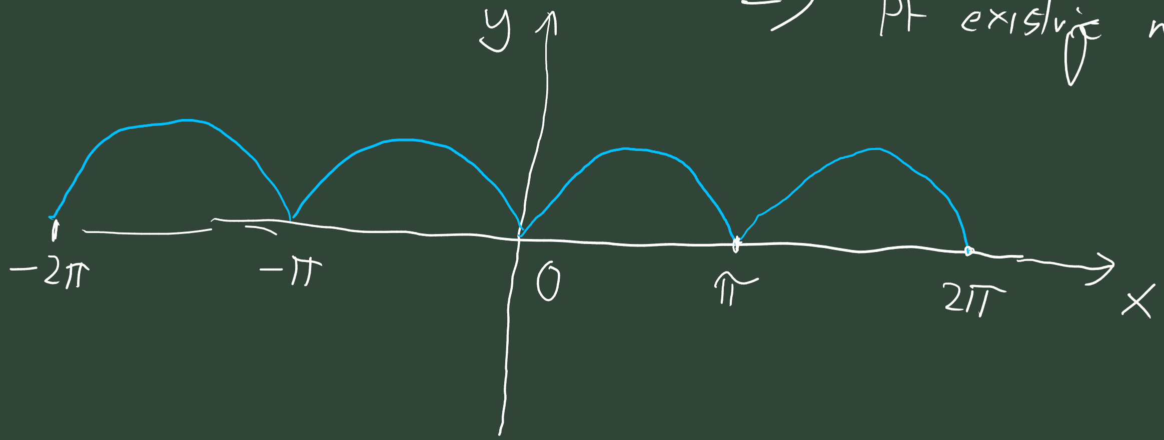
Lepení

Př.

$$\int |\sin x| \, dx = ?$$

$f(x) = |\sin x|$ je spojitá na $\mathbb{R}$

$\Rightarrow$ PF existuje na $\mathbb{R}$



$$I_k = (k\pi, (k+1)\pi), \quad k \in \mathbb{Z}$$

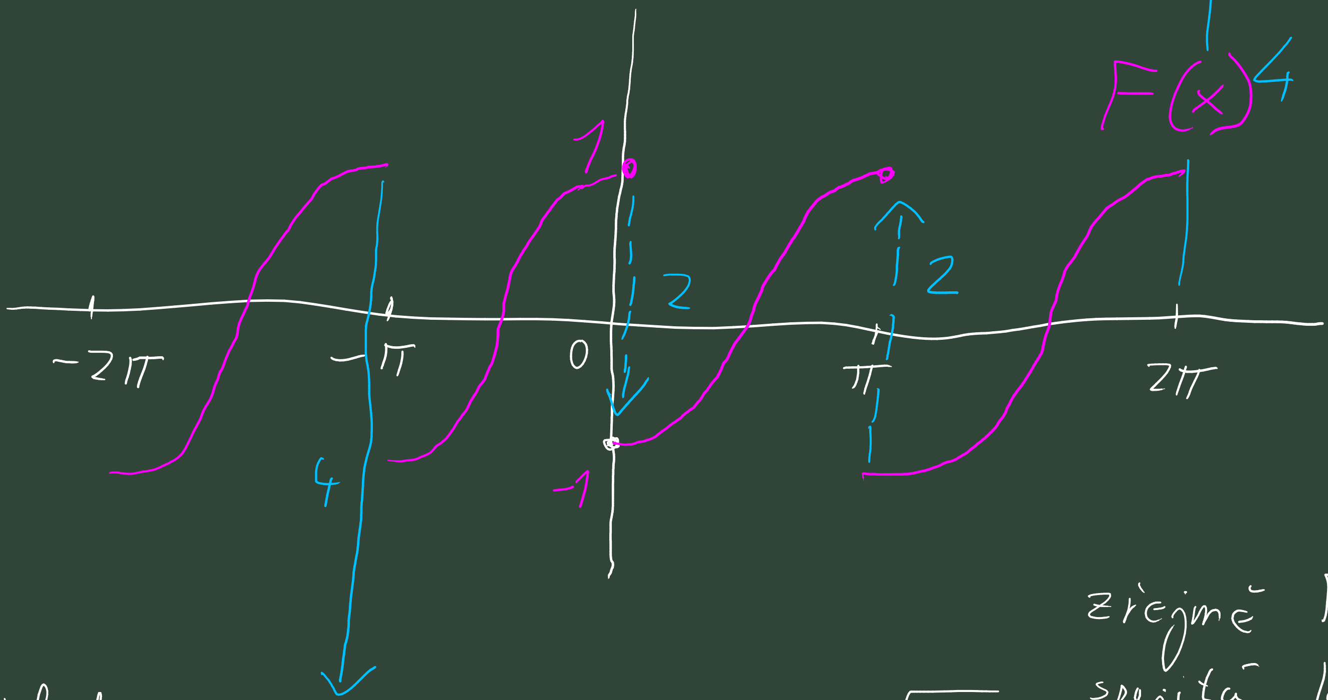
potom

$$f(x) = \begin{cases} \sin x, & x \in I_k, \ k \text{ je sudé} \\ -\sin x, & x \in I_k, \ k \text{ je liché} \end{cases}$$

Najdu PF na každém I_k

$$F(x) = -\cos x \quad , \quad x \in I_k \text{, } k \text{ je sudé}$$

$$F(x) = \cos x \quad , \quad x \in I_k \text{, } k \text{ je lichá}$$



zřejmě $\tilde{F}$ bude spojitá, když $C(k) = 2k$

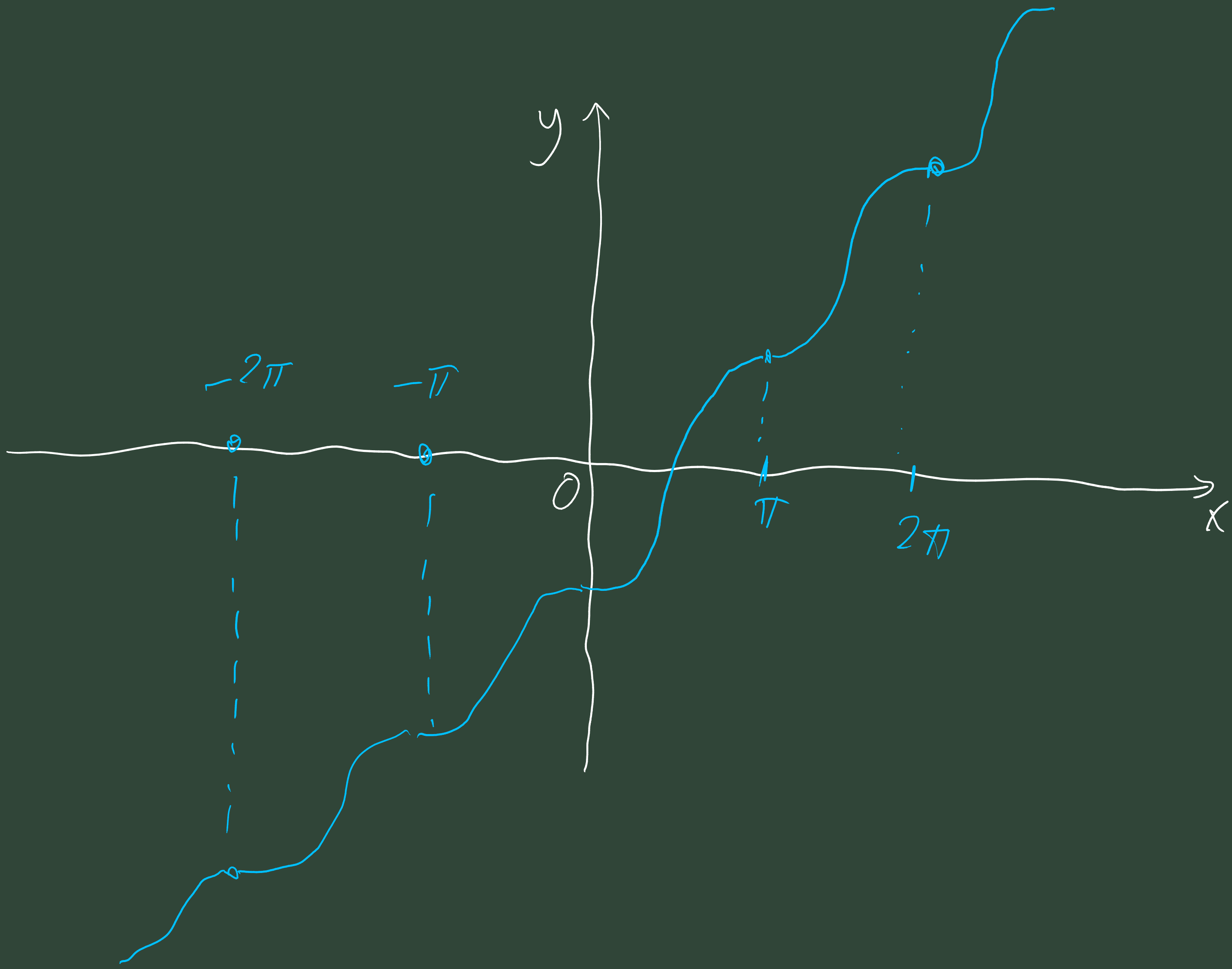
hledám

$$\tilde{F}(x) = F(x) + C(k)$$

$$= F(x) + 2k \quad , \quad x \in I_k$$

$$\int |\sin x|\,dx = \underline{F(x) + 2\left\lfloor \frac{x}{\pi} \right\rfloor + C}$$

$$\left(\begin{array}{l} \text{dodefinuji} \\ \quad F(k\pi) = -1 \end{array} \right)$$



Rozklad na parciální zlomky

$$\int R(x)\,dx \quad , \quad R \ldots \text{racionální funkce}$$

$$\left(R(x) = \frac{P(x)}{Q(x)} \right)$$

polynomy

$$\int \frac{dx}{x^2(x^2+x+1)} = \int \frac{A}{x}\,dx + \int \frac{B}{x^2}\,dx +$$

$$+ \int \frac{Cx+D}{x^2+x+1}\,dx \quad , \quad A, B, C, D \in \mathbb{R}$$

$$\frac{1}{x^2(x^2+x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+x+1}$$

$$\Leftrightarrow$$

$$1 = (Ax+B)(x^2+x+1) + x^2(Cx+D)$$

$$1 = (Ax+B)(x^2+x+1) + x^2(Cx+D)$$

$\boxed{x^3}$: $\quad 0 = A + C \longrightarrow C = 1$

$\boxed{x^2}$: $\quad 0 = A + B + D \longrightarrow D = 0$

$\boxed{x}$: $\quad 0 = A + B \longrightarrow A = -1$

$\boxed{1}$: $\quad \boxed{1 = B}$

$$\int \frac{dx}{x^2(x^2+x+1)} = \int \frac{-1}{x}\,dx + \int \frac{1}{x^2}\,dx + \int \frac{x}{x^2+x+1}\,dx =$$

$$= -\ln|x| - \frac{1}{x} + \int \frac{\frac{1}{2}(2x+1)}{x^2+x+1}\,dx - \int \frac{1/2}{x^2+x+1}\,dx$$

$$= -\ln|x| - \frac{1}{x} + \frac{1}{2}\ln|x^2+x+1| - \int \frac{1/2 \cdot \frac{4}{3}}{\left[\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}\right] \cdot \frac{4}{3}} =$$

$$=$$

$$-\ln|x| - \frac{1}{x} + \frac{1}{2}\ln|x^2+x+1| - \frac{2}{3}\arctan\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)\cdot\frac{\sqrt{3}}{2} + C$$

na $(0,+\infty)$ a na $(-\infty,0)$

Zakrývací metoda

⊕ rychlé

⊖ funguje ve spec. případech

polynomy

$\in \mathbb{N}$

$$\frac{P(x)}{Q(x)(x-\lambda)^n} = \frac{A_1}{x-\lambda} + \frac{A_2}{(x-\lambda)^2} + \dots + \frac{A_n}{(x-\lambda)^n} + \frac{\tilde{P}(x)}{Q(x)}$$

$\in \mathbb{R}$

$$\frac{P(x)}{Q(x)} = A_1(x-\lambda)^{n-1} + A_2(x-\lambda)^{n-2} + \dots + A_n + \frac{\tilde{P}(x)}{Q(x)}(x-\lambda)^n$$

Nechť $Q(\lambda) \neq 0$

$$\frac{P(x)}{Q(x)} = A_1(x-\lambda)^{n-1} + A_2(x-\lambda)^{n-2} + \dots + A_n + \frac{\tilde{P}(x)}{Q(x)}(x-\lambda)^n$$

Nechť $Q(\lambda) \neq 0$

$\lambda = 0$:

$A_n + xA_{n-1} + x^2A_{n-2} + \dots$

$$\boxed{\frac{P(\lambda)}{Q(\lambda)} = A_n}$$

$$\frac{d}{dx^k}\left[x^k A_{n-k}\right] =$$

$$= k!\, A_{n-k}$$

Jak určit ostatní A_k?

<u>Jedna možnost:</u>

$$\frac{\frac{P(x)}{Q(x)} - A_n}{x-\lambda} = A_1(x-\lambda)^{n-2} + A_2(x-\lambda)^{n-3} + \dots + A_{n-1} + \frac{\hat{P}(x)}{Q(x)}(x-\lambda)^{n-1}$$

$$\lim_{x\to\lambda} \frac{\frac{P(x)}{Q(x)} - A_n}{x-\lambda} = A_{n-1}$$

Jiná možnost:

$$\frac{d}{dx}\left[\frac{P(x)}{Q(x)}\right] = A_1(n-1)(x-\lambda)^{n-2} + \cdots + A_{n-1} + \frac{d}{dx}\left[\frac{\hat{P}(x)}{Q(x)}(x-\lambda)^{n}\right]$$

$$\boxed{\left.\frac{d}{dx}\left[\frac{P(x)}{Q(x)}\right]\right|_{x=\lambda} = A_{n-1}}$$

$\vdots$

$$\boxed{\left.\frac{1}{k!}\frac{d^k}{dx^k}\left[\frac{P(x)}{Q(x)}\right]\right|_{x=\lambda} = A_{n-k}}$$

Lze použít jen pokud umím rozdělit jmenovatel $R(x)$ na lin. polynomy $x-\lambda$

např.

$$R(x) = \overline{x^2(x^2+x+1)}^{\,1}$$

① sada 6

$-5x^2+6x+5x^2-6x$

$$\int \frac{x^3+1}{x^3-5x^2+6x}\,dx = \int \frac{x^3+1}{x(x^2-5x+\underset{(-3)\cdot(-2)}{6})}\,dx =$$

$$= \int \frac{x^3+1}{x(x-2)(x-3)}\,dx$$

$$= \int 1\,dx + \int \frac{5x^2-6x+1}{x(x-2)(x-3)}\,dx =$$

$$= x + \int \frac{\overbrace{\left(\frac{5\cdot 0^2-6\cdot 0+1}{(0-2)(0-3)}\right)}^{=\frac{1}{6}}}{x}\,dx + \int \frac{\overbrace{\left(\frac{20-12+1}{2\cdot(-1)}\right)}^{=-\frac{9}{2}}}{x-2}\,dx + \int \frac{\overbrace{\left(\frac{45-18+1}{3\cdot 1}\right)}^{=\frac{28}{3}}}{x-3}\,dx$$

$$= x + \frac{1}{6}\ln|x| - \frac{9}{2}\ln|x-2| + \frac{28}{3}\ln|x-3| + C$$

na intervalech:

$$(-\infty, 0), \quad (0, 2), \quad (2, 3), \quad (3, +\infty)$$

Eulerovy substituce

$$\int \frac{dx}{x\sqrt{x^2-2x-1}} = ?$$

$x^2-2x-1 =$

$= \left(x - \frac{1+\sqrt{2}}{2}\right)\left(x - \frac{1-\sqrt{2}}{2}\right)$

řeším pro

$x \in \left(\frac{1+\sqrt{2}}{2}, +\infty\right)$

$x \in \left(-\infty, \frac{1-\sqrt{2}}{2}\right)$

$$\sqrt{x^2-2x-1} = x+t$$

$$x^2-2x-1 = x^2+2xt+t^2$$

$$-(2+2t)x = t^2+1$$

sub. (2. věta o sub.)

$$\boxed{x = -\frac{1}{2}\,\frac{t^2+1}{t+1}}$$

$t \in (\ldots)$

$t \in (\ldots)$

$$\int \frac{dx}{x\sqrt{x^2-2x-1}} = ?$$

řešim pro
$x \in (1+\sqrt{2}, +\infty)$
$x \in (-\infty, 1-\sqrt{2})$

$$\sqrt{x^2-2x-1} = x+t$$

$$\boxed{x = -\frac{1}{2}\frac{t^2+1}{t+1}}$$

t^2+2t-1

$$dx = -\frac{1}{2}\frac{2t(t+1)-(t^2+1)}{(t+1)^2}\,dt =$$

$$= -\frac{1}{2}\frac{t^2+2t-1}{(t+1)^2}\,dt$$

$$x+t = t-\frac{1}{2}\frac{t^2+1}{t+1} = \frac{1}{2}\frac{2t^2+2t-t^2-1}{t+1} =$$

$$= \frac{1}{2} \quad \frac{t^2+2t-1}{t+1}$$

$$\int \frac{dx}{x\sqrt{x^2-2x-1}} = \int \frac{-\frac{1}{2}\,\frac{\cancel{t^2+2t-1}}{\cancel{(t+1)^2}}\,dt}{-\frac{1}{2}\,\frac{t^2+1}{\cancel{t+1}} \cdot \frac{1}{2} \cdot \frac{\cancel{t^2+2t-1}}{\cancel{t+1}}} =$$

$$= 2\int \frac{dt}{t^2+1} =$$

$$= 2\,\mathrm{arctan}\,t + C$$

$$= 2\,\mathrm{arctan}\left(\sqrt{x^2-2x-1} - x\right) + C$$

na $(1+\sqrt{2}, +\infty)$ a na $(-\infty, 1-\sqrt{2})$

Prestávka: pokračovanie 14:45.

$$\int \frac{x\,dx}{\sqrt{x(1-x)}} = ?$$

$$t = \sqrt{\frac{1-x}{x}} = \sqrt{\frac{1}{x} - 1}$$

$$t^2 = \frac{1}{x} - 1$$

$$\frac{1}{x} = t^2 + 1$$

$$x = \frac{1}{t^2+1} \qquad x \in (0,1)$$

$$t \in (0, +\infty)$$

$$dx = -\frac{2t\,dt}{(t^2+1)^2}$$

$$\int \frac{x\,dx}{\sqrt{\frac{x^2(1-x)}{x}}} = \int \frac{x\,dx}{x\sqrt{\frac{1}{x}-1}} = \int \frac{dx}{\sqrt{\frac{1}{x}-1}} =$$

$$= \int \frac{1}{t} \cdot \frac{-2t\,dt}{(t^2+1)^2} = -2\boxed{\int \frac{dt}{(t^2+1)^2}} =$$

$$= -2 \cdot \frac{1}{2} \operatorname{arctan}(t) - \frac{t}{t^2+1} + C$$

$$= - \operatorname{arctan}\sqrt{\frac{1}{x}-1} - \frac{\sqrt{\frac{1}{x}-1}}{\frac{1}{x}} + C$$

na $(0,1)$

Goniometrické substituce

$$\int R(\sin x, \cos x)\, dx = ?$$

rac. funkce

sub.

$$t = \tan\frac{x}{2}, \qquad x \in (-\pi, \pi) + 2k\pi$$

$$x = 2\arctan t$$

$$dx = \frac{2}{t^2+1}\, dt$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \cos^2\frac{x}{2}\left(1 - t^2\right)$$

$$= \frac{1-t^2}{1+t^2}$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$= 2\cos^2\frac{x}{2}\ t = \frac{2t}{t^2+1}$$

Př.

$$\int \frac{dx}{2\sin x - \cos x + 5} = ?$$

$$2\sin x - \cos x + 5 > -2 - 1 + 5 = 2 > 0$$

$\rightarrow$ PF bude existovat na $\mathbb{R}$

sub.

$$t = \tan\frac{x}{2}$$

$$\int \frac{dx}{2\sin x - \cos x + 5} = \int \frac{\frac{2dt}{t^2+1}}{\frac{4t}{t^2+1} - \frac{1-t^2}{t^2+1} + 5} =$$

$$= \int \frac{2\,dt}{4t - 1 + t^2 + 5t^2 + 5} = \ldots$$

$$= \frac{\sqrt{3}}{9} \arctan\left(\sqrt{3}\left(\tan\frac{x}{2} + \frac{1}{3}\right)\right) + C$$

$\frac{4}{3\sqrt{3}}$ $\qquad$ $\frac{\sqrt{3}}{2}\left(\tan\frac{x}{2} + \frac{1}{2}\right)$

$\frac{3}{5}$ $\qquad$ $\frac{3}{\sqrt{5}}\left(\tan\frac{x}{2} + \frac{1}{3}\right)$

$\frac{3}{\sqrt{5}}$ $\qquad$ $\frac{3}{\sqrt{5}}\left(\tan\frac{x}{2} + \frac{1}{3}\right)$

$$\frac{1}{\sqrt{5}} \qquad \frac{3}{\sqrt{5}}\left(\tan\frac{x}{2}+\frac{1}{3}\right)$$

$$= \int \frac{2\,dt}{6t^2+4t+4} = \int \frac{dt}{3t^2+2t+2} =$$

$$= \frac{1}{3}\int \frac{dt}{t^2+\frac{2t}{3}+\frac{2}{3}} = \frac{1}{3}\int \frac{dt}{\left(t+\frac{1}{3}\right)^2+\frac{5}{9}} =$$

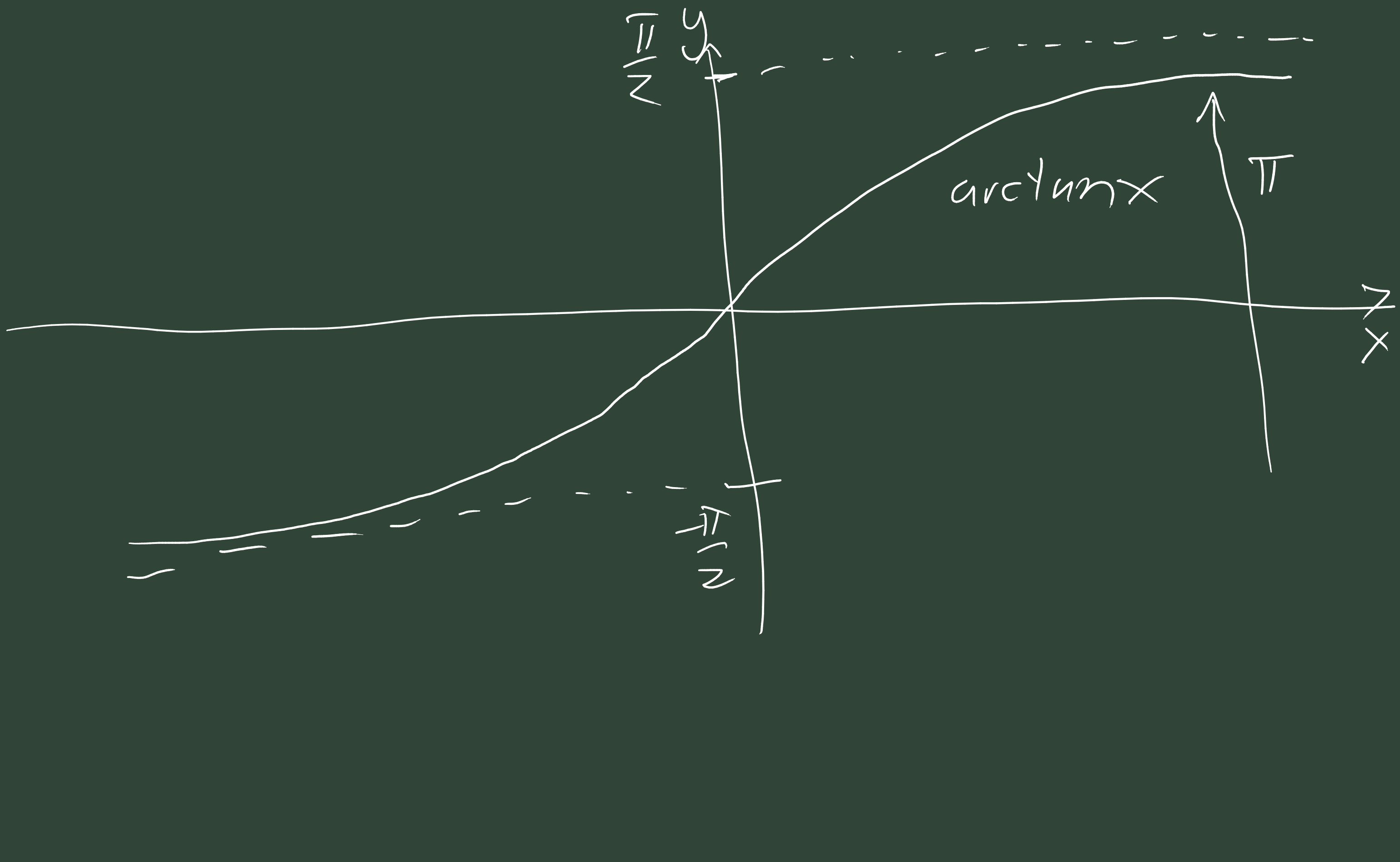
$$= \frac{3}{5}\int \frac{dt}{\frac{9}{5}\left(t+\frac{1}{3}\right)^2+1} = \frac{3}{5}\arctan\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)\cdot$$

$$\cdot\frac{\sqrt{5}}{3}+C = \frac{1}{\sqrt{5}}\arctan\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)+C$$

$$= \frac{1}{\sqrt{5}} \arctan\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right) + C$$

$$= \frac{1}{\sqrt{5}} \arctan\left(\frac{3}{\sqrt{5}}\tan\frac{x}{2} + \frac{1}{\sqrt{5}}\right) + C(k)$$

na $I_k = (-\pi, \pi) + 2k\pi$



$$C(k) = \frac{k\pi}{\sqrt{5}} + C$$

$$R(\sin x, \cos x)$$

II Zápočťák

1) Příklad na limitu (derivaci)

2) PF (jednodušší)