac{\overset{\nearrow 0}{x^{100}-2x+1}}{\underset{\searrow 0}{x^{50}-2x+1}} \overset{L'H\ \text{"}\frac{0}{0}\text{"}}{=}$$

$$= \lim_{x\to 1} \frac{100x^{99}-2}{50x^{49}-2} = \frac{98}{48}$$



Příklad

$$\lim_{x\to\infty} \frac{\sin x^4}{x} = 0$$

$$\lim_{x\to\infty} \frac{\sin x^4}{x} \overset{L'H\ \text{"}\frac{\text{NĚCO}}{\infty}\text{"}}{\cancel{=}} \lim_{x\to\infty} \frac{4x^3\cos x^4}{1}$$

limita neexistuje

Příklad

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\overset{\to -\infty}{\ln x}}{\underset{\to +\infty}{\frac{1}{x}}} \overset{L'H\ \frac{\text{NEČU}}{\infty}}{=}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -\lim_{x\to 0^+} x = 0$$

Příklad

$$\lim_{x\to 0^+} \frac{\overset{\to 1}{\arccos x}}{x} \cancel{\overset{L'H}{=}} \lim_{x\to 0^+} \frac{\frac{-1}{\sqrt{1-x^2}}}{1}$$

$$\lim_{x\to 0^+} \frac{\arccos x}{x} = +\infty \qquad \lim_{x\to 0^+} \frac{\frac{-1}{\sqrt{1-x^2}}}{1} = -1$$

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} \overset{L'H\ \ \text{"}\frac{\text{NĚCO}}{\infty}\text{"}}{=} \lim_{x\to\infty} \frac{\frac{1/2}{\sqrt{x^2+1}}\cdot 2x}{1} =$$

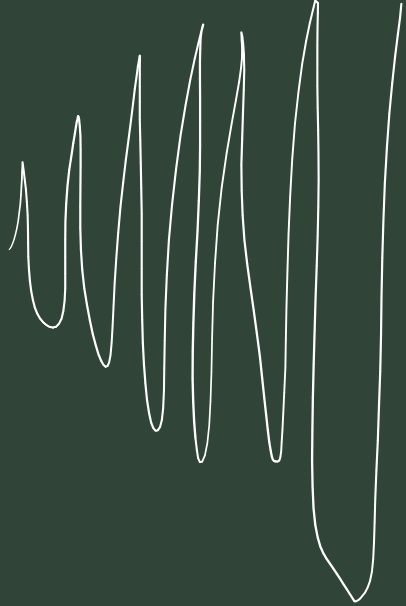
$$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} \overset{L'H\ \ \text{"}\frac{\text{NĚCO}}{\infty}\text{"}}{=}$$

$$= \lim_{x\to\infty} \frac{1}{\frac{1/2}{\sqrt{x^2+1}}\cdot 2x} = \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$$

$$= \lim_{x\to\infty} \frac{\cancel{x}\sqrt{1+\frac{1}{x^2}}}{\cancel{x}} = 1$$

$$\lim_{x \to \infty} x \cos x$$

limite neex.

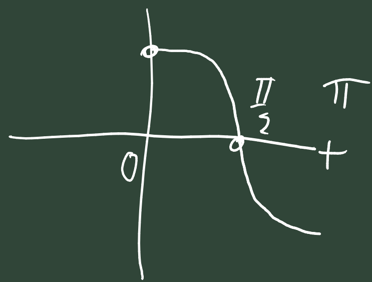


SPOREM

Nechť $\lim_{x \to \infty} x \cos x = A$

$\xrightarrow{\text{Heine}}$ $\lim_{n \to \infty} x_n \cos x_n = A$

$\forall$ posloupnost $x_n \to \infty$



Volba $x_n = 2n\pi$

pak $A = \lim_{n \to \infty} 2n\pi \cdot 1 = +\infty$

Volba $x_n = (2n+1)\pi$

pak $A = \lim_{n \to \infty} 2n\pi(-1) = -\infty$ ↯

prestávka

pokrač. 14:05

Pr.

$$\lim_{x\to 0} \frac{\operatorname{tg} x - x}{x - \sin x} = ?$$

L'H "$\frac{0}{0}$"

$$= \lim_{x\to 0} \frac{\frac{1}{\cos^2 x} - 1}{1 - \cos x} = \underbrace{\lim_{x\to 0} \frac{1}{\cos^2 x}}_{=1} \cdot$$

$$\cdot \lim_{x\to 0} \frac{1-\cos^2 x}{1-\cos x} = \lim_{x\to 0} (1+\cos x) = 2$$

Symboly $O, o, \sim$

Def. f, g funkce def. na okolí $a \in \mathbb{R} \cup \{\pm\infty\}$

i) $f(x) = O(g(x))$, pro $x \to a$

ii) $f(x) = o(g(x))$, —||—

iii) $f(x) \sim g(x)$, —||—

Znamená

i) $\exists C > 0$
$\exists \varepsilon > 0$ tak, že $\forall x \in P(a, \varepsilon)$

$$f(x) \leq C\, g(x)$$

ii) $\lim_{x \to a} \frac{f(x)}{g(x)} = 0$

iii) $\lim_{x \to a} \frac{f(x)}{g(x)} = 1$

$$f(x) = \frac{x^{100} - 2x + 1}{x^{50} - 2x + 1}, \quad x \to \infty$$

chci najít nejmenší $n \in \mathbb{N}$ tak, že

$$f(x) = O(x^n)$$

$$f(x) = \frac{x^{100}}{x^{50}} \left(\frac{1 - 2x^{-99} + x^{-100}}{1 - 2x^{-49} + x^{-50}} \right)$$

$$f(x) = x^{50} \underbrace{\left(\frac{1 - 2x^{-99} + x^{-100}}{1 + 2x^{-49} + x^{-50}} \right)}_{g(x)}$$

na okolí ∞

$$g(x) < 2$$

$$2 = C$$

$\rightarrow$

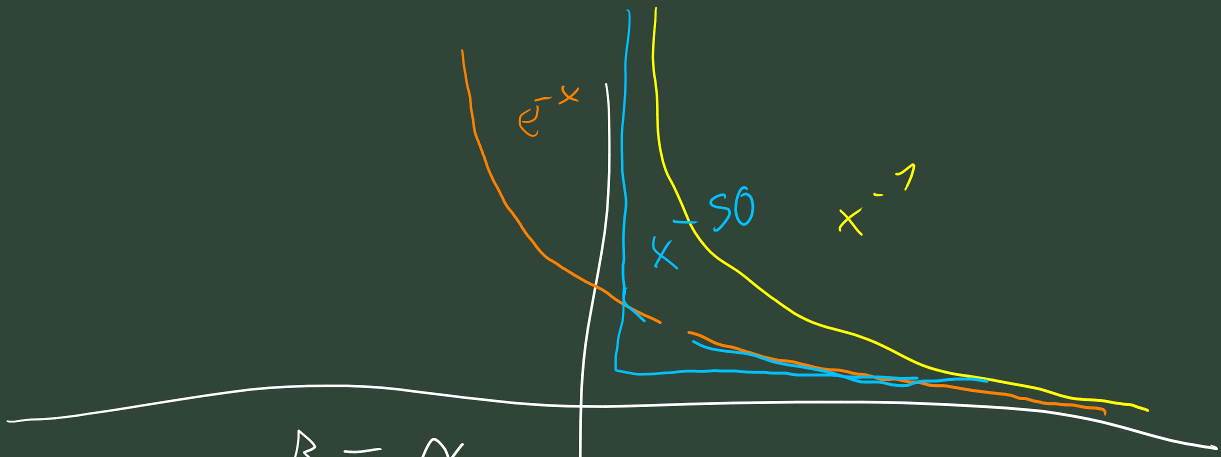
$$f(x) = O\left(x^{50}\right)$$

$$f(x) = o\left(x^{50+\varepsilon}\right), \quad \forall \varepsilon > 0$$

$$f(x) \sim x^{50}$$

Př:

$$e^{-x} = o(x^{\alpha}), \quad x \to \infty$$
$$\forall \alpha < 0$$



$\beta = -\alpha > 0$

Najdu $n \in \mathbb{N}$: $n > \beta$

$$\lim_{x\to\infty} \frac{e^{-x}}{x^{\alpha}} = \lim_{x\to\infty} x^{\beta} e^{-x} = \lim_{x\to\infty} \frac{x^{\beta}}{e^{x}} =$$

$$\overset{\text{L'H } \frac{\text{"NEOO"}}{\infty}}{=} \lim_{x\to\infty} \frac{\beta x^{\beta-1}}{e^{x}} = \lim_{x\to\infty} \frac{\beta(\beta-1)\cdots(\beta-n+1)\overbrace{x^{\beta-n}}^{\to 0}}{\underbrace{e^{x}}_{\to\infty}} =$$

$$\overset{\text{AL}}{=} \frac{0}{+\infty} = 0$$

- $$\underbrace{\lim_{x\to\infty} \frac{p(x)}{e^x} = 0}_{\forall p \text{ polynom}}$$

$$p(x) = o(e^x)$$
$$x \to \infty$$

- $$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = 0$$

$$\ln x = o\left(\frac{1}{x}\right), \quad x \to 0^+$$

- $$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\Longleftrightarrow \qquad \sin x \sim x \quad \text{pro } x \to 0$$

Limity posloupností

$$a_n = \frac{n^2+1}{\sqrt{n^4-6n^2+7}}, \quad n \in \mathbb{N}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\cancel{n^2}\left(1+\overbrace{\frac{1}{n^2}}^{\to 0}\right)}{\cancel{n^2}\sqrt{1-\underbrace{\frac{6}{n^2}}_{\to 0}+\underbrace{\frac{7}{n^4}}_{\to 0}}} =$$

$$= \frac{1+0}{\sqrt{1-0+0}} = 1$$

$$\lim_{n\to\infty} \underbrace{\frac{a^n}{n!}}_{b_n} , \qquad a \in \mathbb{R}\setminus\{0\}$$

$$\frac{b_{n+1}}{b_n} = \frac{\frac{a^{n+1}}{(n+1)!}}{\frac{a^n}{n!}} = \frac{a}{n+1} \xrightarrow[n\to\infty]{} 0$$

$$\Rightarrow \quad \exists\, n_0 \in \mathbb{N} : \quad \frac{b_{n+1}}{b_n} < \frac{1}{2} , \ \forall n \geq n_0$$

$$\Rightarrow \quad b_{n_0+1} < \frac{1}{2} b_{n_0}$$

$$b_{n_0+2} < \frac{1}{2} b_{n_0+1} \quad \ldots$$

$$\Rightarrow 0 < b_{n_0+m} < \underbrace{\left(\frac{1}{2}\right)^m}_{\to 0} b_{n_0}$$

Z věty o dvou strážnících:

$$\lim_{m\to\infty} b_{n_0+m} = 0$$

$$\parallel$$

$$\lim_{n\to\infty} b_n$$

<u>Lemma</u> (podílové krit. pro posloupnosti)

Je-li a_n posloupnost (v $\mathbb{C}$)

a platí-li: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \mathcal{K} < 1$

potom $a_n \to 0$ pro $n \to \infty$

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} \exp\left(\frac{\ln n}{n}\right) =$$

$$\overset{\text{Heine}}{=} \lim_{x\to\infty} \exp\left(\frac{\ln x}{x}\right) \overset{\text{VOLSF(S)}}{=}$$

$$= \exp\left(\lim_{x\to\infty} \frac{\ln x}{x}\right) =$$

$$\overset{\text{L'H "VĚCO/}\infty\text{"}}{=} \exp\left(\underbrace{\lim_{x\to\infty} \frac{\frac{1}{x}}{1}}_{0}\right) = 1$$

Limity rekurentních posloupností

$$a_{n+1} = f(a_n) \quad \forall n \in \mathbb{N}_0, \quad a_0 \in \mathbb{R}$$

f spojité na $\mathbb{R}$

$$\lim_{n\to\infty} a_n = \, ?$$

i) dokážu, že limita $\lim\limits_{n\to\infty} a_n$ existuje a je vlastní

ii) $\exists A = \lim\limits_{n\to\infty} a_n \in \mathbb{R}$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} f(a_n)$$

$$\boxed{A = f(A)}$$

určím A

Př.

$$a_0 > 0 \quad , \quad c > 0$$

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right) , \quad n \in \mathbb{N}_0$$

$$\lim_{n\to\infty} a_n = ?$$

$$f(x) = \frac{1}{2}\left(x + \frac{c}{x}\right) , \quad x > 0$$

$$a_{n+1} = f(a_n)$$

Funkce f je spojitá na $(0, +\infty)$

Potrebuji

1) ukázat, že $\lim_{n\to\infty} a_n = A$ existuje

2) A spočítat z $f(A) = A$

Ad 2) Nechť $A > 0$, $f(A) = A$

potom

$$A = \frac{1}{2}\left(A + \frac{c}{A}\right)$$

$$A = \frac{c}{A} \quad \Big| \cdot A$$

$$A^2 = c$$

$$A = \sqrt{c}$$

2)

$$f(x) = \frac{1}{2}\left(x + \frac{c}{x}\right), \quad x > 0$$

$$\boxed{f(f(f\cdots(f(a_0))))) \to ?}$$

$$f(x) - x = \underbrace{\frac{1}{2}\left(\frac{c}{x} - x\right)}_{g(x)}$$

$$g'(x) = -\frac{1}{2}\left(\frac{c}{x^2} + 1\right) < 0$$

$x \in (0, +\infty)$

$g(x) < 0$ на $x \in (\sqrt{c}, +\infty)$

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right)$$

$$a_n > \sqrt{c}, \quad \forall n > 0$$

$$a_{n+1} = f(a_n) = \frac{a_n}{2} + \frac{1}{2}\frac{c}{a_n} \geq$$

$$\geq \sqrt{a_n \frac{c}{a_n}} = \sqrt{c}$$

$$a_1 \geq a_2 \geq a_3 \geq a_4 \geq \dots \geq \sqrt{c}$$

monotónní posloupnost

$\Rightarrow \lim_{n \to \infty} a_n$ existuje

Např. $\sqrt{2}$

Babylonská metoda

$$a_0 = 1 \quad , \quad c = 2$$

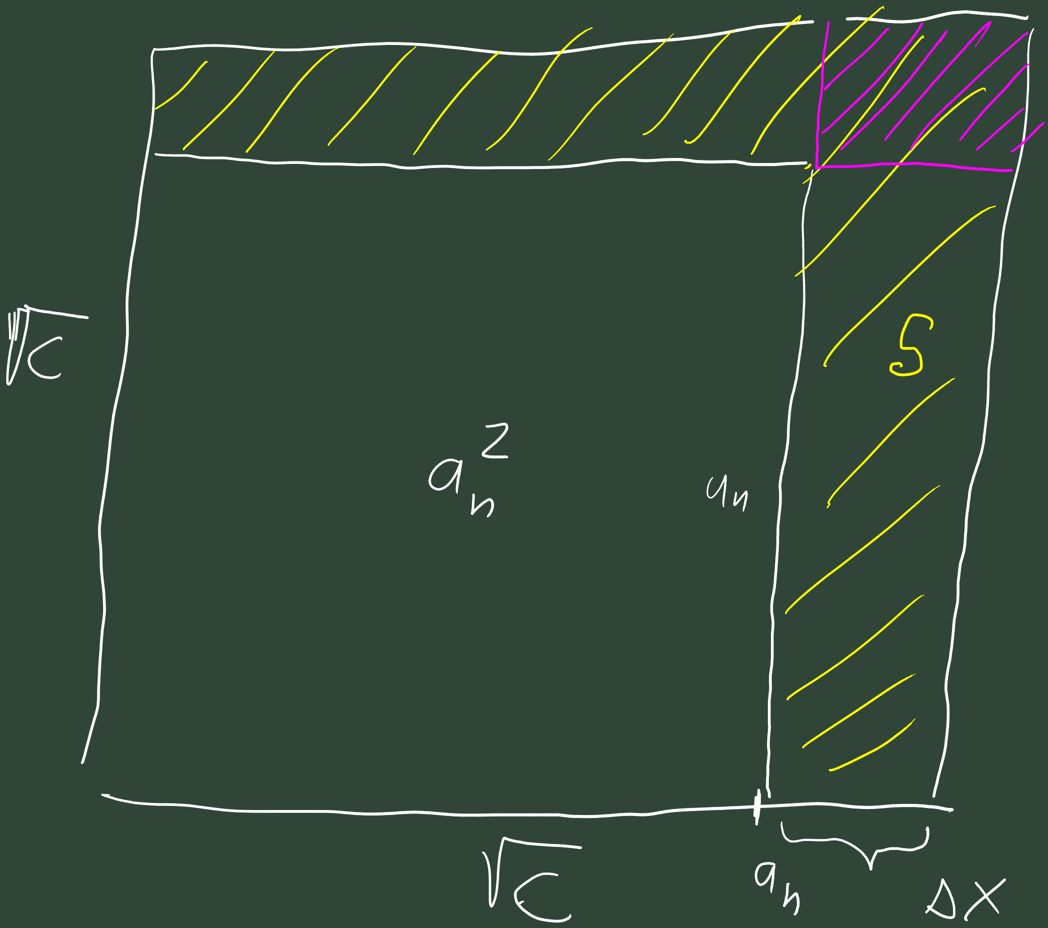
$$a_1 = \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2}$$

$$a_2 = \frac{1}{2}\left(\frac{3}{2} + \frac{4}{3}\right) = \frac{17}{18}$$

$$a_3 = \frac{1}{2} \cdots$$

$$\downarrow$$

$$\sqrt{2}$$



$$S \approx 2\Delta x \cdot a_n$$

$$\Delta x \approx \frac{S}{2a_n} = \frac{C - a_n^2}{2a_n}$$

$$a_{n+1} = a_n + \frac{C - a_n^2}{2a_n} = \frac{1}{2}\left(a_n + \frac{C}{a_n}\right)$$