VOLSF

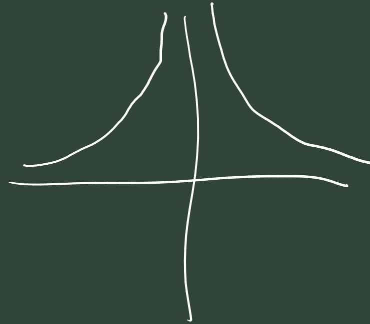
$$\lim_{x \to x_0} f(\varphi(x)) \overset{?}{=} \lim_{y \to y_0} f(y)$$

(S) f je spojitá v y_0

(P) $\exists\, \delta > 0$ tak že $\varphi(x) \neq y_0$ na $P(x_0, \delta)$

$$\lim_{x \to x_0} f(x) = +\infty \quad \left(\text{vesp. } -\infty\right)$$

$$\lim_{x \to 0} \frac{1}{x^2} = +\infty$$



$\frac{0}{0}$, $\infty - \infty$, $0 \cdot \infty$

"ZNÁMÉ" LIMITY

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}, \quad \boxed{\lim_{x\to 0} \frac{e^x - 1}{x} = 1}$$

i) $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} = 1$, ii) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$

iii) $\lim\limits_{x\to 0} \dfrac{\tan x}{x} = 1$, iv) $\lim\limits_{x\to 0} \dfrac{\operatorname{arctan} x}{x} = 1$

Ad i) $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} =$

VOLSF (P)

$1 + x = e^y$

$\ln(1+x) = y = \varphi(x)$

$\lim\limits_{x\to 0} \varphi(x) = \lim\limits_{x\to 0} \ln(1+x) = 0$

Ad i)

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} =$$

VOLSF (P)

$1 + x = e^y$

$\ln(1+x) = y = \varphi(x)$

$\lim\limits_{x\to 0} \varphi(x) = \lim\limits_{x\to 0} \ln(1+x) = 0$

předpoklad (P) splněn

(φ je rostoucí)

$f(y) = \dfrac{y}{e^y - 1}$

$$= \lim_{y\to 0} \frac{\ln(e^y)}{e^y - 1} = \lim_{y\to 0} \frac{y}{e^y - 1} = 1$$

Ad ii)

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} \cdot \frac{1+\cos x}{1+\cos x} =$$

$$= \lim_{x\to 0} \frac{1-\cos^2 x}{x^2(1+\cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x^2}\cdot\frac{1}{1+\cos x} =$$

$$\overset{(AL)}{=} \left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2 \lim_{x\to 0} \frac{1}{1+\cos x} =$$

$$= 1^2 \cdot \frac{1}{2} = \frac{1}{2}$$

Ad iii)

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \underbrace{\frac{\sin x}{x}}_{\downarrow \atop 1} \underbrace{\frac{1}{\cos x}}_{\downarrow \atop 1} = 1$$

Ad iv)

$$\lim_{x\to 0} \frac{\arctan x}{x} = \left| \begin{array}{l} \text{VOLSF (P)} \\ x = \tan y \end{array} \right| = \lim_{y\to 0} \frac{y}{\tan y} = 1$$

DERIVACE

Def. $f : D(f) \longrightarrow \mathbb{R}$, $x_0 \in D(f)$

Pak f má v x_0 derivaci $f'(x_0)$ jestliže

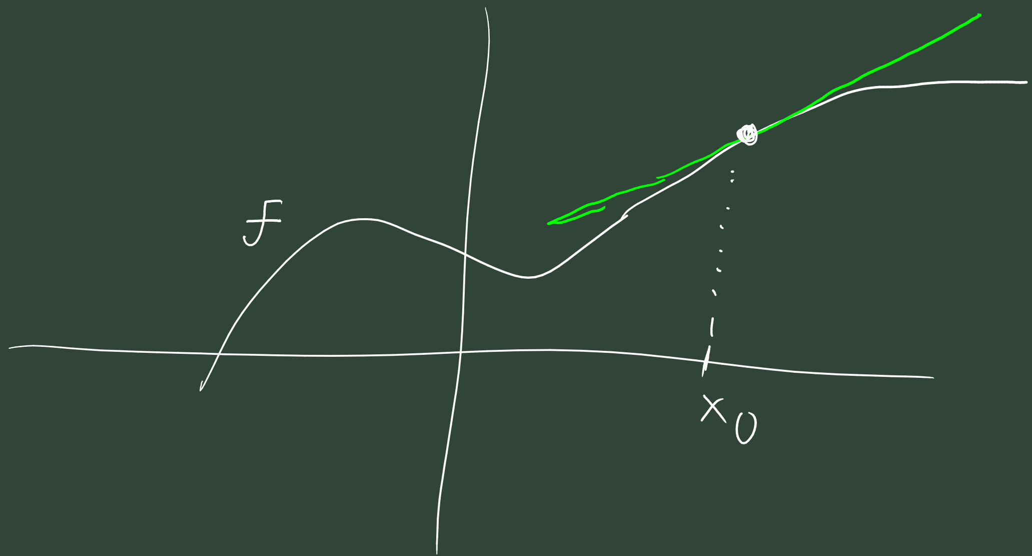
$$f'(x_0) \overset{def}{=} \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Neformálně: když x je "blízko" x_0, potom

$$f'(x_0) \approx \frac{f(x) - f(x_0)}{x - x_0}$$

$$f(x) \approx f(x_0) + (x - x_0)\, f'(x_0)$$



* Když f má derivaci v x_0 pak je v x_0 spojitá.

* $I \subset \mathbb{R}$ ot. interval, $f'(x) > 0$, $\forall x \in I$ pak f je vyzeé rostoucí na I

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\sin x - \sin 0}{x - 0} =$$

$$= \sin' 0$$

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = \lim_{h \to 0} \frac{\ln(1+h) - \ln 1}{h} =$$

$$= \ln'(1)$$

A) Počítání derivací z definice

1) Dokažte, že $\cos' x = -\sin x$

Nacht $x \in \mathbb{R}$

$$\lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} =$$

$$= \lim_{h\to 0} \frac{\cos x \cosh h - \sin x \sinh h - \cos x}{h}$$

AL

$$= \lim_{h\to 0} \cos x \cdot \frac{\cosh h - 1}{h} - \lim_{h\to 0} \sin x \frac{\sinh h}{h}$$

$$= \cos x \underbrace{\lim_{h\to 0} \frac{\cosh h - 1}{h^2}}_{-\frac{1}{2}} \underbrace{\lim_{h\to 0} h}_{0} - \sin x \underbrace{\lim_{h\to 0} \frac{\sinh h}{h}}_{1}$$

$$= 0 - \sin x = -\sin x$$

Spočítat derivaci funkce $f(x) = a^x$

kde $a \in (0, +\infty)$

$$f(x) = \exp(x \ln a)$$

$x \in \mathbb{R}$

$$\lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h \to 0} \frac{a^h - 1}{h} =$$

$$= (\ln a) a^x \lim_{h \to 0} \frac{e^{h \ln a} - 1}{h \ln a} =$$

VOLSF (P)

$y = h \ln a$

$\lim_{h \to 0} h \ln a = 0$

zároveň $h \ln a \neq 0$

pro $h \neq 0$

$$= a^x \ln a \lim_{y \to 0} \frac{e^y - 1}{y} = a^x \ln a$$

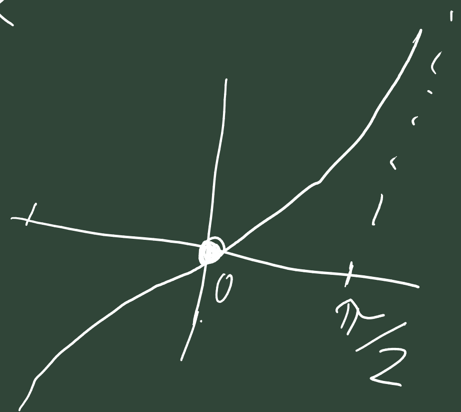
$$(a^x)' = a^x \ln a$$

$$a = e \longrightarrow (e^x)' = e^x \underbrace{\ln e}_{1} = e^x > 0$$

$a < e$, $\ln a < 1$

$a > e$, $\ln a > 1$

$$(\tan x)' = \frac{1}{\cos^2 x} = 1 + \tan^2 x > 0$$



Židovské problémy

(židovské otázky
Jewish problems)

Př. Najděte všechny funkce $F:\mathbb{R}\to\mathbb{R}$ splňující:

$$\forall x_1, x_2 \in \mathbb{R}: \quad \underbrace{F(x_1)-F(x_2)}_{y} \le (x_1-x_2)^2$$

Prohodím x_1, x_2

$$\left(\begin{array}{c} y \le RS \\ -y \le RS \\ \hline |y| \le RS \end{array}\right)$$

$$\forall x_1, x_2 \in \mathbb{R} \quad F(x_2)-F(x_1) \le (x_2-x_1)^2 = (x_1-x_2)^2$$

tedy

$$\forall x_1, x_2 \in \mathbb{R}: \quad |F(x_1)-F(x_2)| \le |x_1-x_2|^2$$

Předpokládejme $x_1 \neq x_2$

Pak vydělím a dostanu

$$0 \le \left| \frac{F(x_1) - F(x_2)}{x_1 - x_2} \right| \le |x_1 - x_2|$$

x_2 pevné, limita $x_1 \longrightarrow x_2$

Věta o dvou strážnících

policajtech

Squeeze theorem

$$\begin{array}{cccccc} & a(x) & \le & b(x) & \le & c(x) \\ x \to x_0 & \downarrow & & & & \downarrow \\ & y & & & & y \end{array}$$

$$\Longrightarrow \quad \lim_{x \to x_0} b(x) = y$$

$$\overset{\nearrow 0}{0} \le \left| \frac{F(x_1) - F(x_2)}{x_1 - x_2} \right| \le |x_1 \overset{\overset{0}{\uparrow}}{-} x_2|$$

Věta o dvou strážnících, $\lim x_1 \to x_2$

$$\lim_{x_1 \to x_2} \left| \frac{F(x_1) - F(x_2)}{x_1 - x_2} \right| = 0$$

$$-|\dots| \le \frac{F(x_1) - F(x_2)}{x_1 - x_2} \le |\dots|$$

$$F'(x_2) = 0$$

x_2 bylo lib. $\Rightarrow$ $F' \equiv 0$ na $\mathbb{R}$

$\Rightarrow$ $F = \text{konst.}$

Naopak, je-li

$$F \equiv F_0$$

Pak splňuje:

$$\forall x_1, x_2 \in \mathbb{R}: \quad \underbrace{F(x_1)}_{F_0} - \underbrace{F(x_2)}_{F_0} \leq (x_1 - x_2)^2$$

Závěr: Funkce F splňuje nerovnost ze zadání, právě když je konstantní.

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h^2}$$

Přestávka, pokrač. v 14:30.

B) Mechanické derivování

Věty: * derivace součtu

$$\frac{d}{dx}\left(f(x)+g(x)\right) = f'(x)+g'(x)$$

$$\frac{d}{dx}\left(\underset{\substack{\uparrow \\ \text{konst}}}{\lambda} f(x)\right) = \lambda f'(x)$$

* derivace součinu

$$\frac{d}{dx}\left(f(x)\,g(x)\right) = f'(x)g(x) + f(x)g'(x)$$

* derivace podílu

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

* derivace složené funkce

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$$

Pokud PS má smysl.

<u>Př.</u>

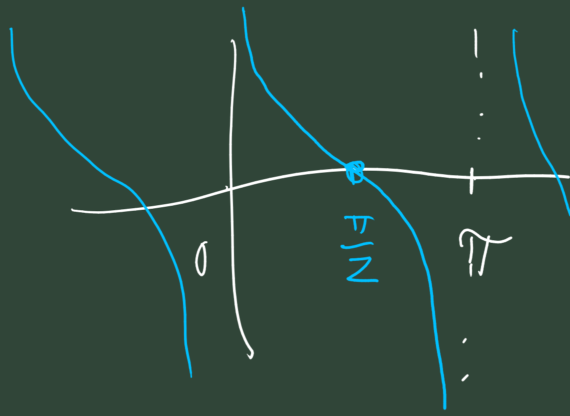
$$f(x) = x \ln x, \quad x \in (0, +\infty)$$

$$f'(x) = (x)' \ln x + x (\ln x)'$$

$$= \ln x + x \cdot \frac{1}{x}$$

$$= 1 + \ln x$$

$$\frac{d}{dx}\left(\operatorname{cotan} x\right) = \, ? \quad , \quad x \in \mathbb{R} \setminus \pi\mathbb{Z}$$



$$\operatorname{cotan} x = \frac{\cos x}{\sin x}$$

$$\frac{d}{dx}\left(\operatorname{cotan} x\right) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) =$$

derivace podílu

$$= \frac{\cos' x \, \sin x - \cos x \, \sin' x}{\sin^2 x}$$

tabulka

$$= \frac{-\sin x \, \sin x - \cos x \, \cos x}{\sin^2 x}$$

$$= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x}$$

$$= -\frac{1}{\sin^2 x} = -1 - \operatorname{cotan}^2 x$$

$\operatorname{cotan}: (0,\pi) \to \mathbb{R}$

$\operatorname{arccotan}: \mathbb{R} \to (0,\pi)$

<u>Př.</u>

$$\frac{d}{dx}(\operatorname{arccotan} x) = ?$$

Derivace inverzní fce:

$f : I \xrightarrow{\text{prosté a na}} J$, f je ryze monotónní

otevřený interval (for I and J)

Pak $f^{-1} : J \to I$

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{f'(f^{-1}(y))}, \text{ pokud } f'(f^{-1}(y)) \neq 0$$

$$\frac{d}{dx}(\cot x) = -1 - \underbrace{\cot^2 x}_{y}$$

$$\rightarrow \frac{d}{dy}(\operatorname{arccot} y) = -\frac{1}{1+y^2}$$

Spočtěte derivace funkcí

i) $\sqrt{x^2+1}$

ii) $\dfrac{x+1}{x-1}$

iii) $\sin(\sin(\sin x))$

$-1 \leq \sin x \leq 1$

iv) $\ln(\ln(\overset{\vee}{\sin x}))$

Řešení

i) $\dfrac{\cancel{x}}{\sqrt{\cancel{x}^2+1}}$ $x \in \mathbb{R}$

ii) $-\dfrac{2}{(x-1)^2}$ $x \neq 1$

iii) $\cos(\sin(\sin x)) \cdot \cos(\sin x) \cdot \cos x \quad , x \in \mathbb{R}$

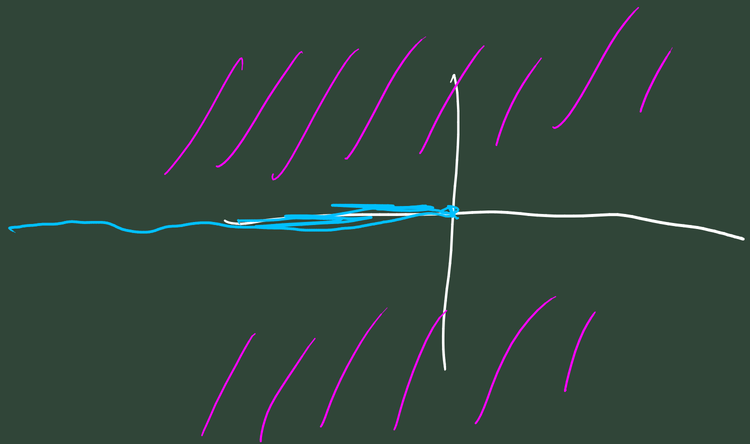
$$f'(x) = \frac{1}{\ln(\sin x)} \cdot \frac{1}{\sin x} \cdot \cos x$$

$x \neq k\pi$, $k \in \mathbb{Z}$

$\sin x > 0$

iv) zadaná funkce má prázdný def. obor

$\Rightarrow$ f' má prázdný def. obor



Výpočet derivace pomocí limity derivací

Věta: Nechť f je reálná funkce definovaná $(\delta>0)$ na okolí $[x_0, x_0+\delta)$ a nechť má f derivaci f' na $(x_0, x_0+\delta)$ a nechť existuje vlastní limita

$$\lim_{x\to x_0^+} f'(x) = A$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$x\in(-1,1)$$

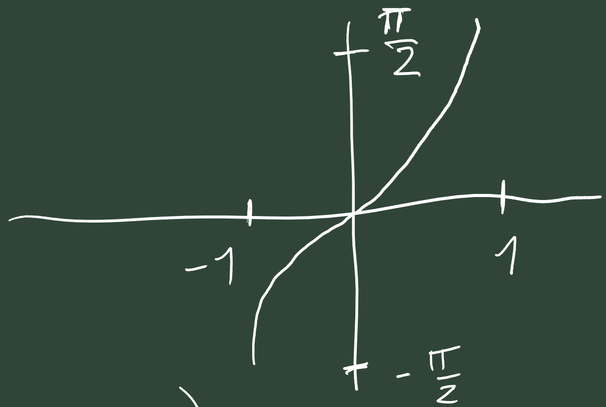
Potom

$$f'_+(x_0) = A$$

(Analogie platí i pro $f'_-(x_0)$, $f'(x_0)$)

Př.

Spočítejte derivaci funkce



$$f(x) = x \arcsin\left(\frac{x^2-1}{x^2+1}\right), \quad x \in \mathbb{R}$$

$$\left(\frac{x^2-1}{x^2+1} = \frac{x^2+1-2}{x^2+1} = 1 - \frac{2}{x^2+1} \begin{matrix}\le 1\\ \ge -1\end{matrix} \right.$$

$$\left. 0 \le \frac{2}{x^2+1} \le 2 \right)$$

$$f'(x) = x' \arcsin(\cdots) + \frac{x}{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)}} \left(\frac{x^2-1}{x^2+1}\right)'$$

$$= x' \arcsin(\cdots) + \frac{x}{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)}} \cdot \frac{(x^2-1)'(x^2+1) - (x^2-1)(x^2+1)'}{(x^2+1)^2}$$

$$= x' \arcsin\left(\cdots\right) + \frac{x}{\sqrt{1-\frac{x^2-1}{x^2+1}}} \cdot \frac{(x^2-1)'(x^2+1)-(x^2-1)(x^2+1)'}{(x^2+1)^2}$$

$$4x$$

$$= x' \arcsin\left(\cdots\right) + \frac{x}{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)^2}} \cdot \frac{2x(x^2+1)-2x(x^2-1)}{(x^2+1)^2}$$

$$\left(1-\left(\frac{x^2-1}{x^2+1}\right)^2 = \frac{(x^2+1)^2-(x^2-1)^2}{(x^2+1)^2} = \frac{4x^2}{(x^2+1)^2}\right)$$

$$= \arcsin\left(\ldots\right) + \frac{x}{\frac{2|x|}{\cancel{x^2+1}}} \cdot \frac{4x}{(x^2+1)^{\cancel{2}}} =$$

$$= \arcsin\left(\ldots\right) + \frac{2|x|}{x^2+1}, \quad \text{platí } x \neq 0$$

$$\underbrace{\frac{x^2-1}{x^2+1}} \in (-1, 1)$$

$$1 - \frac{2}{x^2+1} \in (-1, 1)$$

$$\lim_{x \to 0} \left(\arcsin\left(\frac{x^2-1}{x^2+1}\right) + \frac{2|x|}{x^2+1} \right)$$

$$= \arcsin(-1) + \frac{2 \cdot 0}{0^2+1}$$

$$= -\frac{\pi}{2}$$

$$\Rightarrow$$

$$f'(0) = -\frac{\pi}{2}$$