Numerical Functional Analysis and Optimization, 27(2):161–187, 2006

ISSN: 0163-0563 print/1532-2467 online
DOI: 10.1080/01630560600569999

REDUCED FINITE ELEMENT DISCRETIZATIONS OF THE STOKES AND NAVIER-STOKES EQUATIONS

Petr Knobloch □ *Charles University, Faculty of Mathematics and Physics, Department of Numerical Mathematics, Praha, Czech Republic*

□ *If finite element spaces for the velocity and pressure do not satisfy the Babuška-Brezzi condition, a stable conforming discretization of the Stokes or Navier-Stokes equations can be obtained by enriching the velocity space by suitable functions. Writing any function from the enriched space as a sum of a function from the original space and a function from the supplementary space, the discretization will contain a number of additional terms compared with a conforming discretization for the original pair of spaces. We show that not all these terms are necessary for the solvability of the discrete problem and for optimal convergence properties of the discrete solutions, which is useful for saving computer memory and for establishing a connection to stabilized methods.*

Keywords Convergence; Finite element method; Navier-Stokes equations; Stokes equations.

Mathematics Subject Classification (2000) 65N30; 35Q30.

1. INTRODUCTION

In this paper, we deal with finite element discretizations of the Stokes and Navier-Stokes equations describing a stationary motion of a viscous incompressible fluid. The region occupied by the fluid will be represented by a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with a Lipschitz-continuous boundary $\partial\Omega$.

The Navier-Stokes problem treated in this paper can be formulated as follows. Given a kinematic viscosity ν, an external body force $\boldsymbol{f}$, and a velocity $\boldsymbol{u}_b$ on the boundary of Ω, find the velocity $\boldsymbol{u}$ and pressure p satisfying

$$-\nu\Delta\boldsymbol{u} + (\nabla\boldsymbol{u})\boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \quad \boldsymbol{u} = \boldsymbol{u}_b \quad \text{on } \partial\Omega. \qquad (1.1)$$

Address correspondence to Petr Knobloch, Charles University, Faculty of Mathematics and Physics, Department of Numerical Mathematics, Sokolovská 83, 186 75 Praha 8, Czech Republic; E-mail: knobloch@karlin.mff.cuni.cz

Denoting

$$a(\boldsymbol{u},\boldsymbol{v}) = \nu \int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x, \quad n(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}) = \int_\Omega \boldsymbol{v}\cdot(\nabla\boldsymbol{w})\boldsymbol{u}\,\mathrm{d}x,$$

$$b(\boldsymbol{v},p) = -\int_\Omega p\,\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}x,$$

we can introduce the usual weak formulation of (1.1): Given $\nu > 0$, $\boldsymbol{f} \in H^{-1}(\Omega)^d$ and $\boldsymbol{u}_b \in H^{\frac{1}{2}}(\partial\Omega)^d$, find $\boldsymbol{u} \in H^1(\Omega)^d$ and $p \in L^2_0(\Omega)$ such that

$$\boldsymbol{u} - \tilde{\boldsymbol{u}}_b \in H^1_0(\Omega)^d, \tag{1.2}$$

$$a(\boldsymbol{u},\boldsymbol{v}) + n(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) - b(\boldsymbol{u},q) = \langle \boldsymbol{f},\boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in H^1_0(\Omega)^d,\ \ q \in L^2_0(\Omega), \tag{1.3}$$

where $\tilde{\boldsymbol{u}}_b \in H^1(\Omega)^d$ is any extension of $\boldsymbol{u}_b$. If the flux of $\boldsymbol{u}_b$ through each connected component of $\partial\Omega$ vanishes, then the problem (1.2)–(1.3) has a solution that is unique if ν is sufficiently large and/or $\boldsymbol{f}$ and $\boldsymbol{u}_b$ are sufficiently small (cf., e.g., [11]).

If the convective term in (1.1) can be neglected, we obtain the Stokes equations

$$-\nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \mathrm{div}\,\boldsymbol{u} = 0 \quad \text{in } \Omega, \quad \boldsymbol{u} = \boldsymbol{u}_b \quad \text{on } \partial\Omega. \tag{1.4}$$

The weak formulation of (1.4) is: Given $\nu > 0$, $\boldsymbol{f} \in H^{-1}(\Omega)^d$ and $\boldsymbol{u}_b \in H^{\frac{1}{2}}(\partial\Omega)^d$, find $\boldsymbol{u} \in H^1(\Omega)^d$ and $p \in L^2_0(\Omega)$ such that

$$\boldsymbol{u} - \tilde{\boldsymbol{u}}_b \in H^1_0(\Omega)^d, \tag{1.5}$$

$$a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) - b(\boldsymbol{u},q) = \langle \boldsymbol{f},\boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in H^1_0(\Omega)^d,\ \ q \in L^2_0(\Omega). \tag{1.6}$$

It can be shown that this problem always has a unique solution (cf. [11]).

Introducing some finite element spaces $\mathrm{V}_h \subset H^1_0(\Omega)^d$ and $\mathrm{Q}_h \subset L^2_0(\Omega)$, where h is a discretization parameter tending to zero, we can define a conforming finite element discretization of (1.5)–(1.6): Given an approximation $\tilde{\boldsymbol{u}}_{bh} \in H^1(\Omega)^d$ of $\tilde{\boldsymbol{u}}_b$, find $\boldsymbol{u}_h \in H^1(\Omega)^d$ and $p_h \in \mathrm{Q}_h$ satisfying

$$\boldsymbol{u}_h - \tilde{\boldsymbol{u}}_{bh} \in \mathrm{V}_h, \tag{1.7}$$

$$a(\boldsymbol{u}_h,\boldsymbol{v}_h) + b(\boldsymbol{v}_h,p_h) - b(\boldsymbol{u}_h,q_h) = \langle \boldsymbol{f},\boldsymbol{v}_h\rangle \quad \forall\, \boldsymbol{v}_h \in \mathrm{V}_h,\ \ q_h \in \mathrm{Q}_h. \tag{1.8}$$

In many cases, seemingly reasonable choices of the spaces V_h and Q_h lead to discrete problems that are generally not solvable or whose solutions contain spurious oscillations. One way to suppress these oscillations and to assure the solvability of the discrete problem is to add some stabilizing terms to (1.8), see, for example, [5] or [17]. Another way is to use spaces V_h and Q_h that are stable in the sense of the Babuška-Brezzi condition [cf. (A5) in the next section]. One possibility to construct a stable pair of spaces is to enlarge the velocity space from an unstable pair of spaces by adding some suitable functions. The velocity space V_h then has the form $V_h = V_h^1 \oplus V_h^2$, where V_h^1 typically assures the approximation properties of the space V_h and V_h^2 guarantees the fulfilment of the Babuška-Brezzi condition. In this paper, we shall consider only spaces of this type.

There are many examples of finite element spaces of the mentioned type. The construction of any finite element space is based on a triangulation of Ω, which usually consists of triangles or quadrilaterals in two dimensions and of tetrahedra or hexahedra in three dimensions. The simplest choice for the spaces V_h^1 and Q_h are piecewice constant functions for Q_h and continuous piecewise (bi-, tri-)linear functions for V_h^1. To satisfy the Babuška-Brezzi condition, it suffices to use a space V_h^2 consisting of one vector-valued edge/face-bubble function per each inner edge/face, see [4, 10]. In the triangular/tetrahedral case, spaces Q_h, V_h^1 consisting of continuous piecewise linear functions may be stabilized using V_h^2 consisting of d vector-valued element bubble functions per each element, cf. [1]. In two dimensions, the same space V_h^2 can be used if V_h^1 consists of continuous piecewise quadratic functions and Q_h of discontinuous piecewise linear functions, cf. [8]. A generalization of [1] to the quadrilateral case is described in [15]. Further examples of spaces V_h^1, V_h^2, and Q_h can be found in [11].

Because the spaces V_h^1, V_h^2, and Q_h are assumed to be finite-dimensional, the problem (1.7)–(1.8) can be equivalently written in the matrix form

$$\begin{pmatrix} A_h^{11} & A_h^{12} & (B_h^1)^T \\ A_h^{21} & A_h^{22} & (B_h^2)^T \\ B_h^1 & B_h^2 & 0 \end{pmatrix} \begin{pmatrix} u_h^1 \\ u_h^2 \\ p_h \end{pmatrix} = \begin{pmatrix} f_h^1 \\ f_h^2 \\ g_h \end{pmatrix}, \tag{1.9}$$

where u_h^1, u_h^2, and p_h are coefficient vectors of $\boldsymbol{u}_h - \tilde{\boldsymbol{u}}_{bh}$ and p_h with respect to some bases $\{\boldsymbol{v}_{hi}^1\}_{i=1}^{N_h^1} \subset V_h^1$, $\{\boldsymbol{v}_{hi}^2\}_{i=1}^{N_h^2} \subset V_h^2$ and $\{q_{hi}\}_{i=1}^{N_h} \subset Q_h$, respectively, and

$$A_h^{kl} = \{a(\boldsymbol{v}_{hj}^l, \boldsymbol{v}_{hi}^k)\}_{i=1,\dots,N_h^k,\, j=1,\dots,N_h^l}, \quad B_h^k = \{b(\boldsymbol{v}_{hj}^k, q_{hi})\}_{i=1,\dots,N_h,\, j=1,\dots,N_h^k},$$

for $k, l = 1, 2$. The conforming discretization of the problem (1.2)–(1.3) can also be written in the matrix form (1.9), however, the matrices A_h^{kl} and the vectors f_h^k depend on the unknown velocity $\{u_h^1, u_h^2\}$.

A drawback of the system (1.9) is that the matrices A_h^{12}, A_h^{21}, and A_h^{22} are usually large compared with A_h^{11} although they typically only serve for assuring the unique solvability of (1.9) and do not increase the convergence order of the discrete solution. Thus, in order to save computational memory and computational time, we would like to drop some of these matrices. Similarly, we would like to drop some of the matrices corresponding to the term $n(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h)$ from the conforming discretization of the Navier-Stokes equations. That leads us to the *reduced system*

$$\begin{pmatrix} \widetilde{\mathrm{A}}_h^{11} & 0 & (\mathrm{B}_h^1)^T \\ 0 & \mathrm{A}_h^{22} & (\mathrm{B}_h^2)^T \\ \mathrm{B}_h^1 & \mathrm{B}_h^2 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{u}_h^1 \\ \mathrm{u}_h^2 \\ \mathrm{p}_h \end{pmatrix} = \begin{pmatrix} \tilde{\mathrm{f}}_h^1 \\ 0 \\ \mathrm{g}_h \end{pmatrix}, \tag{1.10}$$

where $\widetilde{\mathrm{A}}_h^{11} = \mathrm{A}_h^{11}$ and $\tilde{\mathrm{f}}_h^1 = \mathrm{f}_h^1$ for the Stokes equations and

$$\widetilde{\mathrm{A}}_h^{11} = \mathrm{A}_h^{11} + \{n(\boldsymbol{u}_h^*, \boldsymbol{v}_{hj}^1, \boldsymbol{v}_{hi}^1)\}_{i,j=1,\dots,N_h^1}, \quad \tilde{\mathrm{f}}_h^1 = \mathrm{f}_h^1 - \{n(\boldsymbol{u}_h^*, \tilde{\boldsymbol{u}}_{bh}, \boldsymbol{v}_{hi}^1)\}_{i=1,\dots,N_h^1}$$

for the Navier-Stokes equations. The function $\boldsymbol{u}_h^*$ is defined as $(\boldsymbol{u}_h - \tilde{\boldsymbol{u}}_{bh})^1 + \tilde{\boldsymbol{u}}_{bh}$, where $(\boldsymbol{u}_h - \tilde{\boldsymbol{u}}_{bh})^1$ is the V_h^1 component of $\boldsymbol{u}_h - \tilde{\boldsymbol{u}}_{bh}$. We shall show that, under usual assumptions, the reduced problems are (locally) uniquely solvable and their solutions (linearly) converge to the weak solution.

Numerical results indicate that the reduced problems provide discrete solutions having almost the same accuracy as the solutions of the original conforming discretizations (cf. [12]). However, the saving of the computer memory due to the use of the reduced discrete problems is substantial, particularly in the three-dimensional case (cf. [12]). Consequently, we save a significant amount of computational operations and computational time. Usually, the discrete solution $\boldsymbol{u}_h$, p_h of the Navier-Stokes equations is computed as the limit of a sequence $\boldsymbol{u}_h^n$, p_h^n of solutions of linearized problems and hence a further advantage of (1.10) is that only the matrix $\widetilde{\mathrm{A}}_h^{11}$ and the vector $\tilde{\mathrm{f}}_h^1$ have to be updated in each step. In addition, because only functions from V_h^1 are used for discretizing the convective term $(\nabla \boldsymbol{u})\boldsymbol{u}$, upwind techniques can be easier applied than for the conforming discretization. As the matrix A_h^{22} in (1.10) is always regular, we can eliminate u_h^2 from (1.10) and pass to the system

$$\begin{pmatrix} \widetilde{\mathrm{A}}_h^{11} & (\mathrm{B}_h^1)^T \\ \mathrm{B}_h^1 & -\mathrm{B}_h^2(\mathrm{A}_h^{22})^{-1}(\mathrm{B}_h^2)^T \end{pmatrix} \begin{pmatrix} \mathrm{u}_h^1 \\ \mathrm{p}_h \end{pmatrix} = \begin{pmatrix} \tilde{\mathrm{f}}_h^1 \\ \mathrm{g}_h \end{pmatrix}. \tag{1.11}$$

As we shall see (cf. Lemma 2.6), the vector u_h^2 need not to be computed because it does not influence the asymptotic convergence behavior of the discrete solution. Numerical computations even show that dropping u_h^2, one can often increase the accuracy of the discrete solution. In many cases, the matrix A_h^{22} is diagonal and hence a practical realization of (1.11) is easy.

The problem (1.11) can be interpreted as a stabilization of a conforming discretization for the unstable pair of spaces V_h^1 and Q_h. For particular choices of the spaces V_h^2 and Q_h, the term $-B_h^2(A_h^{22})^{-1}(B_h^2)^T$ corresponds to some well-known stabilizations or gives rise to some new ones, see [13, 14]. Dropping only some of the terms dropped to obtain (1.10), the reduced discretizations can be interpreted as residual-based stabilizations of the continuity equation (and of the convective term), see again [13, 14]. In some of these cases, we can prove usual convergence orders of the discrete solutions also for higher order finite element spaces. The identification of stabilized methods with suitable reduced (or modified) Galerkin-type discretizations provides a better understanding of their properties and is also helpful for their theoretical investigations (e.g., in the framework of multigrid methods).

To investigate all the reduced discretizations mentioned above at once, we shall consider *general reduced discretizations* where the terms to be dropped are multiplied by arbitrary real numbers. In addition, we shall consider the matrix A_h^{22} multiplied by a positive constant because numerical experiments suggest that such multiplication can lead to a stabilization with respect to ν.

The plan of the remaining part of this paper is as follows. In Section 2, we recall some classical convergence results valid for the conforming discretization (1.7)–(1.8), introduce a general reduced discretization of the Stokes equations, and investigate its properties. Then, in Section 3, we introduce a general reduced discretization of the Navier-Stokes equations and, using the results of Section 2 and the theory of the approximation of branches of nonsingular solutions, we prove an analogous convergence behavior of the discrete solutions as for the Stokes equations. Finally, in Section 4, we investigate the validity of some of the general assumptions made in Sections 2 and 3.

Throughout the paper, we use standard notations that can be found, for example, in [7]. We only mention a few of them. The norm and the seminorm in the Sobolev space $W^{k,p}(\Omega)$ are denoted by $\|\cdot\|_{k,p,\Omega}$ and $|\cdot|_{k,p,\Omega}$, respectively. For $p = 2$, the second index is dropped and we use the notations $H^k(\Omega) \equiv W^{k,2}(\Omega)$, $\|\cdot\|_{k,\Omega}$ and $|\cdot|_{k,\Omega}$. The space $L_0^2(\Omega)$ consists of functions $v \in L^2(\Omega)$ satisfying $\int_\Omega v \,\mathrm{d}x = 0$. The notations C and $\widetilde{C}$ are used to denote generic constants independent of h.

2. DISCRETIZATION OF THE STOKES EQUATIONS

We assume that we are given a family of spaces $V_h^1, V_h^2 \subset H_0^1(\Omega)^d$, $Q_h \subset L_0^2(\Omega)$, where h is a positive parameter tending to zero. We assume that $V_h^1 \cap V_h^2 = \{\mathbf{0}\}$ and denote $V_h \equiv V_h^1 \oplus V_h^2$. Thus, for any $\boldsymbol{v}_h \in V_h$, there exist uniquely determined functions $\boldsymbol{v}_h^1 \in V_h^1$ and $\boldsymbol{v}_h^2 \in V_h^2$ satisfying $\boldsymbol{v}_h^1 + \boldsymbol{v}_h^2 = \boldsymbol{v}_h$.

When there will be no danger of ambiguity, we shall also use the notations $\boldsymbol{v}_h^1$, $\boldsymbol{v}_h^2$ for arbitrary functions belonging to V_h^1 and V_h^2, respectively. We assume that the spaces V_h^1, V_h^2, and Q_h possess the following properties that are valid for any $h > 0$ with the same constant $C > 0$ and the same integer $l \geq 1$.

(A1) There exist operators $r_h \in \mathscr{L}\big(H^2(\Omega)^d \cap H_0^1(\Omega)^d, \mathrm{V}_h^1\big)$ such that

$$\|\boldsymbol{v} - r_h \boldsymbol{v}\|_{1,\Omega} \leq C\, h^m \|\boldsymbol{v}\|_{m+1,\Omega} \quad \forall\, \boldsymbol{v} \in H^{m+1}(\Omega)^d \cap H_0^1(\Omega)^d, \;\; 1 \leq m \leq l.$$

(A2) There exist operators $s_h \in \mathscr{L}\big(H^1(\Omega) \cap L_0^2(\Omega), \mathrm{Q}_h\big)$ such that

$$\|q - s_h q\|_{0,\Omega} \leq C\, h^m \|q\|_{m,\Omega} \quad \forall\, q \in H^m(\Omega) \cap L_0^2(\Omega), \;\; 1 \leq m \leq l.$$

(A3) The spaces V_h^2 satisfy

$$\|\boldsymbol{v}_h^2\|_{0,\Omega} \leq C\, h |\boldsymbol{v}_h^2|_{1,\Omega} \quad \forall\, \boldsymbol{v}_h^2 \in \mathrm{V}_h^2.$$

(A4) The spaces V_h satisfy

$$\|\boldsymbol{v}_h^1\|_{1,\Omega} + \|\boldsymbol{v}_h^2\|_{1,\Omega} \leq C \|\boldsymbol{v}_h\|_{1,\Omega} \quad \forall\, \boldsymbol{v}_h \in \mathrm{V}_h.$$

(A5) The spaces V_h and Q_h satisfy the Babuška-Brezzi condition

$$\sup_{\boldsymbol{v}_h \in \mathrm{V}_h \setminus \{\mathbf{0}\}} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{1,\Omega}} \geq C \|q_h\|_{0,\Omega} \quad \forall\, q_h \in \mathrm{Q}_h.$$

Remark 2.1. The assumptions (A1) and (A2) are standard approximation properties of finite element spaces (cf., e.g., [7]) and the assumption (A5) holds, for example, for the pairs of V_h and Q_h mentioned in the preceding section. The validity of (A3) and (A4) will be investigated in Section 4. Note, however, that the theory we shall present here is valid for general spaces satisfying (A1)–(A5) and not only for spaces constructed by means of the finite element method.

The constant parameter $\nu > 0$, the boundary condition $\boldsymbol{u}_b \in H^{\frac{1}{2}}(\partial\Omega)^d$, its extension $\tilde{\boldsymbol{u}}_b \in H^1(\Omega)^d$ and the approximations $\tilde{\boldsymbol{u}}_{bh} \in H^1(\Omega)^d$ of $\tilde{\boldsymbol{u}}_b$ are assumed to be given and fixed and we suppose that

$$\lim_{h \to 0} \|\tilde{\boldsymbol{u}}_b - \tilde{\boldsymbol{u}}_{bh}\|_{1,\Omega} = 0. \tag{2.1}$$

Then the following theorem holds.

Theorem 2.2. *For any $\boldsymbol{f} \in H^{-1}(\Omega)^d$, the problem* (1.7)–(1.8) *has a unique solution and we have*

$$\lim_{h\to 0}\{\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}+\|p-p_h\|_{0,\Omega}\}=0,$$

where $\boldsymbol{u}$, p is the solution of the problem (1.5)–(1.6). *Moreover, if $\boldsymbol{u}$, $\tilde{\boldsymbol{u}}_b \in H^{m+1}(\Omega)^d$, $p \in H^m(\Omega)$ and $\|\tilde{\boldsymbol{u}}_b-\tilde{\boldsymbol{u}}_{bh}\|_{1,\Omega} \le C\,h^m$ for some $m \in \{1,\ldots,l\}$, then*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}+\|p-p_h\|_{0,\Omega} \le C\,h^m. \tag{2.2}$$

If, in addition, the problem (1.6) *is regular and $\|\tilde{\boldsymbol{u}}_b-\tilde{\boldsymbol{u}}_{bh}\|_{0,\partial\Omega} \le C\,h^{m+1}$, then*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega} \le C\,h^{m+1}. \tag{2.3}$$

Proof. A proof for homogenous boundary conditions can be found in [11]. Using the techniques applied in [9], the proof can be easily generalized to the nonhomogenous case. □

Under the regularity of problem (1.6) mentioned in the above theorem we mean that, for any $\boldsymbol{g} \in L^2(\Omega)^d$, the solution $\boldsymbol{u}_g \in H^1_0(\Omega)^d$, $p_g \in L^2_0(\Omega)$ of

$$a(\boldsymbol{u}_g,\boldsymbol{v})+b(\boldsymbol{v},p_g)-b(\boldsymbol{u}_g,q)=\langle \boldsymbol{g},\boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v}\in H^1_0(\Omega)^d,\ \ q\in L^2_0(\Omega) \tag{2.4}$$

satisfies $\boldsymbol{u}_g \in H^2(\Omega)^d$, $p_g \in H^1(\Omega)$ and

$$\|\boldsymbol{u}_g\|_{2,\Omega}+\|p_g\|_{1,\Omega} \le C\|\boldsymbol{g}\|_{0,\Omega} \tag{2.5}$$

with C independent of $\boldsymbol{g}$.

For any $h>0$ and any $\boldsymbol{v}_h$, $\boldsymbol{w}_h \in \mathrm{V}_h$, we define the bilinear form

$$a_h(\boldsymbol{w}_h,\boldsymbol{v}_h)=a(\boldsymbol{w}_h^1,\boldsymbol{v}_h^1)+\alpha_1 a(\boldsymbol{w}_h^1,\boldsymbol{v}_h^2)+\alpha_2 a(\boldsymbol{w}_h^2,\boldsymbol{v}_h^1)+\alpha_3 a(\boldsymbol{w}_h^2,\boldsymbol{v}_h^2),$$

where α_1, α_2, α_3 are arbitrary real numbers. Further, we replace the functional $\boldsymbol{f}$ from (1.6) by some suitable functional $\boldsymbol{f}_h \in H^{-1}(\Omega)^d$ and choose an arbitrary real number α_4. Then we can introduce the following discrete problem which includes all the particular reduced discretizations of the Stokes equations mentioned in Section 1.

Definition 2.3. The functions $\tilde{\boldsymbol{u}}_h \in H^1(\Omega)^d$ and $\tilde{p}_h \in \mathrm{Q}_h$ are a *discrete solution* of the problem (1.5)–(1.6) if

$$\tilde{\boldsymbol{u}}_h-\tilde{\boldsymbol{u}}_{bh} \in \mathrm{V}_h, \tag{2.6}$$

$$\begin{aligned} &a_h(\tilde{\boldsymbol{u}}_h-\tilde{\boldsymbol{u}}_{bh},\boldsymbol{v}_h)+b(\boldsymbol{v}_h,\tilde{p}_h)-b(\tilde{\boldsymbol{u}}_h,q_h)\\ &\quad=\langle \boldsymbol{f}_h,\boldsymbol{v}_h\rangle-a(\tilde{\boldsymbol{u}}_{bh},\boldsymbol{v}_h^1)-\alpha_4 a(\tilde{\boldsymbol{u}}_{bh},\boldsymbol{v}_h^2) \quad \forall\,\boldsymbol{v}_h\in \mathrm{V}_h,\ \ q_h\in \mathrm{Q}_h. \end{aligned} \tag{2.7}$$

Remark 2.4. For $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$ and $\boldsymbol{f}_h = \boldsymbol{f}$, the discretization (2.6)–(2.7) becomes the conforming discretization (1.7)–(1.8). For $\alpha_1 = \alpha_2 = \alpha_4 = 0$, $\alpha_3 = 1$ and $\boldsymbol{f}_h$ defined by

$$\langle \boldsymbol{f}_h, \boldsymbol{v}_h \rangle = \langle \boldsymbol{f}, \boldsymbol{v}_h^1 \rangle \quad \forall\, \boldsymbol{v}_h \in \mathrm{V}_h, \tag{2.8}$$

the discretization (2.6)–(2.7) can be written in the matrix form (1.10), that is, it is the fully reduced discretization. The relation (2.8) defines a functional $\boldsymbol{f}_h \in [\mathrm{V}_h]'$ that can be extended to $\boldsymbol{f}_h \in H^{-1}(\Omega)^d$ according to the Hahn-Banach theorem. Clearly, $\|\boldsymbol{f} - \boldsymbol{f}_h\|_{[\mathrm{V}_h^1]'} = 0$ so that $\boldsymbol{f}_h$ satisfies the assumption (2.19) of Theorem 2.9 below.

Remark 2.5. For $\alpha_2 = 0$, $\alpha_3 \neq 0$ and $\langle \boldsymbol{f}_h, \boldsymbol{v}_h^1 \rangle = \langle \boldsymbol{f}, \boldsymbol{v}_h^1 \rangle \;\; \forall\, \boldsymbol{v}_h^1 \in \mathrm{V}_h^1$, the discrete problem (2.6)–(2.7) can be formulated in the following way: Find $\boldsymbol{u}_h^* \in H^1(\Omega)^d$ and $\tilde{p}_h \in \mathrm{Q}_h$ such that

$$\begin{aligned} \boldsymbol{u}_h^* - \tilde{\boldsymbol{u}}_{bh} &\in \mathrm{V}_h^1, \\ a(\boldsymbol{u}_h^*, \boldsymbol{v}_h^1) + b(\boldsymbol{v}_h^1, \tilde{p}_h) &= \langle \boldsymbol{f}, \boldsymbol{v}_h^1 \rangle \quad \forall\, \boldsymbol{v}_h^1 \in \mathrm{V}_h^1, \\ b(\boldsymbol{u}_h^*, q_h) &= -b(\bar{\boldsymbol{u}}_h^2, q_h) \quad \forall\, q_h \in \mathrm{Q}_h, \end{aligned}$$

where $\bar{\boldsymbol{u}}_h^2 \in \mathrm{V}_h^2$ is uniquely determined by

$$a\big(\bar{\boldsymbol{u}}_h^2, \boldsymbol{v}_h^2\big) = -\big\langle R_h, \boldsymbol{v}_h^2 \big\rangle \quad \forall\, \boldsymbol{v}_h^2 \in \mathrm{V}_h^2$$

with some functional R_h depending on $\boldsymbol{u}_h^*$, $\tilde{p}_h$ and the data of the discrete problem. Thus, the discrete problem (2.6)–(2.7) with $\alpha_2 = 0$, $\alpha_3 \neq 0$ and $\boldsymbol{f}_h = \boldsymbol{f}$ on V_h^1 corresponds to the conforming discretization (1.7)–(1.8) for the unstable pair of spaces V_h^1 and Q_h with a perturbation of the constraint $b(\boldsymbol{u}_h, q_h) = 0 \;\forall\, q_h \in \mathrm{Q}_h$. If $\alpha_1 = \alpha_3 = \alpha_4 = 1$ and $\boldsymbol{f}_h = \boldsymbol{f}$, then R_h is given by

$$\langle R_h, \boldsymbol{v} \rangle = a(\boldsymbol{u}_h^*, \boldsymbol{v}) + b(\boldsymbol{v}, \tilde{p}_h) - \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall\, \boldsymbol{v} \in H_0^1(\Omega)^d,$$

that is, the reduced discretization can be interpreted as a residual-based stabilization of the continuity equation. For $\alpha_1 = \alpha_2 = \alpha_4 = 0$, $\alpha_3 = 1$ and $\boldsymbol{f}_h$ defined by (2.8), we have $\langle R_h, \boldsymbol{v} \rangle = b(\boldsymbol{v}, \tilde{p}_h)$ and the above formulation corresponds to the matrix form (1.11).

A basic feature of the terms removed from (1.8) in order to obtain the fully reduced discretization [i.e., (2.7) with $\alpha_1 = \alpha_2 = \alpha_4 = 0$ and $\boldsymbol{f}_h$ defined by (2.8)] is that they contain $\boldsymbol{v}_h^2$ or $(\boldsymbol{u}_h - \tilde{\boldsymbol{u}}_{bh})^2$ and are not important for the solvability of (1.7)–(1.8). A motivation for removing these terms is given by the corollary of the following lemma which shows that the asymptotic behavior of $\boldsymbol{u}_h$ with respect to h is the same as the asymptotic behavior of $\boldsymbol{u}_h - (\boldsymbol{u}_h - \tilde{\boldsymbol{u}}_{bh})^2$.

Lemma 2.6. *Consider a sequence* $\boldsymbol{v}_h \in \mathrm{V}_h$ *satisfying*

$$\lim_{h\to 0} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{1,\Omega} = 0 \tag{2.9}$$

for some $\boldsymbol{v} \in H_0^1(\Omega)^d$. *Then we have*

$$\lim_{h\to 0}\{\|\boldsymbol{v} - \boldsymbol{v}_h^1\|_{1,\Omega} + \|\boldsymbol{v}_h^2\|_{1,\Omega}\} = 0. \tag{2.10}$$

If $\boldsymbol{v} \in H^{m+1}(\Omega)^d \cap H_0^1(\Omega)^d$ *and* $\|\boldsymbol{v} - \boldsymbol{v}_h\|_{1,\Omega} \leq C\, h^m$ *for some* $m \in \{1, \dots, l\}$, *it further holds*

$$\|\boldsymbol{v} - \boldsymbol{v}_h^1\|_{1,\Omega} + \|\boldsymbol{v}_h^2\|_{1,\Omega} \leq C\, h^m. \tag{2.11}$$

Assuming, in addition, that $\|\boldsymbol{v} - \boldsymbol{v}_h\|_{0,\Omega} \leq C\, h^{m+1}$, *we also obtain*

$$\|\boldsymbol{v} - \boldsymbol{v}_h^1\|_{0,\Omega} + \|\boldsymbol{v}_h^2\|_{0,\Omega} \leq C\, h^{m+1}. \tag{2.12}$$

Proof. Let $\hat{\boldsymbol{u}} \in H^2(\Omega)^d \cap H_0^1(\Omega)^d$. Then, due to (A3) and (A4), we have for $k = 0, 1$

$$\|\boldsymbol{v}_h^2\|_{k,\Omega} = \|(\boldsymbol{v}_h - r_h\hat{\boldsymbol{u}})^2\|_{k,\Omega} \leq C\, h^{1-k}\|\boldsymbol{v}_h - r_h\hat{\boldsymbol{u}}\|_{1,\Omega}$$

and hence it follows using the triangular inequality that

$$\begin{aligned}&\|\boldsymbol{v} - \boldsymbol{v}_h^1\|_{k,\Omega} + \|\boldsymbol{v}_h^2\|_{k,\Omega}\\ &\quad\leq \|\boldsymbol{v} - \boldsymbol{v}_h\|_{k,\Omega} + C\, h^{1-k}\{\|\boldsymbol{v} - \boldsymbol{v}_h\|_{1,\Omega} + \|\boldsymbol{v} - \hat{\boldsymbol{u}}\|_{1,\Omega} + \|\hat{\boldsymbol{u}} - r_h\hat{\boldsymbol{u}}\|_{1,\Omega}\}.\end{aligned}$$

Using (A1), (2.9) and the density of $H^2(\Omega)^d \cap H_0^1(\Omega)^d$ in $H_0^1(\Omega)^d$, we obtain (2.10). If $\boldsymbol{v} \in H^{m+1}(\Omega)^d \cap H_0^1(\Omega)^d$, we can set $\hat{\boldsymbol{u}} = \boldsymbol{v}$ and (2.11) and (2.12) follow using (A1). □

Corollary 2.7. *The solution* $\boldsymbol{u}_h \equiv \boldsymbol{u}_h^* + \bar{\boldsymbol{u}}_h^2$ *of* (1.7)–(1.8), *where* $\bar{\boldsymbol{u}}_h^2 = (\boldsymbol{u}_h - \tilde{\boldsymbol{u}}_{bh})^2$, *satisfies*

$$\lim_{h\to 0}\{\|\boldsymbol{u} - \boldsymbol{u}_h^*\|_{1,\Omega} + \|\bar{\boldsymbol{u}}_h^2\|_{1,\Omega}\} = 0, \tag{2.13}$$

where $\boldsymbol{u}$ *is the solution of* (1.5)–(1.6). *Under the assumptions of Theorem* 2.2 *leading to* (2.2), *we further have*

$$\|\boldsymbol{u} - \boldsymbol{u}_h^*\|_{1,\Omega} + \|\bar{\boldsymbol{u}}_h^2\|_{1,\Omega} \leq C\, h^m. \tag{2.14}$$

Finally, if $\|\tilde{\boldsymbol{u}}_b - \tilde{\boldsymbol{u}}_{bh}\|_{0,\Omega} \leq C\, h^{m+1}$ *and all assumptions of Theorem* 2.2 *hold, we get*

$$\|\boldsymbol{u} - \boldsymbol{u}_h^*\|_{0,\Omega} + \|\bar{\boldsymbol{u}}_h^2\|_{0,\Omega} \leq C\, h^{m+1}.$$

Remark 2.8. Lemma 2.6 shows that, for a finite element discretization of *any* problem, the V_h^2 component of the discrete solution can be dropped without influencing the asymptotic convergence properties of the discrete solution.

Now let us investigate the properties of the discrete problem (2.6)–(2.7).

Theorem 2.9. *Let the constants* $\alpha_1, \ldots, \alpha_4$ *used in Definition* 2.3 *satisfy* $\alpha_3 > 0$ *and* $|\alpha_1 + \alpha_2| \leq 2\sqrt{\alpha_3}$. *Then, for any* $\boldsymbol{f}_h \in H^{-1}(\Omega)^d$, *the problem* (2.6)–(2.7) *has a unique solution and if*

$$\lim_{h\to 0} \|\boldsymbol{f} - \boldsymbol{f}_h\|_{[V_h]'} = 0 \tag{2.15}$$

for some $\boldsymbol{f} \in H^{-1}(\Omega)^d$, *then we have*

$$\lim_{h\to 0}\{\|\boldsymbol{u} - \tilde{\boldsymbol{u}}_h\|_{1,\Omega} + \|p - \tilde{p}_h\|_{0,\Omega}\} = 0, \tag{2.16}$$

where $\boldsymbol{u}$, p *is the solution of the problem* (1.5)–(1.6) *with* $\boldsymbol{f}$ *from* (2.15). *Further, if* $\boldsymbol{u}, \tilde{\boldsymbol{u}}_b \in H^{m+1}(\Omega)^d$, $p \in H^m(\Omega)$ *and*

$$\|\tilde{\boldsymbol{u}}_b - \tilde{\boldsymbol{u}}_{bh}\|_{1,\Omega} + \|\boldsymbol{f} - \boldsymbol{f}_h\|_{[V_h]'} \leq C\, h^m \tag{2.17}$$

for some $m \in \{1, \ldots, l\}$, *then*

$$\|\boldsymbol{u} - \tilde{\boldsymbol{u}}_h\|_{1,\Omega} + \|p - \tilde{p}_h\|_{0,\Omega} \leq C\, h^m + C\, h(|1 - \alpha_1| + |1 - \alpha_4|). \tag{2.18}$$

If, in addition, the problem (1.6) *is regular and*

$$\|\tilde{\boldsymbol{u}}_b - \tilde{\boldsymbol{u}}_{bh}\|_{0,\partial\Omega} + \|\boldsymbol{f} - \boldsymbol{f}_h\|_{[V_h^1]'} \leq C\, h^{m+1}, \tag{2.19}$$

then we obtain

$$\|\boldsymbol{u} - \tilde{\boldsymbol{u}}_h\|_{0,\Omega} \leq C\, h^{m+1} + C\, h^2(|1 - \alpha_1| + |1 - \alpha_4|). \tag{2.20}$$

Proof. Denoting $\alpha = (\alpha_1 + \alpha_2)/2$, we have for any $\boldsymbol{v}_h \in V_h$

$$\begin{aligned} a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) &= a\big(\boldsymbol{v}_h^1 + \alpha \boldsymbol{v}_h^2, \boldsymbol{v}_h^1 + \alpha \boldsymbol{v}_h^2\big) + (\alpha_3 - \alpha^2)\, a\big(\boldsymbol{v}_h^2, \boldsymbol{v}_h^2\big) \\ &= \nu\big|\boldsymbol{v}_h^1 + \alpha \boldsymbol{v}_h^2\big|_{1,\Omega}^2 + \nu(\alpha_3 - \alpha^2)\big|\boldsymbol{v}_h^2\big|_{1,\Omega}^2 \end{aligned}$$

and hence it follows from the Friedrichs inequality and (A4) that, for some $C > 0$,

$$C\|\boldsymbol{v}_h\|_{1,\Omega}^2 \leq a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \quad \forall\, \boldsymbol{v}_h \in V_h. \tag{2.21}$$

Further, it follows from (A4) that

$$a_h(\boldsymbol{w}_h, \boldsymbol{v}_h) \le C\|\boldsymbol{w}_h\|_{1,\Omega}\|\boldsymbol{v}_h\|_{1,\Omega} \quad \forall\, \boldsymbol{v}_h, \boldsymbol{w}_h \in \mathrm{V}_h. \tag{2.22}$$

Thus, a_h is a continuous V_h-elliptic bilinear form and the proof of the unique solvability of (2.6)–(2.7) can be therefore performed in the same way as for the problem (1.7)–(1.8).

Now let us investigate the convergence behavior of the discrete solution $\tilde{\boldsymbol{u}}_h, \tilde{p}_h$. As in Corollary 2.7, we introduce the functions $\bar{\boldsymbol{u}}_h^2 = (\boldsymbol{u}_h - \tilde{\boldsymbol{u}}_{bh})^2$ and $\boldsymbol{u}_h^* = \boldsymbol{u}_h - \bar{\boldsymbol{u}}_h^2$, where $\boldsymbol{u}_h$ is the solution of (1.7)–(1.8). Then, subtracting (1.8) from (2.7), we obtain for $q_h = 0$ and any $\boldsymbol{v}_h \in \mathrm{V}_h$

$$\begin{aligned} &a_h(\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, \tilde{p}_h - p_h) \\ &\quad = \langle \boldsymbol{f}_h - \boldsymbol{f}, \boldsymbol{v}_h\rangle + (1-\alpha_1)a(\boldsymbol{u}_h^*, \boldsymbol{v}_h^2) + (1-\alpha_2)a(\bar{\boldsymbol{u}}_h^2, \boldsymbol{v}_h^1) \\ &\qquad + (1-\alpha_3)a(\bar{\boldsymbol{u}}_h^2, \boldsymbol{v}_h^2) + (\alpha_1-\alpha_4)a(\tilde{\boldsymbol{u}}_{bh}, \boldsymbol{v}_h^2). \end{aligned} \tag{2.23}$$

Consider any $\hat{\boldsymbol{u}} \in H^2(\Omega)^d$. Then we infer applying (A3) that

$$a(\hat{\boldsymbol{u}}, \boldsymbol{v}_h^2) \le \nu\|\Delta\hat{\boldsymbol{u}}\|_{0,\Omega}\|\boldsymbol{v}_h^2\|_{0,\Omega} \le C\,h|\hat{\boldsymbol{u}}|_{2,\Omega}|\boldsymbol{v}_h^2|_{1,\Omega} \quad \forall\, \boldsymbol{v}_h^2 \in \mathrm{V}_h^2. \tag{2.24}$$

Using the indentity $a(\boldsymbol{u}_h^*, \boldsymbol{v}_h^2) = a(\boldsymbol{u}_h^* - \hat{\boldsymbol{u}}, \boldsymbol{v}_h^2) + a(\hat{\boldsymbol{u}}, \boldsymbol{v}_h^2)$, we get for any $\boldsymbol{v}_h^2 \in \mathrm{V}_h^2$

$$a(\boldsymbol{u}_h^*, \boldsymbol{v}_h^2) \le C\{|\boldsymbol{u}_h^* - \boldsymbol{u}|_{1,\Omega} + |\boldsymbol{u} - \widehat{\boldsymbol{u}}|_{1,\Omega} + h|\hat{\boldsymbol{u}}|_{2,\Omega}\}|\boldsymbol{v}_h^2|_{1,\Omega} \quad \forall\, \hat{\boldsymbol{u}} \in H^2(\Omega)^d$$

and analogously

$$a(\tilde{\boldsymbol{u}}_{bh}, \boldsymbol{v}_h^2) \le C\{|\tilde{\boldsymbol{u}}_{bh} - \tilde{\boldsymbol{u}}_b|_{1,\Omega} + |\tilde{\boldsymbol{u}}_b - \hat{\boldsymbol{u}}_b|_{1,\Omega} + h|\hat{\boldsymbol{u}}_b|_{2,\Omega}\}|\boldsymbol{v}_h^2|_{1,\Omega} \quad \forall\, \hat{\boldsymbol{u}}_b \in H^2(\Omega)^d.$$

Now, denoting

$$\begin{aligned} A_h = {} & \|\boldsymbol{f} - \boldsymbol{f}_h\|_{[\mathrm{V}_h]'} + |\boldsymbol{u} - \boldsymbol{u}_h^*|_{1,\Omega} + |\bar{\boldsymbol{u}}_h^2|_{1,\Omega} + |\tilde{\boldsymbol{u}}_b - \tilde{\boldsymbol{u}}_{bh}|_{1,\Omega} \\ & + |1-\alpha_1| \inf_{\hat{\boldsymbol{u}} \in H^2(\Omega)^d} \{|\boldsymbol{u} - \hat{\boldsymbol{u}}|_{1,\Omega} + h|\hat{\boldsymbol{u}}|_{2,\Omega}\} \\ & + |\alpha_1 - \alpha_4| \inf_{\hat{\boldsymbol{u}}_b \in H^2(\Omega)^d} \{|\tilde{\boldsymbol{u}}_b - \hat{\boldsymbol{u}}_b|_{1,\Omega} + h|\hat{\boldsymbol{u}}_b|_{2,\Omega}\}, \end{aligned}$$

we derive from (2.23) applying (A4) that

$$a_h(\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, \tilde{p}_h - p_h) \le C\,A_h\|\boldsymbol{v}_h\|_{1,\Omega} \quad \forall\, \boldsymbol{v}_h \in \mathrm{V}_h. \tag{2.25}$$

From (2.15), (2.13), (2.1), and the density of $H^2(\Omega)^d$ in $H^1(\Omega)^d$, we deduce that

$$\lim_{h\to 0} A_h = 0, \tag{2.26}$$

and, if $\boldsymbol{u}$, $\tilde{\boldsymbol{u}}_b \in H^{m+1}(\Omega)^d$, $p \in H^m(\Omega)$ and (2.17) holds, it follows from (2.14) that

$$A_h \leq C\,h^m + C\,h(|1-\alpha_1| + |1-\alpha_4|). \tag{2.27}$$

Setting $\boldsymbol{v}_h = \tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h$ in (2.25), we obtain by (1.8), (2.7), and (2.21)

$$\|\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h\|_{1,\Omega} \leq C\,A_h. \tag{2.28}$$

That together with (2.25) and (2.22) gives

$$b(\boldsymbol{v}_h, \tilde{p}_h - p_h) \leq C\,A_h \|\boldsymbol{v}_h\|_{1,\Omega} \quad \forall\, \boldsymbol{v}_h \in \mathrm{V}_h,$$

which implies by (A5)

$$\|\tilde{p}_h - p_h\|_{0,\Omega} \leq C\,A_h. \tag{2.29}$$

Thus, (2.16) and (2.18) follow from (2.26)–(2.29) and Theorem 2.2.

Finally, let us assume that the problem (1.6) is regular. Setting $\boldsymbol{v} = \tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h$ and $q = 0$ in (2.4), we obtain for any $\boldsymbol{g} \in L^2(\Omega)^d$

$$\langle \boldsymbol{g}, \tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h \rangle = a(\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h, \boldsymbol{u}_g - r_h \boldsymbol{u}_g) + a(\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h, r_h \boldsymbol{u}_g) + b(\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h, p_g).$$

Denoting $\bar{\boldsymbol{v}}_h = \tilde{\boldsymbol{u}}_h - \tilde{\boldsymbol{u}}_{bh}$, we have for any $\boldsymbol{v}_h^1 \in \mathrm{V}_h^1$

$$a(\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h, \boldsymbol{v}_h^1) = a_h(\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h, \boldsymbol{v}_h^1) + (1-\alpha_2)\,a(\bar{\boldsymbol{v}}_h^2, \boldsymbol{v}_h^1) - (1-\alpha_2)\,a(\bar{\boldsymbol{u}}_h^2, \boldsymbol{v}_h^1)$$

and hence it follows from (2.23) that

$$a(\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h, \boldsymbol{v}_h^1) = \langle \boldsymbol{f}_h - \boldsymbol{f}, \boldsymbol{v}_h^1 \rangle - b(\boldsymbol{v}_h^1, \tilde{p}_h - p_h) + (1-\alpha_2)\,a(\bar{\boldsymbol{v}}_h^2, \boldsymbol{v}_h^1).$$

Thus, we obtain in view of (1.8), (2.4), and (2.7)

$$\begin{aligned}\langle \boldsymbol{g}, \tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h \rangle = {} & \langle \boldsymbol{f}_h - \boldsymbol{f}, r_h \boldsymbol{u}_g \rangle + a(\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h, \boldsymbol{u}_g - r_h \boldsymbol{u}_g) \\ & - (1-\alpha_2)\,a(\bar{\boldsymbol{v}}_h^2, \boldsymbol{u}_g - r_h \boldsymbol{u}_g) + (1-\alpha_2)\,a(\bar{\boldsymbol{v}}_h^2, \boldsymbol{u}_g) \\ & + b(\boldsymbol{u}_g - r_h \boldsymbol{u}_g, \tilde{p}_h - p_h) + b(\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h, p_g - s_h p_g)\end{aligned}$$

and (2.19), (2.24), (A1), and (A2) imply that

$$\begin{aligned}\langle \boldsymbol{g}, \tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h \rangle \leq {} & C\,h \|\boldsymbol{u}_g\|_{2,\Omega} (h^m + |\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h|_{1,\Omega} + |\bar{\boldsymbol{v}}_h^2|_{1,\Omega} + \|\tilde{p}_h - p_h\|_{0,\Omega}) \\ & + C\,h \|p_g\|_{1,\Omega} |\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h|_{1,\Omega}.\end{aligned}$$

Using (2.5) and the fact that $\boldsymbol{g} \in L^2(\Omega)^d$ is arbitrary, (2.20) follows as a consequence of (2.27)–(2.29), (2.18), (2.11), and (2.3). □

Remark 2.10. It follows from Lemma 2.6 that (2.16) and (2.18) remain valid if $\tilde{\boldsymbol{u}}_h$ is replaced by $\tilde{\boldsymbol{u}}_h^* \equiv (\tilde{\boldsymbol{u}}_h - \tilde{\boldsymbol{u}}_{bh})^1 + \tilde{\boldsymbol{u}}_{bh}$. Moreover, if $\|\tilde{\boldsymbol{u}}_b - \tilde{\boldsymbol{u}}_{bh}\|_{0,\Omega} \leq C\, h^{m+1}$, then (2.20) holds with $\tilde{\boldsymbol{u}}_h^*$ instead of $\tilde{\boldsymbol{u}}_h$ as well.

Remark 2.11. For $\alpha_1 \neq 1$ or $\alpha_4 \neq 1$, we get only a linear convergence in (2.18) because, for usual finite element spaces, the estimate (2.24) cannot be improved (neither for $\hat{\boldsymbol{u}} \in C^\infty(\overline{\Omega})^d$).

Remark 2.12. Let $\boldsymbol{f}_h$ be defined by (2.8) and let $\boldsymbol{f} \in L^2(\Omega)^d$. Then, for any $\boldsymbol{v}_h \in \mathrm{V}_h$, we have $\langle \boldsymbol{f} - \boldsymbol{f}_h, \boldsymbol{v}_h\rangle = \langle \boldsymbol{f}, \boldsymbol{v}_h^2\rangle \leq \|\boldsymbol{f}\|_{0,\Omega}\|\boldsymbol{v}_h^2\|_{0,\Omega}$ and, using (A3) and (A4), we deduce that $\boldsymbol{f}_h$ satisfies (2.17) with $m = 1$. It is also possible to define $\boldsymbol{f}_h$ by applying numerical integration for evaluating $\langle \boldsymbol{f}, \boldsymbol{v}_h\rangle$ (cf., e.g., [7]).

Remark 2.13. For simplicity, in the relations (2.2), (2.3), (2.18), and (2.20), we have not specified the dependence of the constant C on $\boldsymbol{u}$, p, and $\boldsymbol{f}$. Let us mention this dependence in the case when $\tilde{\boldsymbol{u}}_b = \tilde{\boldsymbol{u}}_{bh} = \boldsymbol{0}$ and $\boldsymbol{f}_h = \boldsymbol{f}$. Concerning (2.2) and (2.3), it is known that $C \leq \widetilde{C}(\|\boldsymbol{u}\|_{m+1,\Omega} + \|p\|_{m,\Omega})$, where the constant $\widetilde{C}$ is independent of h, $\boldsymbol{u}$, p, and $\boldsymbol{f}$ (cf., [11], pp. 125–127, Theorems 1.8 and 1.9). It is easy to check that this estimate is also valid for C in (2.18) and (2.20). Particularly, if the problem (1.6) is regular and $\boldsymbol{f} \in L^2(\Omega)^d$, then $\boldsymbol{u} \in H^2(\Omega)^d$, $p \in H^1(\Omega)$ and it follows from (2.5) that

$$\|\boldsymbol{u} - \tilde{\boldsymbol{u}}_h\|_{1,\Omega} + \|p - \tilde{p}_h\|_{0,\Omega} \leq C\, h\|\boldsymbol{f}\|_{0,\Omega}, \quad \|\boldsymbol{u} - \tilde{\boldsymbol{u}}_h\|_{0,\Omega} \leq C\, h^2\|\boldsymbol{f}\|_{0,\Omega},$$

where the constant C is again independent of h, $\boldsymbol{u}$, p, and $\boldsymbol{f}$.

3. DISCRETIZATION OF THE NAVIER-STOKES EQUATIONS

In this section, we retain the assumptions (A1)–(A5) made in Section 2 and assume further that there exists a positive constant α independent of h such that:

(A6) There exist operators $\bar{r}_h \in \mathscr{L}(H_0^1(\Omega)^d, \mathrm{V}_h^1)$ such that

$$\|\boldsymbol{v} - \bar{r}_h\boldsymbol{v}\|_{0,4,\Omega} \leq C\, h^\alpha\|\boldsymbol{v}\|_{1,\Omega} \quad \forall\, \boldsymbol{v} \in H_0^1(\Omega)^d, \tag{3.1}$$

$$\|\boldsymbol{v} - \bar{r}_h\boldsymbol{v}\|_{0,\Omega} + h\|\boldsymbol{v} - \bar{r}_h\boldsymbol{v}\|_{1,\Omega} \leq C\, h^{m+1}\|\boldsymbol{v}\|_{m+1,\Omega}$$
$$\forall\, \boldsymbol{v} \in H^{m+1}(\Omega)^d \cap H_0^1(\Omega)^d,\ \ 0 \leq m \leq l. \tag{3.2}$$

(A7) The spaces V_h^2 satisfy

$$\|\boldsymbol{v}_h^2\|_{0,4,\Omega} \leq C\, h^\alpha|\boldsymbol{v}_h^2|_{1,\Omega} \quad \forall\, \boldsymbol{v}_h^2 \in \mathrm{V}_h^2.$$

Remark 3.1. The assumption (A6) is satisfied for standard finite element spaces, see [2]. The validity of (A7) will be investigated in Section 4.

In the following lemma, we construct an operator z_h having the properties (3.1) and (3.2) and extending the mapping $\boldsymbol{v}_h \to \boldsymbol{v}_h^1$ to functions from $H_0^1(\Omega)^d$. We shall use the operator z_h for establishing an operator formulation of the discrete problem defined on $H_0^1(\Omega)^d \times L_0^2(\Omega)$.

Lemma 3.2. *There exist operators $z_h \in \mathscr{L}(H_0^1(\Omega)^d, H_0^1(\Omega)^d)$ such that*

$$z_h \boldsymbol{v}_h = \boldsymbol{v}_h^1 \quad \forall\, \boldsymbol{v}_h \in \mathrm{V}_h, \tag{3.3}$$

$$\|\boldsymbol{v} - z_h \boldsymbol{v}\|_{0,4,\Omega} \leq C\, h^\alpha \|\boldsymbol{v}\|_{1,\Omega} \quad \forall\, \boldsymbol{v} \in H_0^1(\Omega)^d, \tag{3.4}$$

$$\begin{aligned} &\|\boldsymbol{v} - z_h \boldsymbol{v}\|_{0,\Omega} + h\|\boldsymbol{v} - z_h \boldsymbol{v}\|_{1,\Omega} \leq C\, h^{m+1} \|\boldsymbol{v}\|_{m+1,\Omega} \\ &\qquad \forall\, \boldsymbol{v} \in H^{m+1}(\Omega)^d \cap H_0^1(\Omega)^d, \;\; 0 \leq m \leq l. \end{aligned} \tag{3.5}$$

Proof. Let $i_h \in \mathscr{L}(H_0^1(\Omega)^d, \mathrm{V}_h)$ be the orthogonal projection of $H_0^1(\Omega)^d$ onto V_h, that is,

$$\int_\Omega \nabla(\boldsymbol{v} - i_h \boldsymbol{v}) \cdot \nabla \boldsymbol{v}_h \,\mathrm{d}x = 0 \quad \forall\, \boldsymbol{v} \in H_0^1(\Omega)^d, \;\; \boldsymbol{v}_h \in \mathrm{V}_h,$$

which implies

$$|i_h \boldsymbol{v}|_{1,\Omega} \leq |\boldsymbol{v}|_{1,\Omega} \quad \forall\, \boldsymbol{v} \in H_0^1(\Omega)^d.$$

Using the operator i_h, we can define an operator $z_h \in \mathscr{L}(H_0^1(\Omega)^d, \mathrm{V}_h^1)$ by

$$z_h \boldsymbol{v} = (i_h \boldsymbol{v})^1 + \bar{r}_h(\boldsymbol{v} - i_h \boldsymbol{v}), \quad \boldsymbol{v} \in H_0^1(\Omega)^d,$$

where $(i_h \boldsymbol{v})^1$ denotes the part of $i_h \boldsymbol{v}$ lying in V_h^1. Then (3.3) holds and, for any $\boldsymbol{v} \in H_0^1(\Omega)^d$, we have

$$\boldsymbol{v} - z_h \boldsymbol{v} = (\boldsymbol{v} - i_h \boldsymbol{v}) - \bar{r}_h(\boldsymbol{v} - i_h \boldsymbol{v}) + (i_h \boldsymbol{v} - \bar{r}_h i_h \boldsymbol{v})^2.$$

Applying (3.2), (A3), (A4), and Friedrichs' inequality, we obtain for $k = 0, 1$

$$h^k \|\boldsymbol{v} - z_h \boldsymbol{v}\|_{k,\Omega} \leq C\, h \|\boldsymbol{v} - i_h \boldsymbol{v}\|_{1,\Omega} + C\, h \|i_h \boldsymbol{v} - \bar{r}_h i_h \boldsymbol{v}\|_{1,\Omega} \leq \widetilde{C} h \|\boldsymbol{v}\|_{1,\Omega} \tag{3.6}$$

and hence (3.5) holds with $m = 0$. Analogously, using (3.1) and (A7), we also get (3.4). Finally, in view of (3.3), we have for any $\boldsymbol{v} \in H_0^1(\Omega)^d$

$$\boldsymbol{v} - z_h \boldsymbol{v} = (\boldsymbol{v} - \bar{r}_h \boldsymbol{v}) - z_h(\boldsymbol{v} - \bar{r}_h \boldsymbol{v})$$

and hence, using (3.6) and (3.2), we obtain for $k = 0, 1$ and any $\boldsymbol{v} \in H^{m+1}(\Omega)^d \cap H_0^1(\Omega)^d$ with $m \in \{1, \ldots, l\}$

$$h^k \|\boldsymbol{v} - z_h \boldsymbol{v}\|_{k,\Omega} \le C\, h \|\boldsymbol{v} - \bar{r}_h \boldsymbol{v}\|_{1,\Omega} \le \widetilde{C} h^{m+1} \|\boldsymbol{v}\|_{m+1,\Omega}. \qquad \square$$

A conforming discretization of the Navier-Stokes equations can be obtained from (1.2)–(1.3) analogously as the conforming discretization (1.7)–(1.8) of the Stokes equations from (1.5)–(1.6). To define a general reduced discretization of the Navier-Stokes equations, we shall again use the bilinear form a_h and the approximative functional $\boldsymbol{f}_h$ introduced in the preceding section and we shall assume that $\alpha_3 > 0$ and $|\alpha_1 + \alpha_2| \le 2\sqrt{\alpha_3}$. In addition, we replace the nonlinear term $n(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h)$ from the conforming discretization by the term

$$\begin{aligned} n_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) = {} & n(\boldsymbol{u}_h^*, \boldsymbol{u}_h^*, \boldsymbol{v}_h^1) + \gamma_1 n(\boldsymbol{u}_h^*, \boldsymbol{u}_h^*, \boldsymbol{v}_h^2) + \gamma_2 n(\boldsymbol{u}_h^*, \bar{\boldsymbol{u}}_h^2, \boldsymbol{v}_h^1) \\ & + \gamma_3 n(\boldsymbol{u}_h^*, \bar{\boldsymbol{u}}_h^2, \boldsymbol{v}_h^2) + \gamma_4 n(\bar{\boldsymbol{u}}_h^2, \boldsymbol{u}_h, \boldsymbol{v}_h), \end{aligned}$$

where $\boldsymbol{u}_h^* = (\boldsymbol{u}_h - \tilde{\boldsymbol{u}}_{bh})^1 + \tilde{\boldsymbol{u}}_{bh}$, $\bar{\boldsymbol{u}}_h^2 = (\boldsymbol{u}_h - \tilde{\boldsymbol{u}}_{bh})^2$ and $\gamma_1, \ldots, \gamma_4$ are arbitrary real numbers. For $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 1$, we have $n_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) = n(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h)$. Finally, we again choose an arbitrary real number α_4.

Definition 3.3. The functions $\boldsymbol{u}_h \in H^1(\Omega)^d$ and $p_h \in Q_h$ are a *discrete solution* of the problem (1.2)–(1.3) if

$$\boldsymbol{u}_h - \tilde{\boldsymbol{u}}_{bh} \in V_h, \tag{3.7}$$

$$\begin{aligned} & a_h(\boldsymbol{u}_h - \tilde{\boldsymbol{u}}_{bh}, \boldsymbol{v}_h) + n_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) - b(\boldsymbol{u}_h, q_h) \\ & \quad = \langle \boldsymbol{f}_h, \boldsymbol{v}_h \rangle - a(\tilde{\boldsymbol{u}}_{bh}, \boldsymbol{v}_h^1) - \alpha_4 a(\tilde{\boldsymbol{u}}_{bh}, \boldsymbol{v}_h^2) \quad \forall\, \boldsymbol{v}_h \in V_h,\ q_h \in Q_h. \end{aligned} \tag{3.8}$$

Remark 3.4. The discrete problem (3.7)–(3.8) can be interpreted analogously as the problem (2.6)–(2.7) in Remarks 2.4 and 2.5. Particularly, for $\alpha_1 = \alpha_2 = \alpha_4 = 0$, $\alpha_3 = 1$, $\gamma_i = 0$, $i = 1, \ldots, 4$, and $\boldsymbol{f}_h$ satisfying (2.8), the discretization (3.7)–(3.8) can be written in the matrix form (1.10).

Remark 3.5. Using the operator z_h, we can define $\boldsymbol{f}_h$ by

$$\langle \boldsymbol{f}_h, \boldsymbol{v} \rangle = \langle \boldsymbol{f}, z_h \boldsymbol{v} \rangle \quad \forall\, \boldsymbol{v} \in H_0^1(\Omega)^d.$$

Then $\boldsymbol{f}_h$ satisfies (2.8) and, if $\boldsymbol{f} \in L^2(\Omega)^d$, we have $\|\boldsymbol{f} - \boldsymbol{f}_h\|_{-1,\Omega} \le C\, h \|\boldsymbol{f}\|_{0,\Omega}$.

For investigating the convergence behavior of the solutions of (3.7)–(3.8), it is convenient to establish operator formulations of both the problem (3.7)–(3.8) and the weak formulation (1.2)–(1.3). We shall proceed in a similar way as in [11], [3], or [6]. We define the spaces

$$\mathrm{X} = H_0^1(\Omega)^d \times L_0^2(\Omega), \quad \widehat{\mathrm{X}} = H^1(\Omega)^d \times L_0^2(\Omega), \quad \widehat{\mathrm{X}}_h = H^1(\Omega)^d \times \mathrm{Q}_h$$

and we equip the space $\widehat{\mathrm{X}}$ (containing both X and $\widehat{\mathrm{X}}_h$) with a norm $\|\cdot\|_{\widehat{\mathrm{X}}}$, which is some of the usual norms of a Cartesian product of normed spaces. Further we introduce operators $\mathrm{P} \in \mathscr{L}(H^1(\Omega)^d, \widehat{\mathrm{X}})$ and $\mathrm{R} \in \mathscr{L}(\widehat{\mathrm{X}}, H^1(\Omega)^d)$ defined by

$$\mathrm{P}\boldsymbol{u} = (\boldsymbol{u}, 0), \quad \mathrm{R}(\boldsymbol{u}, p) = \boldsymbol{u} \quad \forall\, \boldsymbol{u} \in H^1(\Omega)^d, \; p \in L_0^2(\Omega). \tag{3.9}$$

According to the preceding sections, there exist uniquely determined operators $\mathrm{T} : H^{-1}(\Omega)^d \to \widehat{\mathrm{X}}$ and $\mathrm{T}_h : H^{-1}(\Omega)^d \to \widehat{\mathrm{X}}_h$ such that, for any $\boldsymbol{f}$, $\boldsymbol{f}_h \in H^{-1}(\Omega)^d$, $(\boldsymbol{u}, p) = \mathrm{T}\boldsymbol{f}$ is the solution of (1.5)–(1.6) and $(\tilde{\boldsymbol{u}}_h, \tilde{p}_h) = \mathrm{T}_h\boldsymbol{f}_h$ is the solution of (2.6)–(2.7). In addition, we define linear operators T^0, $\mathrm{T}_h^0 : H^{-1}(\Omega)^d \to \mathrm{X}$ corresponding to the respective problems with homogeneous Dirichlet boundary conditions, that is, for any $\boldsymbol{f}$, $\boldsymbol{f}_h \in H^{-1}(\Omega)^d$, $(\boldsymbol{u}, p) = \mathrm{T}^0\boldsymbol{f}$ is the solution of (1.6) and $(\tilde{\boldsymbol{u}}_h, \tilde{p}_h) = \mathrm{T}_h^0\boldsymbol{f}_h$ with $\tilde{p}_h \in \mathrm{Q}_h$ is the solution of (2.6)–(2.7) with $\tilde{\boldsymbol{u}}_{bh} = \boldsymbol{0}$. It is easy to verify that

$$\mathrm{T}\boldsymbol{f} - \mathrm{T}\tilde{\boldsymbol{f}} = \mathrm{T}^0(\boldsymbol{f} - \tilde{\boldsymbol{f}}) = \mathrm{T}^0\boldsymbol{f} - \mathrm{T}^0\tilde{\boldsymbol{f}} \quad \forall\, \boldsymbol{f}, \tilde{\boldsymbol{f}} \in H^{-1}(\Omega)^d \tag{3.10}$$

and

$$\mathrm{T}_h\boldsymbol{f} - \mathrm{T}_h\tilde{\boldsymbol{f}} = \mathrm{T}_h^0(\boldsymbol{f} - \tilde{\boldsymbol{f}}) = \mathrm{T}_h^0\boldsymbol{f} - \mathrm{T}_h^0\tilde{\boldsymbol{f}} \quad \forall\, \boldsymbol{f}, \tilde{\boldsymbol{f}} \in H^{-1}(\Omega)^d. \tag{3.11}$$

In the space $\widehat{\mathrm{X}}$, the Dirichlet boundary conditions will be represented by

$$\widehat{\mathrm{U}}_b = (\tilde{\boldsymbol{u}}_b, 0), \quad \widehat{\mathrm{U}}_{bh} = (\tilde{\boldsymbol{u}}_{bh}, 0).$$

We recall that the functions $\tilde{\boldsymbol{u}}_b$ and $\tilde{\boldsymbol{u}}_{bh}$ are fixed and satisfy (2.1). Moreover, from now on, the functionals $\boldsymbol{f}$, $\boldsymbol{f}_h \in H^{-1}(\Omega)^d$ from (1.3) and (3.8) will be assumed to be fixed as well and to satisfy

$$\lim_{h\to 0} \|\boldsymbol{f} - \boldsymbol{f}_h\|_{-1,\Omega} = 0. \tag{3.12}$$

To describe the nonlinear terms in (1.3) and (3.8) we introduce operators G, $\mathrm{G}_h : \mathrm{X} \to H^{-1}(\Omega)^d$ defined for any $\mathrm{U} \equiv (\boldsymbol{u}, p) \in \mathrm{X}$ and $\boldsymbol{v} \in H_0^1(\Omega)^d$ by

$$\begin{aligned}
\langle \mathrm{G}(\mathrm{U}), \boldsymbol{v}\rangle &= \langle \boldsymbol{f}, \boldsymbol{v}\rangle - n(\boldsymbol{u} + \tilde{\boldsymbol{u}}_b, \boldsymbol{u} + \tilde{\boldsymbol{u}}_b, \boldsymbol{v}),\\
\langle \mathrm{G}_h(\mathrm{U}), \boldsymbol{v}\rangle &= \langle \boldsymbol{f}_h, \boldsymbol{v}\rangle - n(z_h\boldsymbol{u} + \tilde{\boldsymbol{u}}_{bh}, z_h\boldsymbol{u} + \tilde{\boldsymbol{u}}_{bh}, \boldsymbol{v})\\
&\quad + (1 - \gamma_1)\, n(z_h\boldsymbol{u} + \tilde{\boldsymbol{u}}_{bh}, z_h\boldsymbol{u} + \tilde{\boldsymbol{u}}_{bh}, \boldsymbol{v} - z_h\boldsymbol{v})
\end{aligned}$$

$$- \gamma_2 n(z_h \boldsymbol{u} + \tilde{\boldsymbol{u}}_{bh}, \boldsymbol{u} - z_h \boldsymbol{u}, z_h \boldsymbol{v})$$
$$- \gamma_3 n(z_h \boldsymbol{u} + \tilde{\boldsymbol{u}}_{bh}, \boldsymbol{u} - z_h \boldsymbol{u}, \boldsymbol{v} - z_h \boldsymbol{v}) - \gamma_4 n(\boldsymbol{u} - z_h \boldsymbol{u}, \boldsymbol{u} + \tilde{\boldsymbol{u}}_{bh}, \boldsymbol{v}).$$

Note that $\mathrm{G}_h = \mathrm{G}$ for $\boldsymbol{f}_h = \boldsymbol{f}$, $\tilde{\boldsymbol{u}}_{bh} = \tilde{\boldsymbol{u}}_b$ and $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 1$. Further, for $\widehat{\mathrm{U}}_h = (\boldsymbol{u}_h, p_h)$ with $\boldsymbol{u}_h$ satisfying (3.7), we have

$$\langle \mathrm{G}_h(\widehat{\mathrm{U}}_h - \widehat{\mathrm{U}}_{bh}), \boldsymbol{v}_h \rangle = \langle \boldsymbol{f}_h, \boldsymbol{v}_h \rangle - n_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) \quad \forall\, \boldsymbol{v}_h \in \mathrm{V}_h.$$

Thus, $\widehat{\mathrm{U}} \equiv (\boldsymbol{u}, p) \in \widehat{\mathrm{X}}$ is a solution of (1.2)–(1.3) if and only if $\widehat{\mathrm{U}} - \widehat{\mathrm{U}}_b \in \mathrm{X}$ and $\widehat{\mathrm{U}} = \mathrm{TG}(\widehat{\mathrm{U}} - \widehat{\mathrm{U}}_b)$. Similarly, $\widehat{\mathrm{U}}_h \equiv (\boldsymbol{u}_h, p_h) \in \widehat{\mathrm{X}}_h$ is a solution of (3.7)–(3.8) if and only if $\widehat{\mathrm{U}}_h - \widehat{\mathrm{U}}_{bh} \in \mathrm{X}$ and $\widehat{\mathrm{U}}_h = \mathrm{T}_h \mathrm{G}_h(\widehat{\mathrm{U}}_h - \widehat{\mathrm{U}}_{bh})$. Therefore, defining operators $\mathrm{F}, \mathrm{F}_h : \mathrm{X} \to \mathrm{X}$ by

$$\mathrm{F}(\mathrm{U}) = \mathrm{TG}(\mathrm{U}) - \mathrm{U} - \widehat{\mathrm{U}}_b, \quad \mathrm{F}_h(\mathrm{U}) = \mathrm{T}_h \mathrm{G}_h(\mathrm{U}) - \mathrm{U} - \widehat{\mathrm{U}}_{bh} \quad \forall\, \mathrm{U} \in \mathrm{X},$$

we have

$$\widehat{\mathrm{U}} \equiv (\boldsymbol{u}, p) \in \widehat{\mathrm{X}} \text{ solves } (1.2)\text{–}(1.3) \Leftrightarrow \widehat{\mathrm{U}} - \widehat{\mathrm{U}}_b \in \mathrm{X},\ \ \mathrm{F}(\widehat{\mathrm{U}} - \widehat{\mathrm{U}}_b) = 0,$$
$$\widehat{\mathrm{U}}_h \equiv (\boldsymbol{u}_h, p_h) \in \widehat{\mathrm{X}}_h \text{ solves } (3.7)\text{–}(3.8) \Leftrightarrow \widehat{\mathrm{U}}_h - \widehat{\mathrm{U}}_{bh} \in \mathrm{X},\ \ \mathrm{F}_h(\widehat{\mathrm{U}}_h - \widehat{\mathrm{U}}_{bh}) = 0.$$

Now let us investigate the properties of the above operators.

Theorem 3.6. *The operators* T, T^0, T_h, *and* T_h^0 *are continuous and the operators* T^0 *and* T_h^0 *are in addition linear. Further we have*

$$\|\mathrm{T}_h^0\|_{\mathscr{L}(H^{-1}(\Omega)^d, X)} \leq C \tag{3.13}$$

(with C *independent of* h*) and*

$$\lim_{h \to 0} \|\mathrm{T}\boldsymbol{g} - \mathrm{T}_h \boldsymbol{g}\|_{\widehat{X}} = 0 \quad \forall\, \boldsymbol{g} \in H^{-1}(\Omega)^d. \tag{3.14}$$

If the problem (1.6) *is regular, then*

$$\|\mathrm{RT}^0 - \mathrm{RT}_h^0\|_{\mathscr{L}(L^2(\Omega)^d, L^2(\Omega)^d)} + h\|\mathrm{T}^0 - \mathrm{T}_h^0\|_{\mathscr{L}(L^2(\Omega)^d, X)} \leq C\, h^2. \tag{3.15}$$

If $\mathrm{T}\boldsymbol{g} \in H^{m+1}(\Omega)^d \times H^m(\Omega)$ *for some* $\boldsymbol{g} \in H^{-1}(\Omega)^d$ *and if* $\tilde{\boldsymbol{u}}_b \in H^{m+1}(\Omega)^d$ *and* $\|\tilde{\boldsymbol{u}}_b - \tilde{\boldsymbol{u}}_{bh}\|_{1,\Omega} \leq C\, h^m$, *where* $m \in \{1, \ldots, l\}$, *then*

$$\|\mathrm{T}\boldsymbol{g} - \mathrm{T}_h \boldsymbol{g}\|_{\widehat{X}} \leq C\, h^m + C\, h(|1 - \alpha_1| + |1 - \alpha_4|). \tag{3.16}$$

Moreover, if $\mathrm{TG}(\mathrm{U}) \in H^{m+1}(\Omega)^d \times H^m(\Omega)$ *for some* $\mathrm{U} \equiv (\boldsymbol{u}, p) \in X$ *with* $\boldsymbol{u} \in H^{m+1}(\Omega)^d$ *and if* $\tilde{\boldsymbol{u}}_b \in H^{m+1}(\Omega)^d$ *and* (2.17) *holds, then we have*

$$\|\mathrm{TG}(\mathrm{U}) - \mathrm{T}_h \mathrm{G}_h(\mathrm{U})\|_{\widehat{X}} \leq C\, h^m + C\, h(|1 - \alpha_1| + |1 - \alpha_4| + |1 - \gamma_1|). \tag{3.17}$$

If, in addition, the problem (1.6) *is regular,* $\|\tilde{\boldsymbol{u}}_b - \tilde{\boldsymbol{u}}_{bh}\|_{0,\Omega} \le C\,h^{m+1}$ *and* (2.19) *holds, then we obtain*

$$\|\mathrm{RTG(U)} - \mathrm{RT}_h\mathrm{G}_h\mathrm{(U)}\|_{0,\Omega} \le C\,h^{m+1} + C\,h^2(|1-\alpha_1| + |1-\alpha_4| + |1-\gamma_1|). \tag{3.18}$$

Proof. Let us consider (2.6)–(2.7) with $\tilde{\boldsymbol{u}}_{bh} = \mathbf{0}$. Setting $\boldsymbol{v}_h = \tilde{\boldsymbol{u}}_h$ and $q_h = \tilde{p}_h$ in (2.7) and applying (2.21), we obtain $\|\tilde{\boldsymbol{u}}_h\|_{1,\Omega} \le C\|\boldsymbol{f}_h\|_{-1,\Omega}$. From (2.7), (2.22), and (A5), we then deduce that also $\|\tilde{p}_h\|_{0,\Omega} \le C\|\boldsymbol{f}_h\|_{-1,\Omega}$ and hence we obtain (3.13). The continuity of T_h follows from (3.11). Because the Babuška-Brezzi condition (A5) also holds for the spaces $H_0^1(\Omega)^d$ and $L_0^2(\Omega)$ (cf. [11], p. 81), the continuity of T and T^0 can be proven analogously. The convergence statements (3.14)–(3.16) immediately follow from Theorem 2.9 and Remark 2.13.

It remains to show the validity of (3.17) and (3.18). In view of the continuous imbedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$, we have for any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in H^1(\Omega)^d$

$$n(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) \le \sqrt{d}\|\boldsymbol{u}\|_{0,4,\Omega}\|\boldsymbol{v}\|_{0,4,\Omega}|\boldsymbol{w}|_{1,\Omega} \le C\|\boldsymbol{u}\|_{1,\Omega}\|\boldsymbol{v}\|_{1,\Omega}|\boldsymbol{w}|_{1,\Omega} \tag{3.19}$$

(cf. [11], p. 284, Lemma 2.1). Further, integrating by parts, we get

$$n(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) = -n(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) - \int_\Omega (\boldsymbol{v}\cdot\boldsymbol{w})\operatorname{div}\boldsymbol{u}\,\mathrm{d}x \quad \forall\, \boldsymbol{u}, \boldsymbol{w} \in H^1(\Omega)^d, \ \ \boldsymbol{v} \in H_0^1(\Omega)^d. \tag{3.20}$$

Using this relation, we obtain for any $\boldsymbol{u}$, $\bar{\boldsymbol{u}}$, $\boldsymbol{w}$, $\bar{\boldsymbol{w}} \in H^1(\Omega)^d$ and $\boldsymbol{v} \in H_0^1(\Omega)^d$

$$n(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) - n(\bar{\boldsymbol{u}}, \bar{\boldsymbol{w}}, \boldsymbol{v}) = -\int_\Omega \boldsymbol{v}\cdot(\boldsymbol{w} - \bar{\boldsymbol{w}})\operatorname{div}\boldsymbol{u}\,\mathrm{d}x - n(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} - \bar{\boldsymbol{w}}) + n(\boldsymbol{u} - \bar{\boldsymbol{u}}, \bar{\boldsymbol{w}}, \boldsymbol{v}). \tag{3.21}$$

Thus, owing to (3.19) and the imbeddings $H^1(\Omega) \hookrightarrow L^4(\Omega)$ and $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$, we derive for any $\boldsymbol{u} \in H^2(\Omega)^d$, $\bar{\boldsymbol{u}} \in H^1(\Omega)^d$ and $\boldsymbol{v} \in H_0^1(\Omega)^d$

$$n(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - n(\bar{\boldsymbol{u}}, \bar{\boldsymbol{u}}, \boldsymbol{v}) \le C(\|\boldsymbol{u} - \bar{\boldsymbol{u}}\|_{0,\Omega}\|\boldsymbol{u}\|_{2,\Omega} + \|\boldsymbol{u} - \bar{\boldsymbol{u}}\|_{1,\Omega}^2)\|\boldsymbol{v}\|_{1,\Omega}.$$

Therefore, we get for any $\boldsymbol{u} \in H^2(\Omega)^d \cap H_0^1(\Omega)^d$ and $\boldsymbol{v} \in H_0^1(\Omega)^d$

$$\begin{aligned} &n(\boldsymbol{u} + \tilde{\boldsymbol{u}}_b, \boldsymbol{u} + \tilde{\boldsymbol{u}}_b, \boldsymbol{v}) - n(z_h\boldsymbol{u} + \tilde{\boldsymbol{u}}_{bh}, z_h\boldsymbol{u} + \tilde{\boldsymbol{u}}_{bh}, \boldsymbol{v}) \\ &\quad \le C(\|\boldsymbol{u} - z_h\boldsymbol{u}\|_{0,\Omega} + \|\tilde{\boldsymbol{u}}_b - \tilde{\boldsymbol{u}}_{bh}\|_{0,\Omega})\|\boldsymbol{u} + \tilde{\boldsymbol{u}}_b\|_{2,\Omega}\|\boldsymbol{v}\|_{1,\Omega} \\ &\qquad + C(\|\boldsymbol{u} - z_h\boldsymbol{u}\|_{1,\Omega}^2 + \|\tilde{\boldsymbol{u}}_b - \tilde{\boldsymbol{u}}_{bh}\|_{1,\Omega}^2)\|\boldsymbol{v}\|_{1,\Omega}. \end{aligned}$$

Further, we have

$$\begin{aligned}
n(z_h\boldsymbol{u}+\tilde{\boldsymbol{u}}_{bh}, z_h\boldsymbol{u}+\tilde{\boldsymbol{u}}_{bh}, \boldsymbol{v}) &= -n(\boldsymbol{u}-z_h\boldsymbol{u}, z_h\boldsymbol{u}+\tilde{\boldsymbol{u}}_{bh}, \boldsymbol{v}) - n(\tilde{\boldsymbol{u}}_b-\tilde{\boldsymbol{u}}_{bh}, z_h\boldsymbol{u}+\tilde{\boldsymbol{u}}_{bh}, \boldsymbol{v})\\
&\quad + n(\boldsymbol{u}+\tilde{\boldsymbol{u}}_b, z_h\boldsymbol{u}+\tilde{\boldsymbol{u}}_{bh}, \boldsymbol{v})\\
&\leq C\|z_h\boldsymbol{u}+\tilde{\boldsymbol{u}}_{bh}\|_{1,\Omega}\big(\|\boldsymbol{u}-z_h\boldsymbol{u}\|_{1,\Omega}\|\boldsymbol{v}\|_{1,\Omega}\\
&\quad + \|\tilde{\boldsymbol{u}}_b-\tilde{\boldsymbol{u}}_{bh}\|_{1,\Omega}\|\boldsymbol{v}\|_{1,\Omega} + \|\boldsymbol{u}+\tilde{\boldsymbol{u}}_b\|_{2,\Omega}\|\boldsymbol{v}\|_{0,\Omega}\big),\\
n(z_h\boldsymbol{u}+\tilde{\boldsymbol{u}}_{bh}, \boldsymbol{u}-z_h\boldsymbol{u}, \boldsymbol{v}) &= -n(\boldsymbol{u}-z_h\boldsymbol{u}, \boldsymbol{u}-z_h\boldsymbol{u}, \boldsymbol{v}) - n(\boldsymbol{u}+\tilde{\boldsymbol{u}}_b, \boldsymbol{v}, \boldsymbol{u}-z_h\boldsymbol{u})\\
&\quad - \int_\Omega \boldsymbol{v}\cdot(\boldsymbol{u}-z_h\boldsymbol{u})\mathrm{div}(\boldsymbol{u}+\tilde{\boldsymbol{u}}_b)\mathrm{d}x\\
&\quad - n(\tilde{\boldsymbol{u}}_b-\tilde{\boldsymbol{u}}_{bh}, \boldsymbol{u}-z_h\boldsymbol{u}, \boldsymbol{v})\\
&\leq C(\|\boldsymbol{u}-z_h\boldsymbol{u}\|_{1,\Omega}^2 + \|\boldsymbol{u}+\tilde{\boldsymbol{u}}_b\|_{2,\Omega}\|\boldsymbol{u}-z_h\boldsymbol{u}\|_{0,\Omega}\\
&\qquad + \|\tilde{\boldsymbol{u}}_b-\tilde{\boldsymbol{u}}_{bh}\|_{1,\Omega}\,\|\boldsymbol{u}-z_h\boldsymbol{u}\|_{1,\Omega})\|\boldsymbol{v}\|_{1,\Omega},\\
n(\boldsymbol{u}-z_h\boldsymbol{u}, \boldsymbol{u}+\tilde{\boldsymbol{u}}_{bh}, \boldsymbol{v}) &= n(\boldsymbol{u}-z_h\boldsymbol{u}, \boldsymbol{u}+\tilde{\boldsymbol{u}}_b, \boldsymbol{v}) - n(\boldsymbol{u}-z_h\boldsymbol{u}, \tilde{\boldsymbol{u}}_b-\tilde{\boldsymbol{u}}_{bh}, \boldsymbol{v})\\
&\leq C(\|\boldsymbol{u}-z_h\boldsymbol{u}\|_{0,\Omega}\,\|\boldsymbol{u}+\tilde{\boldsymbol{u}}_b\|_{2,\Omega}\\
&\qquad + \|\boldsymbol{u}-z_h\boldsymbol{u}\|_{1,\Omega}\,\|\tilde{\boldsymbol{u}}_b-\tilde{\boldsymbol{u}}_{bh}\|_{1,\Omega})\|\boldsymbol{v}\|_{1,\Omega}.
\end{aligned}$$

Applying (3.3) and (3.5), we obtain

$$\begin{aligned}
\|\mathrm{G}(\mathrm{U})-\mathrm{G}_h(\mathrm{U})\|_{[\mathrm{V}_h^1]'} &\leq \|\boldsymbol{f}-\boldsymbol{f}_h\|_{[\mathrm{V}_h^1]'} + C(h^{2m}\,\|\boldsymbol{u}\|_{m+1,\Omega}^2 + \|\tilde{\boldsymbol{u}}_b-\tilde{\boldsymbol{u}}_{bh}\|_{1,\Omega}^2)\\
&\quad + C(h^{m+1}\,\|\boldsymbol{u}\|_{m+1,\Omega} + \|\tilde{\boldsymbol{u}}_b-\tilde{\boldsymbol{u}}_{bh}\|_{0,\Omega})\|\boldsymbol{u}+\tilde{\boldsymbol{u}}_b\|_{2,\Omega}
\end{aligned}$$

and

$$\begin{aligned}
\|\mathrm{G}(\mathrm{U})-\mathrm{G}_h(\mathrm{U})\|_{[\mathrm{V}_h]'} &\leq \|\boldsymbol{f}-\boldsymbol{f}_h\|_{[\mathrm{V}_h]'} + C\,h\,|1-\gamma_1|\,\|\boldsymbol{u}+\tilde{\boldsymbol{u}}_b\|_{2,\Omega}^2\\
&\quad + C(h^m\,\|\boldsymbol{u}\|_{m+1,\Omega} + \|\tilde{\boldsymbol{u}}_b-\tilde{\boldsymbol{u}}_{bh}\|_{1,\Omega})(\|\boldsymbol{u}\|_{2,\Omega} + \|\widetilde{\boldsymbol{u}}_b\|_{2,\Omega} + \|\widetilde{\boldsymbol{u}}_{bh}\|_{1,\Omega}),
\end{aligned}$$

and (3.17) and (3.18) follow applying Theorem 2.9. □

Theorem 3.7. *The operators* G *and* G_h *are* C^1 *mappings and the Frechét derivative* DG(U) *is compact for any* U ∈ X. *Moreover, we have for any* U, $\widetilde{\mathrm{U}}$ ∈ X

$$\|\mathrm{DG}_h(\mathrm{U})-\mathrm{DG}_h(\widetilde{\mathrm{U}})\|_{\mathscr{L}(\mathrm{X},H^{-1}(\Omega)^d)} \leq C\|\mathrm{U}-\widetilde{\mathrm{U}}\|_{\hat{\mathrm{X}}}, \tag{3.22}$$

$$\lim_{h\to 0}\|\mathrm{G}(\mathrm{U})-\mathrm{G}_h(\mathrm{U})\|_{-1,\Omega} = 0, \tag{3.23}$$

$$\lim_{h\to 0}\|\mathrm{DG}(\mathrm{U})-\mathrm{DG}_h(\mathrm{U})\|_{\mathscr{L}(\mathrm{X},H^{-1}(\Omega)^d)} = 0, \tag{3.24}$$

where the constant C is independent of h. If $\tilde{\boldsymbol{u}}_b \in H^2(\Omega)^d$, then we obtain for any $\mathrm{U} \equiv (\boldsymbol{u}, p) \in \mathrm{X}$ *with* $\boldsymbol{u} \in H^2(\Omega)^d$

$$\|\mathrm{DG}(\mathrm{U}) - \mathrm{DG}_h(\mathrm{U})\|_{\mathscr{L}(\mathrm{X},H^{-1}(\Omega)^d)} \leq C\, h(\|\boldsymbol{u}\|_{2,\Omega} + \|\tilde{\boldsymbol{u}}_b\|_{2,\Omega}) + C\|\tilde{\boldsymbol{u}}_b - \tilde{\boldsymbol{u}}_{bh}\|_{1,\Omega}. \tag{3.25}$$

Proof. Consider any $\boldsymbol{v} \in H_0^1(\Omega)^d$ and any U, $\widetilde{\mathrm{U}}$, $\mathrm{W} \in \mathrm{X}$ with $\mathrm{U} = (\boldsymbol{u}, p)$, $\widetilde{\mathrm{U}} = (\tilde{\boldsymbol{u}}, \tilde{p})$ and $\mathrm{W} = (\boldsymbol{w}, q)$. The Fréchet derivatives of G and G_h are given by

$$\begin{aligned}
\langle \mathrm{DG}(\mathrm{U})[\mathrm{W}], \boldsymbol{v}\rangle &= -n(\boldsymbol{u} + \tilde{\boldsymbol{u}}_b, \boldsymbol{w}, \boldsymbol{v}) - n(\boldsymbol{w}, \boldsymbol{u} + \tilde{\boldsymbol{u}}_b, \boldsymbol{v}),\\
\langle \mathrm{DG}_h(\mathrm{U})[\mathrm{W}], \boldsymbol{v}\rangle &= -n(z_h\boldsymbol{u} + \tilde{\boldsymbol{u}}_{bh}, z_h\boldsymbol{w}, \boldsymbol{v}) - n(z_h\boldsymbol{w}, z_h\boldsymbol{u} + \tilde{\boldsymbol{u}}_{bh}, \boldsymbol{v})\\
&\quad + (1-\gamma_1)n(z_h\boldsymbol{u} + \tilde{\boldsymbol{u}}_{bh}, z_h\boldsymbol{w}, \boldsymbol{v} - z_h\boldsymbol{v})\\
&\quad + (1-\gamma_1)n(z_h\boldsymbol{w}, z_h\boldsymbol{u} + \tilde{\boldsymbol{u}}_{bh}, \boldsymbol{v} - z_h\boldsymbol{v})\\
&\quad - \gamma_2\, n(z_h\boldsymbol{u} + \tilde{\boldsymbol{u}}_{bh}, \boldsymbol{w} - z_h\boldsymbol{w}, z_h\boldsymbol{v}) - \gamma_2\, n(z_h\boldsymbol{w}, \boldsymbol{u} - z_h\boldsymbol{u}, z_h\boldsymbol{v})\\
&\quad - \gamma_3\, n(z_h\boldsymbol{u} + \tilde{\boldsymbol{u}}_{bh}, \boldsymbol{w} - z_h\boldsymbol{w}, \boldsymbol{v} - z_h\boldsymbol{v}) - \gamma_3\, n(z_h\boldsymbol{w}, \boldsymbol{u} - z_h\boldsymbol{u}, \boldsymbol{v} - z_h\boldsymbol{v})\\
&\quad - \gamma_4\, n(\boldsymbol{u} - z_h\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) - \gamma_4\, n(\boldsymbol{w} - z_h\boldsymbol{w}, \boldsymbol{u} + \tilde{\boldsymbol{u}}_{bh}, \boldsymbol{v})
\end{aligned}$$

and hence we obtain using (3.19) and (3.5)

$$\langle \mathrm{DG}_h(\mathrm{U})[\mathrm{W}] - \mathrm{DG}_h(\widetilde{\mathrm{U}})[\mathrm{W}], \boldsymbol{v}\rangle \leq C\|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{1,\Omega}\, \|\boldsymbol{w}\|_{1,\Omega}\, \|\boldsymbol{v}\|_{1,\Omega},$$

which gives (3.22). The same estimate holds for the operator G and hence both the operators are C^1 mappings. Applying (3.20) and (3.19) and using the imbedding $H^1(\Omega) \subsetneq L^4(\Omega)$, we obtain for any $\boldsymbol{v} \in H_0^1(\Omega)^d$

$$\begin{aligned}
\langle \mathrm{DG}(\mathrm{U})[\mathrm{W}], \boldsymbol{v}\rangle &= n(\boldsymbol{u} + \tilde{\boldsymbol{u}}_b, \boldsymbol{v}, \boldsymbol{w}) - n(\boldsymbol{w}, \boldsymbol{u} + \tilde{\boldsymbol{u}}_b, \boldsymbol{v})\\
&+ \int_\Omega (\boldsymbol{v}\cdot\boldsymbol{w})\mathrm{div}(\boldsymbol{u} + \tilde{\boldsymbol{u}}_b)\mathrm{d}x \leq C\|\boldsymbol{u} + \tilde{\boldsymbol{u}}_b\|_{1,\Omega}\|\boldsymbol{w}\|_{0,4,\Omega}\|\boldsymbol{v}\|_{1,\Omega}.
\end{aligned}$$

Thus, we have

$$\|\mathrm{DG}(\mathrm{U})[\mathrm{W}]\|_{-1,\Omega} \leq C\|\boldsymbol{u} + \tilde{\boldsymbol{u}}_b\|_{1,\Omega}\|\boldsymbol{w}\|_{0,4,\Omega}$$

and because the imbedding $H^1(\Omega) \subsetneq L^4(\Omega)$ is compact, we deduce that DG(U) is a compact operator. Applying (3.21) and (3.19), we obtain for any $\boldsymbol{u}$, $\bar{\boldsymbol{u}}$, $\boldsymbol{w}$, $\bar{\boldsymbol{w}} \in H^1(\Omega)^d$, and $\boldsymbol{v} \in H_0^1(\Omega)^d$

$$n(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) - n(\bar{\boldsymbol{u}}, \bar{\boldsymbol{w}}, \boldsymbol{v}) \leq C(\|\boldsymbol{w} - \bar{\boldsymbol{w}}\|_{0,4,\Omega}\|\boldsymbol{u}\|_{1,\Omega} + \|\boldsymbol{u} - \bar{\boldsymbol{u}}\|_{0,4,\Omega}\, \|\bar{\boldsymbol{w}}\|_{1,\Omega})\|\boldsymbol{v}\|_{1,\Omega}.$$

Using this inequality and (3.19), we get

$$
\begin{aligned}
&\langle \mathrm{G}(\mathrm{U}) - \mathrm{G}_h(\mathrm{U}), \boldsymbol{v}\rangle \\
&\quad \le \langle \boldsymbol{f} - \boldsymbol{f}_h, \boldsymbol{v}\rangle + C\,\|z_h\boldsymbol{u} + \tilde{\boldsymbol{u}}_{bh}\|_{1,\Omega}^2\,\|\boldsymbol{v} - z_h\boldsymbol{v}\|_{0,4,\Omega} \\
&\qquad + C(\|\boldsymbol{u} - z_h\boldsymbol{u}\|_{0,4,\Omega} + \|\tilde{\boldsymbol{u}}_b - \tilde{\boldsymbol{u}}_{bh}\|_{1,\Omega})(\|\boldsymbol{u}\|_{1,\Omega} + \|\tilde{\boldsymbol{u}}_b\|_{1,\Omega} + \|\tilde{\boldsymbol{u}}_{bh}\|_{1,\Omega})\|\boldsymbol{v}\|_{1,\Omega}, \\
&\langle \mathrm{DG}(\mathrm{U})[\mathrm{W}] - \mathrm{DG}_h(\mathrm{U})[\mathrm{W}], \boldsymbol{v}\rangle \\
&\quad \le C\|z_h\boldsymbol{u} + \tilde{\boldsymbol{u}}_{bh}\|_{1,\Omega}\,\|\boldsymbol{w}\|_{1,\Omega}\,\|\boldsymbol{v} - z_h\boldsymbol{v}\|_{0,4,\Omega} + C(\|\boldsymbol{u}\|_{1,\Omega} + \|\tilde{\boldsymbol{u}}_b\|_{1,\Omega} + \|\tilde{\boldsymbol{u}}_{bh}\|_{1,\Omega}) \\
&\qquad \times \|\boldsymbol{w} - z_h\boldsymbol{w}\|_{0,4,\Omega}\,\|\boldsymbol{v}\|_{1,\Omega} + C(\|\boldsymbol{u} - z_h\boldsymbol{u}\|_{0,4,\Omega} + \|\tilde{\boldsymbol{u}}_b - \tilde{\boldsymbol{u}}_{bh}\|_{1,\Omega})\|\boldsymbol{w}\|_{1,\Omega}\|\boldsymbol{v}\|_{1,\Omega}
\end{aligned}
$$

and (3.23) and (3.24) follow from (3.4), (3.12), and (2.1). Finally, let $\boldsymbol{u}$, $\tilde{\boldsymbol{u}}_b \in H^2(\Omega)^d$. Then, analogously as in the proof of Theorem 3.6, we derive using (3.20) and (3.21)

$$
\begin{aligned}
&\langle \mathrm{DG}(\mathrm{U})[\mathrm{W}] - \mathrm{DG}_h(\mathrm{U})[\mathrm{W}], \boldsymbol{v}\rangle \\
&\quad \le C(\|\boldsymbol{v} - z_h\boldsymbol{v}\|_{0,\Omega}\|\boldsymbol{w}\|_{1,\Omega} + \|\boldsymbol{w} - z_h\boldsymbol{w}\|_{0,\Omega}\|\boldsymbol{v}\|_{1,\Omega})\|\boldsymbol{u} + \tilde{\boldsymbol{u}}_b\|_{2,\Omega} \\
&\qquad + C(\|\boldsymbol{u} - z_h\boldsymbol{u}\|_{1,\Omega} + \|\tilde{\boldsymbol{u}}_b - \tilde{\boldsymbol{u}}_{bh}\|_{1,\Omega})\|\boldsymbol{w}\|_{1,\Omega}\,\|\boldsymbol{v}\|_{1,\Omega}
\end{aligned}
$$

and (3.25) follows using (3.5). □

Theorem 3.8. *The operators* F *and* F_h *are* C^1 *mappings satisfying*

$$\|\mathrm{DF}_h(\mathrm{U}) - \mathrm{DF}_h(\widetilde{\mathrm{U}})\|_{\mathscr{L}(\mathrm{X},\mathrm{X})} \le C\|\mathrm{U} - \widetilde{\mathrm{U}}\|_{\widehat{\mathrm{X}}} \quad \forall\,\mathrm{U}, \widetilde{\mathrm{U}} \in \mathrm{X}, \tag{3.26}$$

$$\lim_{h\to 0}\|\mathrm{F}(\mathrm{U}) - \mathrm{F}_h(\mathrm{U})\|_{\widehat{\mathrm{X}}} = 0 \qquad \forall\,\mathrm{U} \in \mathrm{X}, \tag{3.27}$$

$$\lim_{h\to 0}\|\mathrm{DF}(\mathrm{U}) - \mathrm{DF}_h(\mathrm{U})\|_{\mathscr{L}(\mathrm{X},\mathrm{X})} = 0 \quad \forall\,\mathrm{U} \in \mathrm{X}, \tag{3.28}$$

where the constant C is independent of h.

Proof. Using (3.10), (3.11), and Theorems 3.6 and 3.7, we infer that, for any $\mathrm{U} \in \mathrm{X}$,

$$\mathrm{DF}(\mathrm{U}) = \mathrm{T}^0\mathrm{DG}(\mathrm{U}) - \mathrm{I}, \quad \mathrm{DF}_h(\mathrm{U}) = \mathrm{T}_h^0\mathrm{DG}_h(\mathrm{U}) - \mathrm{I},$$

where $\mathrm{I} : \mathrm{X} \to \mathrm{X}$ is the identity operator. Thus, (3.26) immediately follows from (3.13) and (3.22). Further, using (3.11), we obtain

$$\mathrm{F}(\mathrm{U}) - \mathrm{F}_h(\mathrm{U}) = (\mathrm{T}\,\mathrm{G}(\mathrm{U}) - \mathrm{T}_h\,\mathrm{G}(\mathrm{U})) + \mathrm{T}_h^0(\mathrm{G}(\mathrm{U}) - \mathrm{G}_h(\mathrm{U})) - (\widehat{\mathrm{U}}_b - \widehat{\mathrm{U}}_{bh})$$

and (3.27) follows applying (3.14), (3.13), (3.23), and (2.1). Finally, we have

$$\mathrm{DF}(\mathrm{U}) - \mathrm{DF}_h(\mathrm{U}) = (\mathrm{T}^0 - \mathrm{T}_h^0)\mathrm{DG}(\mathrm{U}) + \mathrm{T}_h^0(\mathrm{DG}(\mathrm{U}) - \mathrm{DG}_h(\mathrm{U})).$$

According to (3.13) and (3.24), the second term on the right-hand side tends to zero and, employing the compactness of DG(U), (3.13), and (3.14), it can be shown by contradiction that the first term converges to zero as well. □

The properties of the operators F and F_h make it possible to investigate the existence and convergence of the solutions of (3.7)–(3.8) by applying the abstract theory of the approximation of branches of nonsingular solutions developed by Brezzi, Rappaz, and Raviart [6]. Here we shall use a particular result of this theory, which is formulated in the following theorem. Let us recall that $\widetilde{\mathrm{U}} \in \mathrm{X}$ is a nonsingular solution of the equation $\mathrm{F}(\mathrm{U}) = 0$ if $\mathrm{F}(\widetilde{\mathrm{U}}) = 0$ and $\mathrm{DF}(\widetilde{\mathrm{U}})$ is a topological isomorphism of X. Analogously, we say that $\widehat{\mathrm{U}} = (\boldsymbol{u}, p)$ is a nonsingular solution of (1.2)–(1.3) if $\widehat{\mathrm{U}} - \widehat{\mathrm{U}}_b$ is a nonsingular solution of $\mathrm{F}(\mathrm{U}) = 0$, and that $\widehat{\mathrm{U}}_h = (\boldsymbol{u}_h, p_h)$ is a nonsingular solution of (3.7)–(3.8) if $\widehat{\mathrm{U}}_h - \widehat{\mathrm{U}}_{bh}$ is a nonsingular solution of the equation $\mathrm{F}_h(\mathrm{U}) = 0$.

Theorem 3.9. *Let* X *be a Banach space and* $\mathrm{F} : \mathrm{X} \to \mathrm{X}$ *a* C^1 *operator. Let* $\{\mathrm{F}_h\}_h$ *be a family of* C^1 *operators* $\mathrm{F}_h : \mathrm{X} \to \mathrm{X}$ *satisfying* (3.26)–(3.28). *Then, for any nonsingular solution* $\widetilde{\mathrm{U}} \in \mathrm{X}$ *of the equation* $\mathrm{F}(\mathrm{U}) = 0$, *there exist constants* $h_0 > 0$ *and* $R > 0$ *such that, for* $h \in (0, h_0)$, *the equation* $\mathrm{F}_h(\mathrm{U}) = 0$ *has a solution that is unique in the ball*

$$\mathscr{B}(\widetilde{\mathrm{U}}, R) = \{\mathrm{V} \in \mathrm{X},\ \|\widetilde{\mathrm{U}} - \mathrm{V}\|_{\mathrm{X}} \le R\}.$$

Moreover, these unique solutions $\widetilde{\mathrm{U}}_h \in \mathscr{B}(\widetilde{\mathrm{U}}, R)$ *are nonsingular and satisfy*

$$\|\widetilde{\mathrm{U}} - \widetilde{\mathrm{U}}_h\|_{\mathrm{X}} \le C\|\mathrm{F}_h(\widetilde{\mathrm{U}})\|_{\mathrm{X}} \qquad \forall\ h \in (0, h_0), \tag{3.29}$$

where the constant C *is independent of* h.

Proof. The theorem is a consequence of Lemma 3.3, inequality (3.15), and Theorem 3.1 from [11], pp. 301–302. □

Now we can easily prove the following existence and convergence result for the problem (3.7)–(3.8).

Theorem 3.10. *Let the constants* α_1, α_2, α_3 *used for defining* a_h *satisfy* $\alpha_3 > 0$ *and* $|\alpha_1 + \alpha_2| \le 2\sqrt{\alpha_3}$ *and let* $\boldsymbol{u}$, p *be a nonsingular solution of the problem* (1.2)–(1.3). *Then there exist constants* $h_0 > 0$ *and* $R > 0$ *such that, for* $h \in (0, h_0)$, *the problem* (3.7)–(3.8) *has a solution* $\boldsymbol{u}_h$, p_h *that is unique in the ball*

$$\{(\bar{\boldsymbol{u}}, \bar{p}) \in H^1(\Omega)^d \times L^2_0(\Omega), \quad \|\boldsymbol{u} - \bar{\boldsymbol{u}}\|_{1,\Omega} + \|p - \bar{p}\|_{0,\Omega} \le R\}.$$

Moreover, these unique solutions are nonsingular and satisfy

$$\lim_{h\to 0}\{\,\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}+\|p-p_h\|_{0,\Omega}\}=0. \tag{3.30}$$

In addition, if, for some $m\in\{1,\dots,l\}$, $\boldsymbol{u}$, $\tilde{\boldsymbol{u}}_b\in H^{m+1}(\Omega)^d$, $p\in H^m(\Omega)$ *and* (2.17) *holds, then we have for any* $h\in(0,h_0)$

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}+\|p-p_h\|_{0,\Omega}\le C\,h^m+C\,h(|1-\alpha_1|+|1-\alpha_4|+|1-\gamma_1|). \tag{3.31}$$

Proof. The solvability and the convergence (3.30) immediately follow from Theorems 3.8 and 3.9 and from the assumption (2.1). For proving the estimate (3.31), we apply the relation

$$\mathrm{F}_h(\widetilde{\mathrm{U}})=\mathrm{F}_h(\widetilde{\mathrm{U}})-\mathrm{F}(\widetilde{\mathrm{U}})=\mathrm{T}_h\,\mathrm{G}_h(\widetilde{\mathrm{U}})-\mathrm{T}\,\mathrm{G}(\widetilde{\mathrm{U}})+\widehat{\mathrm{U}}_b-\widehat{\mathrm{U}}_{bh},$$

valid with $\widetilde{\mathrm{U}}=(\boldsymbol{u}-\tilde{\boldsymbol{u}}_b,p)$, and the relations (3.29) and (3.17). □

For proving an improved convergence of the velocity in the L^2 norm, we need an extension of the operator $\mathrm{DF(U)}$ defined on $L^2(\Omega)^d\times L^2_0(\Omega)$. Such an extension can be easily constructed if $\mathrm{R\,U}+\tilde{\boldsymbol{u}}_b\in W^{1,4}(\Omega)^d$, where R is the operator defined in (3.9). Because we are now only interested in the behavior of the velocity, we shall drop the pressure space $L^2_0(\Omega)$ in the following.

First, assuming that $\tilde{\boldsymbol{u}}_b\in W^{1,4}(\Omega)^d$, we introduce an operator $\mathrm{H}: W^{1,4}(\Omega)^d\to\mathscr{L}(L^2(\Omega)^d,H^{-1}(\Omega)^d)$ defined for any $\boldsymbol{u}\in W^{1,4}(\Omega)^d$, $\boldsymbol{w}\in L^2(\Omega)^d$ and $\boldsymbol{v}\in H^1_0(\Omega)^d$ by

$$\langle\mathrm{H}(\boldsymbol{u})\boldsymbol{w},\boldsymbol{v}\rangle=n(\boldsymbol{u}+\tilde{\boldsymbol{u}}_b,\boldsymbol{v},\boldsymbol{w})-n(\boldsymbol{w},\boldsymbol{u}+\tilde{\boldsymbol{u}}_b,\boldsymbol{v})+\int_\Omega(\boldsymbol{v}\cdot\boldsymbol{w})\mathrm{div}(\boldsymbol{u}+\tilde{\boldsymbol{u}}_b)\mathrm{d}x.$$

Because $W^{1,4}(\Omega)\hookrightarrow C(\overline{\Omega})$, the operator H is well defined. Moreover, according to (3.20), we have for any $\mathrm{U},\mathrm{W}\in\mathrm{X}$ with $\mathrm{U}=(\boldsymbol{u},p)$, $\mathrm{W}=(\boldsymbol{w},q)$ and $\boldsymbol{u}\in W^{1,4}(\Omega)^d$

$$\mathrm{H}(\boldsymbol{u})\boldsymbol{w}=\mathrm{DG(U)[W]}. \tag{3.32}$$

Now, we define an operator $\mathrm{B}: W^{1,4}(\Omega)^d\to\mathscr{L}(L^2(\Omega)^d,L^2(\Omega)^d)$ by

$$\mathrm{B}(\boldsymbol{u})=\mathrm{RT}^0\mathrm{H}(\boldsymbol{u})-\mathrm{I}\quad\forall\,\boldsymbol{u}\in W^{1,4}(\Omega)^d,$$

where $\mathrm{I}: L^2(\Omega)^d\to L^2(\Omega)^d$ is the identity operator. Then we have for any $\mathrm{U}\in\mathrm{X}$ with $\mathrm{U}=(\boldsymbol{u},p)$ and $\boldsymbol{u}\in W^{1,4}(\Omega)^d$

$$\mathrm{B}(\boldsymbol{u})\boldsymbol{w}=\mathrm{R\,DF(U)[P}\boldsymbol{w}]\quad\forall\,\boldsymbol{w}\in H^1_0(\Omega)^d, \tag{3.33}$$

where P is the operator defined in (3.9), and hence $\mathrm{B}(\boldsymbol{u})$ can be used as the above-mentioned extension of the operator $\mathrm{DF}(\mathrm{U})$. The following property of the operator B is crucial for our further proceeding.

Lemma 3.11. *Let* $\mathrm{U} \equiv (\boldsymbol{u}, p) \in \mathrm{X}$, $\boldsymbol{u} \in W^{1,4}(\Omega)^d$, *be such that* $\mathrm{DF}(\mathrm{U})$ *is a one-to-one mapping. Then* $\mathrm{B}(\boldsymbol{u})^{-1}$ *exists and is continuous.*

Proof. Consider any $\boldsymbol{w} \in L^2(\Omega)^d$ and assume that $\mathrm{B}(\boldsymbol{u})\boldsymbol{w} = \mathbf{0}$. Then $\boldsymbol{w} = \mathrm{R}\,\mathrm{T}^0\,\mathrm{H}(\boldsymbol{u})\boldsymbol{w} \in H_0^1(\Omega)^d$ and hence, by (3.33), $\mathrm{R}\,\mathrm{DF}(\mathrm{U})[\mathrm{P}\boldsymbol{w}] = \mathbf{0}$. Therefore, $\mathrm{DF}(\mathrm{U})[\mathrm{P}\boldsymbol{w}] = (\mathbf{0}, q)$ for some $q \in L_0^2(\Omega)$ and because $\mathrm{DG}(\mathrm{U})[\mathrm{W}]$ depends on W only through RW, we infer that $\mathrm{DF}(\mathrm{U})[\mathrm{W}] = 0$ for $\mathrm{W} = (\boldsymbol{w}, q)$. Therefore, $\mathrm{W} = 0$, which means that $\mathrm{B}(\boldsymbol{u})$ is a one-to-one mapping. Now, assuming that $\mathrm{B}(\boldsymbol{u})^{-1}$ is not continuous, there exists a sequence $\{\boldsymbol{w}_n\}_{n=1}^{\infty} \subset L^2(\Omega)^d$ such that $\|\boldsymbol{w}_n\|_{0,\Omega} = 1$ and $\|\mathrm{B}(\boldsymbol{u})\boldsymbol{w}_n\|_{0,\Omega} \leq 1/n$ for any $n \in \mathbb{N}$. Because the space $H^1(\Omega)$ is compactly embedded into $L^2(\Omega)$, the operator $\mathrm{R}\,\mathrm{T}^0\,\mathrm{H}(\boldsymbol{u}) \in \mathscr{L}(L^2(\Omega)^d, H_0^1(\Omega)^d)$ is compact as an operator from $\mathscr{L}(L^2(\Omega)^d, L^2(\Omega)^d)$. Therefore, the sequence $\boldsymbol{w}_n = \mathrm{R}\,\mathrm{T}^0\,\mathrm{H}(\boldsymbol{u})\boldsymbol{w}_n - \mathrm{B}(\boldsymbol{u})\boldsymbol{w}_n$ contains a subsequence, which we again denote by $\boldsymbol{w}_n$, such that $\boldsymbol{w}_n \to \boldsymbol{w}$ for some $\boldsymbol{w} \in L^2(\Omega)^d$. Clearly, $\mathrm{B}(\boldsymbol{u})\boldsymbol{w} = \lim_{n\to\infty} \mathrm{B}(\boldsymbol{u})\boldsymbol{w}_n = \mathbf{0}$ and hence $\boldsymbol{w} = \mathbf{0}$. That is however in contradiction with $\|\boldsymbol{w}\|_{0,\Omega} = \lim_{n\to\infty} \|\boldsymbol{w}_n\|_{0,\Omega} = 1$. Therefore, $\mathrm{B}(\boldsymbol{u})^{-1}$ is continuous and the lemma is proven. □

Now we can investigate the convergence of the velocity in the L^2 norm.

Theorem 3.12. *Let the constants* α_1, α_2, α_3 *used for defining* a_h *satisfy* $\alpha_3 > 0$ *and* $|\alpha_1 + \alpha_2| \leq 2\sqrt{\alpha_3}$ *and let* $\boldsymbol{u} \in H^{m+1}(\Omega)^d$, $p \in H^m(\Omega)$, $m \in \{1, \dots, l\}$, *be a nonsingular solution of the problem* (1.2)–(1.3). *Let* $\boldsymbol{u}_h$, p_h *be the solution of the problem* (3.7)–(3.8) *from Theorem 3.5. If the problem* (1.6) *is regular,* (2.17) *and* (2.19) *hold,* $\|\tilde{\boldsymbol{u}}_b - \tilde{\boldsymbol{u}}_{bh}\|_{0,\Omega} \leq C\,h^{m+1}$ *and* $\tilde{\boldsymbol{u}}_b \in H^{m+1}(\Omega)^d$, *then we have for any* $h \in (0, h_0)$

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} \leq C\,h^{m+1} + C\,h^2(|1-\alpha_1| + |1-\alpha_4| + |1-\gamma_1|). \tag{3.34}$$

Proof. Denote $\widehat{\mathrm{U}} = (\boldsymbol{u}, p)$, $\widehat{\mathrm{U}}_h = (\boldsymbol{u}_h, p_h)$, $\widetilde{\mathrm{U}} = \widehat{\mathrm{U}} - \widehat{\mathrm{U}}_b$, $\widetilde{\mathrm{U}}_h = \widehat{\mathrm{U}}_h - \widehat{\mathrm{U}}_{bh}$, $\tilde{\boldsymbol{u}} = \mathrm{R}\,\widetilde{\mathrm{U}}$ and $\tilde{\boldsymbol{u}}_h = \mathrm{R}\,\widetilde{\mathrm{U}}_h$. Then $\tilde{\boldsymbol{u}} \in H^{m+1}(\Omega)^d \cap H_0^1(\Omega)^d$, $\tilde{\boldsymbol{u}}_h \in H_0^1(\Omega)^d$ and

$$\mathrm{F}(\widetilde{\mathrm{U}}) = 0, \quad \mathrm{F}_h(\widetilde{\mathrm{U}}_h) = 0. \tag{3.35}$$

Using (3.32), we obtain

$$\mathrm{B}(\tilde{\boldsymbol{u}})[\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_h] = \mathrm{R}(\mathrm{T}^0\,\mathrm{DG}(\widetilde{\mathrm{U}})[\widetilde{\mathrm{U}} - \widetilde{\mathrm{U}}_h] - \widetilde{\mathrm{U}} + \widetilde{\mathrm{U}}_h)$$

and, in view of (3.11) and (3.35), we infer that

$$\begin{aligned}\mathrm{T}^0\,\mathrm{DG}(\widetilde{\mathrm{U}})[\widetilde{\mathrm{U}}-\widetilde{\mathrm{U}}_h]-\widetilde{\mathrm{U}}+\widetilde{\mathrm{U}}_h &= (\mathrm{T}^0-\mathrm{T}^0_h)\mathrm{DG}(\widetilde{\mathrm{U}})[\widetilde{\mathrm{U}}-\widetilde{\mathrm{U}}_h]\\ &\quad+\mathrm{T}^0_h\{\mathrm{DG}(\widetilde{\mathrm{U}})[\widetilde{\mathrm{U}}-\widetilde{\mathrm{U}}_h]-\mathrm{DG}_h(\widetilde{\mathrm{U}})[\widetilde{\mathrm{U}}-\widetilde{\mathrm{U}}_h]\}\\ &\quad+\mathrm{T}^0_h\{\mathrm{DG}_h(\widetilde{\mathrm{U}})[\widetilde{\mathrm{U}}-\widetilde{\mathrm{U}}_h]-\mathrm{G}_h(\widetilde{\mathrm{U}})+\mathrm{G}_h(\widetilde{\mathrm{U}}_h)\}\\ &\quad-\{\mathrm{T}\,\mathrm{G}(\widetilde{\mathrm{U}})-\mathrm{T}_h\,\mathrm{G}_h(\widetilde{\mathrm{U}})\}+\{\widehat{\mathrm{U}}_b-\widehat{\mathrm{U}}_{bh}\}.\end{aligned}$$

Because $\mathrm{DG}(\widetilde{\mathrm{U}})[\mathrm{W}] = -(\nabla \boldsymbol{w})\boldsymbol{u} - (\nabla \boldsymbol{u})\boldsymbol{w} \in L^2(\Omega)^d$ for any $\mathrm{W} \equiv (\boldsymbol{w}, q) \in \mathrm{X}$, it follows from (3.15) that

$$\|\mathrm{R}(\mathrm{T}^0-\mathrm{T}^0_h)\mathrm{DG}(\widetilde{\mathrm{U}})[\mathrm{W}]\|_{0,\Omega} \le C\,h^2\,\|\mathrm{DG}(\widetilde{\mathrm{U}})[\mathrm{W}]\|_{0,\Omega} \le \widetilde{C}\,h^2\,\|\boldsymbol{u}\|_{2,\Omega}\,\|\boldsymbol{w}\|_{1,\Omega}.$$

Further, it is easy to show that

$$\|\mathrm{DG}_h(\widetilde{\mathrm{U}})[\widetilde{\mathrm{U}}-\widetilde{\mathrm{U}}_h]-\mathrm{G}_h(\widetilde{\mathrm{U}})+\mathrm{G}_h(\widetilde{\mathrm{U}}_h)\|_{-1,\Omega} \le C\|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_h\|^2_{1,\Omega}.$$

Thus, we deduce from (3.13), (3.25), (3.18), (3.31), and the assumptions of the theorem that

$$\|\mathrm{B}(\tilde{\boldsymbol{u}})[\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_h]\|_{0,\Omega} \le C\,h^{m+1} + C\,h^2(|1-\alpha_1|+|1-\alpha_4|+|1-\gamma_1|).$$

According to Lemma 3.11, we have $\|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_h\|_{0,\Omega} \le C\|\mathrm{B}(\tilde{\boldsymbol{u}})[\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_h]\|_{0,\Omega}$ and the theorem is proven. □

Remark 3.13. It follows from Lemma 2.6 that (3.30), (3.31), and (3.34) remain valid if $\boldsymbol{u}_h$ is replaced by $\boldsymbol{u}^*_h \equiv (\boldsymbol{u}_h - \tilde{\boldsymbol{u}}_{bh})^1 + \tilde{\boldsymbol{u}}_{bh}$.

4. VALIDITY OF THE ASSUMPTIONS (A3), (A4), AND (A7)

In this section, we assume that Ω is a bounded domain with a polygonal resp. polyhedral boundary, which makes it possible to introduce a family of triangulations $\mathcal{T}_h$ of Ω consisting of polygonal resp. polyhedral elements T (e.g., triangles, squares, tetrahedra, or hexahedra). The parameter h represents the largest diameter of the elements of $\mathcal{T}_h$. We assume that there exists a reference element $\widehat{T}$ such that, for each element T, we can introduce a regular one-to-one mapping $F_T : \widehat{T} \to T$ with $F_T(\widehat{T}) = T$. Moreover, we assume that the triangulations $\mathcal{T}_h$ possess the usual compatibility properties (cf. [7]) and that they are shape regular in the sense that

$$|F_T|_{1,\infty,\widehat{T}} \le C\,h_T, \quad |F_T^{-1}|_{1,\infty,T} \le C\,h_T^{-1} \quad \forall\, T \in \mathcal{T}_h, \quad h > 0,$$

where $h_T = \mathrm{diam}(T)$ and the constant C is independent of T and h. Thus, denoting for any element T and any $v \in L^1(T)$

$$\hat{v}_T = v \circ F_T,$$

we have

$$C\, h_T^d \|\hat{v}_T\|_{0,p,\hat{T}}^p \leq \|v\|_{0,p,T}^p \leq \widetilde{C} h_T^d \|\hat{v}_T\|_{0,p,\hat{T}}^p \quad \forall\, v \in L^p(T),\ \ T \in \mathcal{T}_h,\ \ p \geq 1, \tag{4.1}$$

$$C\, h_T^{d-2} |\hat{v}_T|_{1,\hat{T}}^2 \leq |v|_{1,T}^2 \leq \widetilde{C}\, h_T^{d-2} |\hat{v}_T|_{1,\hat{T}}^2 \quad \forall\, v \in H^1(T),\ \ T \in \mathcal{T}_h. \tag{4.2}$$

Using the mappings F_T, we can introduce general finite element spaces

$$\mathrm{W}_h^1 = \{v \in H^1(\Omega),\ \hat{v}_T \in \widehat{W}^1\ \forall\, T \in \mathcal{T}_h\},$$
$$\mathrm{W}_h^2 = \{v \in H^1(\Omega), \hat{v}_T \in \widehat{W}^2\ \forall\, T \in \mathcal{T}_h\},$$

where $\widehat{W}^1$, $\widehat{W}^2 \subset H^1(\widehat{T})$ are some fixed spaces defined on the reference element. The following two lemmas give sufficient conditions for the validity of (A3), (A4), and (A7).

Lemma 4.1. *Let $\widehat{W}^1$ be a finite-dimensional subspace of $H^1(\widehat{T})$ and let $\widehat{W}^2$ be a closed subspace of $H^1(\widehat{T})$ such that $\widehat{W}^1 \cap \widehat{W}^2 = \{0\}$. Then*

$$\|u\|_{1,\Omega} + \|v\|_{1,\Omega} \leq C\, \|u + v\|_{1,\Omega} \quad \forall\, u \in \mathrm{W}_h^1,\ v \in \mathrm{W}_h^2.$$

Proof. Because, in a finite-dimensional space, any bounded sequence contains a convergent subsequence, it is easy to show by contradiction that

$$0 < \widehat{C}_1 \equiv \inf_{\hat{u} \in \widehat{W}^1, \|\hat{u}\|_{1,\hat{T}} = 1}\ \inf_{\hat{v} \in \widehat{W}^2} \|\widehat{u} + \widehat{v}\|_{1,\hat{T}}.$$

This implies that $\widehat{C}_1 \|\widehat{u}\|_{1,\hat{T}} \leq \|\widehat{u} + \widehat{v}\|_{1,\hat{T}}\ \forall\, \widehat{u} \in \widehat{W}^1$, $\widehat{v} \in \widehat{W}^2$. According to [16], p. 18, Theorem 1.5, we have $\|\widehat{u}\|_{1,\hat{T}} \leq \widehat{C}_2 |\widehat{u}|_{1,\hat{T}}\ \forall\, \widehat{u} \in H^1(\widehat{T}) \cap L_0^2(\widehat{T})$, which implies that $\widehat{C}_1\, |\widehat{u}|_{1,\hat{T}} \leq \widehat{C}_2\, |\widehat{u} + \widehat{v}|_{1,\hat{T}}$ for any $\widehat{u} \in \widehat{W}^1 \cap L_0^2(\widehat{T})$, $\widehat{v} \in \widehat{W}^2 \cap L_0^2(\widehat{T})$ and hence for any $\widehat{u} \in \widehat{W}^1$, $\widehat{v} \in \widehat{W}^2$. Applying (4.1) and (4.2), we get $\|u\|_{1,\Omega} \leq C \|u + v\|_{1,\Omega}\ \forall\, u \in \mathrm{W}_h^1$, $v \in \mathrm{W}_h^2$ and the lemma follows. □

Lemma 4.2. *Let $\widehat{W}^2$ be a closed subspace of $H^1(\widehat{T})$ such that $1 \notin \widehat{W}^2$. Then, for any $p \in [2, 2\, d/(d-2))$, we have*

$$\|u\|_{0,p,\Omega} \leq C\, h^{1 + d(\frac{1}{p} - \frac{1}{2})} |u|_{1,\Omega} \quad \forall\, u \in \mathrm{W}_h^2.$$

Proof. Because the space $H^1(\widehat{T})$ is compactly imbedded into $L^p(\widehat{T})$ (cf. [16], p. 106, Theorem 6.1), it can be shown similarly as in the proof of Theorem 1.5 from [16], p. 18, that $\|\widehat{u}\|_{0,p,\hat{T}} \leq C\, |\widehat{u}|_{1,\hat{T}}\ \forall\, \widehat{u} \in \widehat{W}^2$. Then the lemma follows using (4.1) and (4.2). □

Remark 4.3. Note that it is not assumed that $\widehat{W}^2$ is finite-dimensional. This makes it possible to use spaces V_h^2 locally adapted to achieve a proper stabilization. Particularly, one may think of defining V_h^2 in the framework of residual-free bubbles techniques.

Remark 4.4. The above results show that the assumptions (A3), (A4), and (A7) are satisfied for the examples of the spaces V_h^1 and V_h^2 mentioned in Section 1.

ACKNOWLEDGMENT

The work is a part of the research project MSM 0021620839 financed by MSMT.

REFERENCES

1. D.N. Arnold, F. Brezzi, and M. Fortin (1984). A stable finite element for the Stokes equations. *Calcolo* 21:337–344.
2. C. Bernardi (1989). Optimal finite-element interpolation on curved domains. *SIAM J. Numer. Anal.* 26:1212–1240.
3. C. Bernardi, B. Métivet, and B. Pernaud–Thomas (1995). Couplage des équations de Navier–Stokes et de la chaleur: le modèle et son approximation par éléments finis. M^2AN 29:871–921.
4. C. Bernardi and G. Raugel (1985). Analysis of some finite elements for the Stokes problem. *Math. Comput.* 44:71–79.
5. F. Brezzi and M. Fortin (1991). *Mixed and Hybrid Finite Element Methods.* Springer-Verlag, New York.
6. F. Brezzi, J. Rappaz, and P.-A. Raviart (1980). Finite dimensional approximation of nonlinear problems. Part I: Branches of nonsingular solutions. *Numer. Math.* 36:1–25.
7. P.G. Ciarlet (1991). Basic error estimates for elliptic problems. In *Handbook of Numerical Analysis, Finite Element Methods, Pt. 1.* (P. G. Ciarlet, J. L. Lions, eds.), Vol. 2. North-Holland, Amsterdam, pp. 17–351.
8. M. Crouzeix and P.-A. Raviart (1973). Conforming and nonconforming finite element methods for solving the stationary Stokes equations I. *RAIRO* 7(R-3):33–76.
9. G.J. Fix, M.D. Gunzburger, and J.S. Peterson (1983). On finite element approximations of problems having inhomogeneous essential boundary conditions. *Comput. Math. Appl.* 9:687–700.
10. M. Fortin. (1981). Old and new finite elements for incompressible flows. *Int. J. Numer. Methods Fluids* 1:347–364.
11. V. Girault and P.-A. Raviart (1986). *Finite Element Methods for Navier-Stokes Equations.* Springer-Verlag, Berlin.
12. P. Knobloch and L. Tobiska (1998). A new approach to derive stable discretizations of the Stokes and Navier-Stokes equations. In *Proceedings of ENUMATH 97* (H. G. Bock, G. Kanschat, R. Rannacher, F. Brezzi, R. Glowinski, Y. A. Kuznetsov, J. Periaux, eds.), World Scientific, Singapore, pp. 413–420.
13. P. Knobloch and L. Tobiska (2000). Stabilization methods of bubble type for the Q_1/Q_1-element applied to the incompressible Navier-Stokes equations. M^2AN 34:85–107.
14. P. Knobloch and L. Tobiska (2000). Modified FE discretizations of incompressible flow problems and their relationship to stabilized methods. In *Numerical Mathematics and Advanced Applications* (P. Neittaanmäki, T. Tiihonen, P. Tarvainen, eds.), World Scientific, Singapore, pp. 571–578.
15. P. Mons and G. Rogé (1992). L'élément Q_1-bulle/Q_1. M^2AN 26:507–521.
16. J. Nečas (1967). *Les Méthodes Directes en Théorie des Équations Elliptiques.* Academia, Praha.
17. H.-G. Roos, M. Stynes, and L. Tobiska (1996). *Numerical Methods for Singularly Perturbed Differential Equations: Convection–Diffusion and Flow Problems.* Springer-Verlag, Berlin.