Let T be a triangle with the vertices a_1, a_2, a_3. Let λ_1, λ_2, λ_3 be the barycentric coordinates with respect to the vertices of T. Denote

$$P_T = \mathrm{span}\{\lambda_1, \lambda_2, \lambda_3, \lambda_1\, \lambda_2\, \lambda_3\}\,, \qquad \Sigma_T = \{\Phi_i\}_{i=1}^4\,,$$

where Φ_i are linear forms defined by

$$\Phi_i(v) = v(a_i)\,, \quad i = 1, 2, 3\,, \qquad \Phi_4(v) = \frac{1}{|T|}\int_T v\,\mathrm{d}x$$

For any $v \in C(T)$. Prove that the triple (T, P_T, Σ_T) is a finite element.

Let $\mathscr{T}_h$ be a triangulation of $\Omega := (0,1)^2$ consisting of triangles satisfying the assumptions $(\mathscr{T}_h1)$–$(\mathscr{T}_h5)$ introduced during the FEM1 course. Let the finite element (T, P_T, Σ_T) introduced above be assigned to each element of the triangulation $\mathscr{T}_h$ and formulate the corresponding finite element space X_h and the set Σ_h of the degrees of freedom of X_h. Characterize the dimension of X_h and describe the basis functions of X_h. Find out whether $X_h \subset C(\overline{\Omega})$.

Let $\Pi_h : C(\overline{\Omega}) \to X_h$ be the interpolation oparator. Assuming that the triangulations are regular, derive estimates of the interpolation error with respect to the L^2 norm and H^1 norm.

Consider the Poisson equation in Ω with homogenous Dirichlet boundary conditions:

$$-\Delta\, u = f \quad \text{in } \Omega\,, \qquad u = 0 \quad \text{on } \partial\Omega\,, \tag{1}$$

where $f \in L^2(\Omega)$ is a given function. Formulate a discretization of (1) based on the space X_h and prove estimates of the error of the discrete solution with respect to the L^2 norm and H^1 norm.