European Congress on Computational Methods in Applied Sciences and Engineering
ECCOMAS 2000
Barcelona, 11-14 September 2000
©ECCOMAS

ON BUBBLE–BASED MODIFICATIONS OF THE NONCONFORMING P_1 ELEMENT FOR SOLVING CONVECTION–DIFFUSION EQUATIONS

Petr Knobloch* **and Lutz Tobiska**†

*Institute of Numerical Mathematics, Faculty of Mathematics and Physics
Charles University, Malostranské nám. 25, 11800 Praha 1, Czech Republic
e-mail: knobloch@karlin.mff.cuni.cz

†Institut für Analysis und Numerik, Otto–von–Guericke–Universität Magdeburg
Postfach 4120, 39016 Magdeburg, Germany
e-mail: Lutz.Tobiska@mathematik.uni-magdeburg.de

Key words: Convection–Diffusion Equation, Streamline Diffusion Method, Finite Element Method, Nonconforming Finite Elements, Error Estimates.

Abstract. *We consider nonconforming streamline diffusion finite element discretizations for solving convection–diffusion problems. Using these discretizations with the nonconforming P_1 element, the properties of the discrete solutions are much worse than in the conforming case. We show that an improvement can be attained by modifying the nonconforming P_1 element using suitable general nonconforming bubble functions in such a way that the resulting space satisfies the patch test of order 2. In this way, we obtain a class of new nonconforming first order finite element spaces. We also derive a subclass of these general spaces for which the patch test of order 3 holds and hence the optimal convergence order 3/2 can be established. We give a rigorous convergence analysis and present various numerical results which demonstrate the robustness of the new method.*

1 INTRODUCTION

We consider the convection–diffusion equation

$$-\varepsilon\,\Delta\,u + \boldsymbol{b}\cdot\nabla u + c\,u = f \quad \text{in } \Omega, \qquad\qquad u = u_b \quad \text{on } \partial\Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with a polygonal boundary $\partial\Omega$, $\varepsilon > 0$ is constant, $\boldsymbol{b} \in W^{2,\infty}(\Omega)^2$, $c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$ and $u_b \in H^{3/2}(\partial\Omega)$. We assume that

$$c - \frac{1}{2}\,\mathrm{div}\,\boldsymbol{b} \geq c_0\,,$$

where c_0 is a positive constant. This assumption guarantees that (1) admits a unique solution for all positive values of the parameter ε.

We are mainly interested in cases when the convective part $\boldsymbol{b}\cdot\nabla u$ dominates the diffusive part $\varepsilon\,\Delta\,u$, i.e., when $\varepsilon \ll 1$. The solutions of convection dominated convection–diffusion equations typically contain inner and boundary layers which are difficult to approximate numerically unless the computational mesh is sufficiently fine. Standard Galerkin finite element methods applied on meshes which are not fine enough produce unphysical oscillations and therefore, various stabilized methods have been developed. In this paper we concentrate on the streamline diffusion method[1,2] which is known to combine good stability properties with a high accuracy outside the layers. The properties of the streamline diffusion method were intensively studied during the past decade and nowadays its convergence properties are well understood in the case of conforming finite element approximations[2,3,4,5,6].

However, the properties of the streamline diffusion method are much less clear if nonconforming finite elements are applied. Nonconforming finite element methods are very attractive for approximating incompressible materials since they usually fulfil a Babuška–Brezzi condition and discretely divergence–free bases can often easily be constructed. Moreover, implementations on parallel MIMD–machines are more effective for nonconforming finite element methods than for conforming elements[7,8,9]. However, since the finite element functions are discontinuous across the edges of the triangulation, theoretical investigations of nonconforming finite element methods involve various additional difficulties in comparison with the conforming case.

Particularly, in the case of the streamline diffusion method, the nonconformity causes that the coercivity of the respective bilinear form depends on the type of discretization used for the convective term. Moreover, error analysis requires ε–uniform error estimates of consistency errors and additional terms involving jumps of finite element functions across element edges. Therefore, special techniques are necessary to recover the optimal convergence order 3/2 in the streamline diffusion norm when using first order approximation finite element spaces. In[10,11] the mentioned difficulties have been overcome by adding some special jump terms to the standard streamline diffusion finite element method. However, a drawback of the jump terms is that they are difficult to implement. An alternate way to get an ε–uniform consistency error estimate was found in[12] where superconvergence properties on uniform meshes were used.

In this paper, we consider general triangular meshes and we look for nonconforming first order finite element spaces for which ε–uniform error estimates in the streamline diffusion norm with a positive convergence order can be established without modifying the discretization. We shall start with the P_1^{nc} element consisting of piecewise linear functions which are continuous in the midpoints of inner edges of the triangulation. Using this element in a streamline diffusion finite element method, the properties of the discrete solution are much worse than in the conforming case. We shall show that an improvement can be attained by modifying the P_1^{nc} element using suitable nonconforming bubble functions in such a way that the resulting space satisfies the patch test of order 2. This property makes it possible to establish better estimates of the consistency errors. We only require that the nonconforming bubble functions possess some rather general properties so that we obtain a class of new finite element spaces. Each of these finite element spaces can be represented by the direct sum of a subspace of $H_0^1(\Omega)$ and a space consisting of modified P_1^{nc} functions. Particularly, we derive the nonconforming P_1^{mod} element of[13], for which the patch test of order 3 holds and the optimal convergence order 3/2 can be proved.

The paper is organized in the following way. First, in Section 2, we summarize the necessary notation. Then, in Section 3, we establish a weak formulation of (1) and describe the nonconforming streamline diffusion finite element method considered in this paper. In Section 4, we present the error analysis. Section 5 is devoted to the construction of general nonconforming first order finite element spaces satisfying the patch test of order 2. In Section 6, we consider a subclass of the general finite element spaces satisfying the patch test of order 3 which we denote as the P_1^{mod} element. We also give an example of piecewise cubic basis functions which are used in all our numerical experiments. Finally, in Section 7, we present numerical results which demonstrate the good behaviour of discretizations employing the P_1^{mod} element. The numerical results support the optimal convergence order 3/2 and indicate that discretizations using the new finite element are very robust. The results are qualitatively much better than for the P_1^{nc} element, and inner and boundary layers are detected very accurately. Moreover, the iterative solver used to compute the discrete solutions converges much faster than for the P_1^{nc} element.

2 NOTATION

We assume that we are given a family $\{\mathcal{T}_h\}$ of triangulations of the domain Ω consisting of closed triangular elements K having the usual compatibility properties (see e.g.[14]) and satisfying $h_K \equiv \mathrm{diam}(K) \le h$ for any $K \in \mathcal{T}_h$. We assume that the family of triangulations is regular, i.e., there exists a constant C independent of h such that

$$\frac{h_K}{\varrho_K} \le C \qquad \forall\ K \in \mathcal{T}_h,\ h > 0\,,$$

where ϱ_K is the maximum diameter of circles inscribed into K.

We denote by $\mathcal{E}_h$ the set of the edges E of $\mathcal{T}_h$. The set of inner edges will be denoted by $\mathcal{E}_h^i$ and the set of boundary edges by $\mathcal{E}_h^b$. Further, we denote by h_E the length of the edge E, by C_E the midpoint of E, by $x_{E,1}$, $x_{E,2}$ the end points of E and by $\lambda_{E,1}$, $\lambda_{E,2}$ the barycentric coordinates on E with respect to $x_{E,1}$, $x_{E,2}$, respectively. The union of the elements adjacent to an edge E will be denoted by S_E. For any edge E, we choose a fixed unit normal vector $\boldsymbol{n}_E$ to E. If $E \subset \partial\Omega$, then $\boldsymbol{n}_E$ coincides with the outer normal vector to $\partial\Omega$. Consider any $E \in \mathcal{E}_h^i$ and let K, $\widetilde{K}$ be the two elements possessing the edge E denoted such that $\boldsymbol{n}_E$ points into $\widetilde{K}$. If v is a function belonging to the space

$$H^{2,h}(\Omega) = \{v \in L^2(\Omega)\,;\ v|_K \in H^2(K) \quad \forall\ K \in \mathcal{T}_h\}\,,$$

then we define the jump of v across E by

$$[|v|]_E = (v|_K)|_E - (v|_{\widetilde{K}})|_E\,. \tag{2}$$

If $E \in \mathcal{E}_h^b$, then we set

$$[|v|]_E = v|_E\,,$$

which is the jump defined by (2) with v extended by zero outside Ω.

In the following sections, we shall need the spaces

$$\widetilde{\mathrm{V}}_h^{conf} = \{v_h \in C(\overline{\Omega})\,;\ v_h|_K \in P_1(K) \quad \forall\ K \in \mathcal{T}_h\}\,, \qquad \mathrm{V}_h^{conf} = \widetilde{\mathrm{V}}_h^{conf} \cap H_0^1(\Omega)\,,$$
$$\mathrm{V}_h^{nc} = \{v_h \in L^2(\Omega)\,;\ v_h|_K \in P_1(K) \quad \forall\ K \in \mathcal{T}_h\,,\ \int_E [|v_h|]_E \,\mathrm{d}\sigma = 0 \quad \forall\ E \in \mathcal{E}_h\}\,,$$

and we shall denote by $i_h : H^2(\Omega) \to \widetilde{\mathrm{V}}_h^{conf}$ the Lagrange interpolation operator. It is well known that

$$|v - i_h v|_{m,K} \le C\, h_K^{2-m}\, |v|_{2,K} \qquad \forall\ v \in H^2(K),\ K \in \mathcal{T}_h,\ m = 0, 1\,. \tag{3}$$

We denote by $\{\zeta_E\}_{E \in \mathcal{E}_h^i}$ the usual basis in V_h^{nc}, i.e., each ζ_E is piecewise linear, equals 1 on E and vanishes in the midpoints of all edges different from E.

Throughout the paper we use standard notation $L^p(\Omega)$, $W^{k,p}(\Omega)$, $H^k(\Omega) = W^{k,2}(\Omega)$, $C^k(\overline{\Omega})$, etc. for the usual function spaces, see e.g.[14]. The norm and seminorm in the Sobolev space $W^{k,p}(\Omega)$ will be denoted by $\|\cdot\|_{k,p,\Omega}$ and $|\cdot|_{k,p,\Omega}$, respectively, and we set $\|\cdot\|_{k,\Omega} = \|\cdot\|_{k,2,\Omega}$ and $|\cdot|_{k,\Omega} = |\cdot|_{k,2,\Omega}$. Further, we define a discrete analogue of $|\cdot|_{1,\Omega}$ by

$$|v|_{1,h} = \left(\sum_{K \in \mathcal{T}_h} |v|_{1,K}^2\right)^{1/2}.$$

The scalar product in the spaces $L^2(G)$ and $L^2(G)^2$ will be denoted by $(\cdot,\cdot)_G$ and we set $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$. Finally, we use the notation C, $\widetilde{C}$ to denote generic constants independent of h and ε.

3 WEAK FORMULATION AND DISCRETE PROBLEM

Since $u_b \in H^{3/2}(\partial\Omega)$, there exists an extension $\widetilde{u}_b \in H^2(\Omega)$ of u_b. Applying standard techniques, we derive the following weak formulation of the equation (1):

Find $u \in H^1(\Omega)$ such that $u - \widetilde{u}_b \in H^1_0(\Omega)$ and

$$a(u,v) = (f,v) \qquad \forall\ v \in H^1_0(\Omega)\,,$$

where

$$a(u,v) = \varepsilon\,(\nabla u, \nabla v) + (\boldsymbol{b}\cdot\nabla u, v) + (c\,u, v)\,.$$

This weak formulation has a unique solution.

We shall approximate the space $H^1_0(\Omega)$ by a nonconforming first order finite element space V_h and at this stage we only assume that

$$\mathrm{V}_h^{conf} \subset \mathrm{V}_h \subset H^{2,h}(\Omega)\,. \tag{4}$$

The inclusion $\mathrm{V}_h^{conf} \subset \mathrm{V}_h$ assures first order approximation properties of V_h with respect to $|\cdot|_{1,h}$ when $h \to 0$. The inclusion $\mathrm{V}_h \subset H^{2,h}(\Omega)$ makes it possible to introduce a streamline diffusion stabilization.

To establish a finite element discretization of (1), we first introduce the bilinear forms

$$a_h^d(u,v) = \varepsilon \sum_{K\in\mathcal{T}_h} (\nabla u, \nabla v)_K\,,$$

$$a_h^s(u,v) = \frac{1}{2} \sum_{K\in\mathcal{T}_h} \left[(\boldsymbol{b}\cdot\nabla u, v)_K - (\boldsymbol{b}\cdot\nabla v, u)_K - (\operatorname{div}\boldsymbol{b}, u\,v)_K\right],$$

which respectively correspond to the diffusive and convective terms from the equation (1). The bilinear form a_h^s is skew–symmetric if $\operatorname{div}\boldsymbol{b} = 0$. That gives rise to the notation a_h^{skew} below. Further, we define a streamline diffusion term by

$$a_h^{sd}(u,v) = \sum_{K\in\mathcal{T}_h} (-\varepsilon\,\Delta\,u + \boldsymbol{b}\cdot\nabla u + c\,u,\ \delta_K\,\boldsymbol{b}\cdot\nabla v)_K\,,$$

where $\delta_K \geq 0$ is a control parameter. Now, denoting

$$\begin{aligned} a_h^{skew}(u,v) &= a_h^d(u,v) + a_h^s(u,v) + (c\,u,v) + a_h^{sd}(u,v)\,, \\ l_h(v) &= (f,v) + \sum_{K\in\mathcal{T}_h} (f,\ \delta_K\,\boldsymbol{b}\cdot\nabla v)_K\,, \end{aligned}$$

the streamline diffusion finite element method investigated in this paper reads:

Find $u_h \in \mathrm{V}_h \oplus \widetilde{\mathrm{V}}_h^{conf}$ such that $u_h - i_h\widetilde{u}_b \in \mathrm{V}_h$ and

$$a_h^{skew}(u_h, v_h) = l_h(v_h) \qquad \forall\ v_h \in \mathrm{V}_h\,. \tag{5}$$

Using standard arguments (cf.[14], Chapter III), we deduce that there exist constants μ_1, μ_2 independent of h such that, for any $v_h \in \mathrm{V}_h$ and $K \in \mathcal{T}_h$, we have

$$\|\Delta\, v_h\|_{0,K} \le \mu_1\, h_K^{-1}\, |v_h|_{1,K}\,, \qquad |v_h|_{1,K} \le \mu_2\, h_K^{-1}\, \|v_h\|_{0,K}\,. \tag{6}$$

We assume that the control parameter δ_K satisfies

$$0 \le \delta_K \le \min\left\{\frac{c_0}{2\,\|c\|^2_{0,\infty,K}}, \frac{h_K^2}{2\,\varepsilon\,\mu_1^2}\right\}. \tag{7}$$

Since the streamline diffusion stabilization is of importance in convection dominated regions only, we admit $\delta_K = 0$ in (7). A possible choice of δ_K is

$$\delta_K = \begin{cases} \kappa_K\, h_K & \text{if } h_K > \varepsilon, \\ 0 & \text{if } h_K \le \varepsilon, \end{cases} \tag{8}$$

where κ_K satisfies

$$0 < \kappa_0 \le \kappa_K \le \min\left\{\frac{c_0}{2\,\|c\|^2_{0,\infty,K}\, h_K}, \frac{h_K}{2\,\varepsilon\,\mu_1^2}\right\}. \tag{9}$$

The following result implies that the discrete problem (5) has a unique solution.

Theorem 1 *Under the assumption (7), the bilinear form a_h^{skew} is coercive, i.e.,*

$$a_h^{skew}(v_h, v_h) \ge \frac{1}{2}\, |||v_h|||^2 \qquad \forall\ v_h \in \mathrm{V}_h\,, \tag{10}$$

where the streamline diffusion norm $|||\cdot|||$ is defined by

$$|||v||| = \left(\sum_{K\in\mathcal{T}_h} \{\varepsilon\,|v|^2_{1,K} + c_0\,\|v\|^2_{0,K} + \delta_K\,\|\boldsymbol{b}\cdot\nabla v\|^2_{0,K}\}\right)^{1/2}.$$

Proof. See[13]. ♣

Remark 1 We shall also discuss the convergence properties of the discrete problem (5) when the convective term from (1) is discretized using the convective bilinear form

$$a_h^c(u,v) = \sum_{K\in\mathcal{T}_h} (\boldsymbol{b}\cdot\nabla u, v)_K\,.$$

Thus, the bilinear form a_h^{skew} in (5) is replaced by

$$a_h^{conv}(u,v) = a_h^d(u,v) + a_h^c(u,v) + (c\,u, v) + a_h^{sd}(u,v)\,. \tag{11}$$

Unfortunately, in general, a result similar to (10) does not hold for this bilinear form.

4 ERROR ANALYSIS

In this section we assume that the weak solution of (1) satisfies $u \in H^2(\Omega)$. If we use a conforming finite element space $\mathrm{V}_h \subset H_0^1(\Omega)$ in the discrete problem (5), then the weak solution u also solves the discrete problem, which leads to the well–known Galerkin orthogonality. If, however, we use a nonconforming space V_h, then the discrete functions have jumps across edges and (5) is not longer valid for the weak solution. We only obtain

$$a_h^{skew}(u, v_h) = l_h(v_h) + r_h^d(u, v_h) + r_h^s(u, v_h) \qquad \forall\ v_h \in \mathrm{V}_h\,,$$

where the consistency errors r_h^d, r_h^s, are given by

$$\begin{aligned} r_h^d(u, v_h) &= \varepsilon \sum_{E\in\mathcal{E}_h} \int_E \frac{\partial u}{\partial \boldsymbol{n}_E}\, [|v_h|]_E \,\mathrm{d}\sigma\,, \\ r_h^s(u, v_h) &= -\frac{1}{2} \sum_{E\in\mathcal{E}_h} \int_E (\boldsymbol{b}\cdot\boldsymbol{n}_E)\, u\, [|v_h|]_E \,\mathrm{d}\sigma\,. \end{aligned}$$

The behaviour of the consistency errors is crucial for the convergence properties of the discrete problem and it is desirable to design such spaces V_h that the consistency errors are 'small'.

First, let us formulate a convergence result valid for a general first order finite element space V_h satisfying the patch test of order 1.

Theorem 2 *Let the weak solution of (1) belong to $H^2(\Omega)$ and let the assumptions (4) and (7) be fulfilled. In addition, let the space V_h satisfy*

$$\int_E [|v_h|]_E \,\mathrm{d}\sigma = 0 \qquad \forall\ v_h \in \mathrm{V}_h,\ E \in \mathcal{E}_h\,. \tag{12}$$

Then the discrete solution u_h satisfies

$$|||u - u_h||| \le C\, h \left(\sum_{K\in\mathcal{T}_h} \gamma_K\, |u|_{2,K}^2 \right)^{1/2} + C \left(\sum_{E\in\mathcal{E}_h} \gamma_E\, \|u\|_{2,S_E}^2 \right)^{1/2}, \tag{13}$$

where

$$\gamma_K = \varepsilon + h_K^2 + \delta_K + (\max\{\varepsilon, \delta_K\})^{-1}\, h_K^2\,, \qquad \gamma_E = \min\left\{ \frac{h_E^2}{\varepsilon}, 1 \right\}. \tag{14}$$

Proof. See[13]. ♣

Estimate (13) is optimal with respect to the discretization parameter h since, for any fixed ε, it assures the convergence order 1. However, an ε–uniform estimate can only be obtained if we replace γ_E by 1, which leads to the convergence order 0. Numerical experiments for $\mathrm{V}_h = \mathrm{V}_h^{nc}$ really confirm this pessimistic prediction (see Section 7), which suggests that it is generally a property of the method and not a consequence of an unaccurate estimation. There are two basic ways how the properties of the method can be improved: either we can modify the discretization (cf. e.g.[10,11]) or we can try to improve the properties of the space V_h. Here we consider the latter possibility.

The second term on the right–hand side of (13), which causes the bad behaviour of this estimate, comes from the estimation of the consistency error $r_h^s(u,v_h)$ and of the term

$$\sum_{E\in\mathcal{E}_h} \int_E (\boldsymbol{b}\cdot\boldsymbol{n}_E)\, w\, [|v_h|]_E \,\mathrm{d}\sigma\,, \qquad w = i_h u - u\,, \tag{15}$$

which arises in consequence of an integration by parts applied to the bilinear form a_h^s. Let us explain how the consistency error $r_h^s(u,v_h)$ and the term (15) are estimated. We shall need projection operators $\mathcal{M}_E^k : L^2(E)\rightarrow P_k(E)$, $k\geq 0$, defined by

$$\int_E q\,\mathcal{M}_E^k v\,\mathrm{d}\sigma = \int_E q\,v\,\mathrm{d}\sigma \qquad \forall\ q\in P_k(E),\ v\in L^2(E),\ E\in\mathcal{E}_h\,.$$

According to[15], Lemma 3, there exists a constant C independent of E and h such that

$$\left|\int_E \varphi\,(v-\mathcal{M}_E^k v)\,\mathrm{d}\sigma\right| \leq C\,h_E^{k+1}\,|\varphi|_{1,K}\,|v|_{k+1,K} \tag{16}$$

for all $K\in\mathcal{T}_h$, $E\subset K$, $\varphi\in H^1(K)$ and $v\in H^{k+1}(K)$.

First, let us estimate the term (15). The property (12) allows us to subtract any constant function from $(\boldsymbol{b}\cdot\boldsymbol{n}_E)\,w$. Particularly, we can subtract $\mathcal{M}_E^0((\boldsymbol{b}\cdot\boldsymbol{n}_E)\,w)$, which in view of (16) and (3) gives

$$\begin{aligned}\int_E (\boldsymbol{b}\cdot\boldsymbol{n}_E)\,w\,[|v_h|]_E\,\mathrm{d}\sigma &= \int_E [(\boldsymbol{b}\cdot\boldsymbol{n}_E)\,w - \mathcal{M}_E^0((\boldsymbol{b}\cdot\boldsymbol{n}_E)\,w)]\,[|v_h|]_E\,\mathrm{d}\sigma\\ &\leq C\,h_E\,\|w\|_{1,S_E}\,|v_h|_{1,S_E} \leq \widetilde{C}\,h_E^2\,|u|_{2,S_E}\,|v_h|_{1,S_E}\,.\end{aligned} \tag{17}$$

Using (6), we derive

$$\int_E (\boldsymbol{b}\cdot\boldsymbol{n}_E)\,w\,[|v_h|]_E\,\mathrm{d}\sigma \leq C\,h_E\,|u|_{2,S_E}\,\gamma_E^{1/2}\,(\varepsilon\,|v_h|_{1,S_E}^2 + c_0\,\|v_h\|_{0,S_E}^2)^{1/2}\,,$$

which implies that

$$\sum_{E\in\mathcal{E}_h}\int_E (\boldsymbol{b}\cdot\boldsymbol{n}_E)\,w\,[|v_h|]_E\,\mathrm{d}\sigma \leq C\,h\left(\sum_{E\in\mathcal{E}_h}\gamma_E\,|u|_{2,S_E}^2\right)^{1/2} |||v_h|||\,.$$

The consistency error $r_h^s(u,v_h)$ can be estimated in a similar way. We have

$$\begin{aligned}\int_E (\boldsymbol{b}\cdot\boldsymbol{n}_E)\,u\,[|v_h|]_E\,\mathrm{d}\sigma &= \int_E [(\boldsymbol{b}\cdot\boldsymbol{n}_E)\,u - \mathcal{M}_E^0((\boldsymbol{b}\cdot\boldsymbol{n}_E)\,u)]\,[|v_h|]_E\,\mathrm{d}\sigma\\ &\leq C\,h_E\,\|u\|_{1,S_E}\,|v_h|_{1,S_E}\end{aligned} \tag{18}$$

and hence we only get

$$r_h^s(u,v_h) \le C \left(\sum_{E\in\mathcal{E}_h} \gamma_E \, \|u\|_{1,S_E}^2 \right)^{1/2} |||v_h||| \, .$$

As we see, it is the consistency error r_h^s which deteriorates the convergence order, uniform in ε, in the estimate (13). To obtain the same estimate of $r_h^s(u,v_h)$ as for the term (15), it would be sufficient if we could replace $\mathcal{M}_E^0$ by $\mathcal{M}_E^1$ in (18). Thus, it suffices to require that V_h satisfies the patch test of order 2, i.e.,

$$\int_E [|v_h|]_E \, q \, \mathrm{d}\sigma = 0 \qquad \forall \; v_h \in \mathrm{V}_h, \; q \in P_1(E), \; E \in \mathcal{E}_h \, . \tag{19}$$

Then, instead of (18), we get

$$\begin{aligned} \int_E (\boldsymbol{b}\cdot\boldsymbol{n}_E)\, u \, [|v_h|]_E \, \mathrm{d}\sigma &= \int_E [(\boldsymbol{b}\cdot\boldsymbol{n}_E)\, u - \mathcal{M}_E^1((\boldsymbol{b}\cdot\boldsymbol{n}_E)\, u)] \, [|v_h|]_E \, \mathrm{d}\sigma \\ &\le C \, h_E^2 \, \|u\|_{2,S_E} \, |v_h|_{1,S_E} \end{aligned} \tag{20}$$

and hence

$$r_h^s(u,v_h) \le C \, h \left(\sum_{E\in\mathcal{E}_h} \gamma_E \, \|u\|_{2,S_E}^2 \right)^{1/2} |||v_h||| \, .$$

Thus, we obtain the following result.

Theorem 3 *Let the weak solution of (1) belong to $H^2(\Omega)$ and let the assumptions (4) and (7) be fulfilled. In addition, let the space V_h satisfy (19). Then the discrete solution u_h satisfies*

$$|||u-u_h||| \le C \, h \left(\sum_{K\in\mathcal{T}_h} \gamma_K \, |u|_{2,K}^2 \right)^{1/2} + C \, h \left(\sum_{E\in\mathcal{E}_h} \gamma_E \, \|u\|_{2,S_E}^2 \right)^{1/2} , \tag{21}$$

where γ_K and γ_E are given by (14).

Remark 2 If we consider the discrete problem (5) with a_h^{skew} replaced by a_h^{conv} defined in (11), there is no consistency error induced by the convective term. Therefore, assuming coercivity of a_h^{conv}, the estimate (21) also holds if only the assumptions of Theorem 2 are satisfied.

Theorem 3 shows that, for a first order finite element space V_h satisfying (19) and for δ_K defined by (8) with bounded κ_K satisfying (9), we have the estimate

$$|||u-u_h||| \le C \, h \, \|u\|_{2,\Omega} \, ,$$

where the constant C is independent of ε (for ε bounded by some $\varepsilon_0 > 0$). This is a substantial improvement in comparison with Theorem 2. However, the question is whether a nonconforming space V_h satisfying (19) and having a structure convenient for practical computations can be constructed. This question will be discussed in the following section.

5 CONSTRUCTION OF A SPACE V_h SATISFYING THE PATCH TEST OF ORDER 2

Our aim is to construct a nonconforming first order finite element space V_h satisfying the patch test of order 2 expressed by (19). The simplest nonconforming finite element space, the space V_h^{nc}, does not possess this property and our idea is to construct the desired space V_h by modifying functions from V_h^{nc}. Precisely, for any function $v_h \in \mathrm{V}_h^{nc}$, we want to find some 'simple' function b_h such that $v_h + b_h$ satisfies the patch test of order 2. Since the patch test of order 1 holds for the space V_h^{nc}, it also has to be valid for the function b_h. Thus, given any $v_h \in \mathrm{V}_h^{nc}$, we look for a function b_h satisfying

$$\int_E [|b_h|]_E \,\mathrm{d}\sigma = 0\,, \qquad \int_E [|b_h|]_E \,\lambda_{E,1}\,\mathrm{d}\sigma = -\int_E [|v_h|]_E \,\lambda_{E,1}\,\mathrm{d}\sigma \qquad \forall\; E \in \mathcal{E}_h\,. \tag{22}$$

To fulfil (22), it seems to be natural to seek the function b_h in the form

$$b_h = \sum_{E' \in \mathcal{E}_h} \alpha_{E'}\,\varphi_{E'}\,, \tag{23}$$

where the functions $\varphi_{E'}$ satisfy

$$\int_E [|\varphi_{E'}|]_E \,\mathrm{d}\sigma = 0 \qquad \forall\; E \in \mathcal{E}_h\,, \tag{24}$$

$$\int_E [|\varphi_{E'}|]_E \,\lambda_{E,1}\,\mathrm{d}\sigma = 0 \qquad \forall\; E \in \mathcal{E}_h \setminus \{E'\}\,, \tag{25}$$

$$\int_{E'} [|\varphi_{E'}|]_{E'} \,\lambda_{E',1}\,\mathrm{d}\sigma \neq 0\,. \tag{26}$$

Then (22) holds for b_h defined by (23) with

$$\alpha_E = -\frac{\int_E [|v_h|]_E \,\lambda_{E,1}\,\mathrm{d}\sigma}{\int_E [|\varphi_E|]_E \,\lambda_{E,1}\,\mathrm{d}\sigma}\,, \qquad E \in \mathcal{E}_h\,. \tag{27}$$

Replacing any function $v_h \in \mathrm{V}_h^{nc}$ by a function $v_h + b_h$ with b_h defined by (23) and (27), we obtain a space

$$\mathrm{V}_h^{mod} = \{v_h \in \mathrm{V}_h^{nc} \oplus \mathrm{B}_h\,;\; \int_E [|v_h|]_E \,q\,\mathrm{d}\sigma = 0 \;\; \forall\; q \in P_1(E),\, E \in \mathcal{E}_h\}\,, \tag{28}$$

where

$$\mathrm{B}_h = \mathrm{span}\{\varphi_E\}_{E \in \mathcal{E}_h}\,. \tag{29}$$

We use the notation V_h^{mod} since this space consists of *modified* functions from V_h^{nc}. The space V_h^{mod} has several properties common with V_h^{nc}: it is a nonconforming first order finite element space which has the same dimension as V_h^{nc} and whose degrees of freedom are associated with edges. However, as we know from the preceding section, the space V_h^{mod} allows us to derive a better error estimate than the space V_h^{nc}.

The functions φ_E can be constructed in various ways. Here, we shall consider functions φ_E having supports contained in the elements (or element) adjacent to the respective edge E and we shall show how to construct such functions using a fixed bubble function $\widehat{b} \in H^1(\widehat{K})$ defined on the standard reference element $\widehat{K}$. We denote by $\widehat{E}$ one of the edges of $\widehat{K}$ and by $\widehat{x}_1$, $\widehat{x}_2$ the end points of $\widehat{E}$, and we shall assume that the function $\widehat{b}$ has the following properties:

$$\widehat{b}|_{\partial\widehat{K}\setminus\widehat{E}} = 0\,, \qquad \|\widehat{b}\|_{0,\widehat{E}} \neq 0\,, \qquad \int_{\widehat{E}} \widehat{b}\,\mathrm{d}\widehat{\sigma} = 0\,, \tag{30}$$

$$\gamma \equiv \frac{1}{|\widehat{E}|}\int_{\widehat{E}} \widehat{b}\,\widehat{\lambda}_1\,\mathrm{d}\widehat{\sigma} > 0 \quad \text{for } \widehat{\lambda}_1 \in P_1(\widehat{E}) \text{ with } \widehat{\lambda}_1(\widehat{x}_1) = 1,\ \widehat{\lambda}_1(\widehat{x}_2) = 0. \tag{31}$$

Now, for any $K \in \mathcal{T}_h$ and any edge E of K, we introduce a nonconforming bubble function

$$b_{K,E} = \begin{cases} \widehat{b} \circ F_K^{-1} & \text{in } K, \\ 0 & \text{in } \Omega \setminus K, \end{cases}$$

where $F_K : \widehat{K} \to K$ is a unique regular affine mapping satisfying $F_K(\widehat{K}) = K$, $F_K(\widehat{x}_1) = x_{E,1}$ and $F_K(\widehat{x}_2) = x_{E,2}$ (cf. e.g.[14]). Then, the functions φ_E can be constructed in the following way. If $E \in \mathcal{E}_h^b$ and K is the element adjacent to E, we set

$$\varphi_E = 2\,b_{K,E}\,.$$

If $E \in \mathcal{E}_h^i$ and K, $\widetilde{K}$ are the two elements adjacent to E and chosen in such a way that $\boldsymbol{n}_E$ points into $\widetilde{K}$, we define

$$\varphi_E = b_{K,E} - b_{\widetilde{K},E}\,.$$

Clearly, the functions φ_E have their supports in the elements (or element) adjacent to E and it is easy to check that

$$\int_E [|\varphi_E|]_E\,\lambda_{E,1}\,\mathrm{d}\sigma = 2\,\gamma\,h_E \qquad \forall\ E \in \mathcal{E}_h\,.$$

In addition, each function φ_E vanishes on all edges different from E. Thus, we see that the functions φ_E satisfy (24)–(26).

Unfortunately, there is a severe drawback of the above–defined space V_h^{mod} compared to the space V_h^{nc}. A stiffness matrix built up using the basis $\{\zeta_E\}_{E\in\mathcal{E}_h^i}$ of V_h^{nc} typically contains 5 nonzero entries in each row. However, constructing the stiffness matrix using the basis functions of V_h^{mod} defined as modified functions ζ_E, we generally obtain 27 nonzero entries in each row. The reason is that the supports of the basis functions from V_h^{mod} lie in six elements whereas supports of basis functions of V_h^{nc} consist of two elements only.

An easy remedy for the mentioned drawback of the space V_h^{mod} is to enlarge the space B_h used in the definition (28) of V_h^{mod}. Since now we consider functions φ_E constructed using the functions $b_{K,E}$, the enlargement is very easy. Instead of defining B_h by (29), we simply set

$$\mathrm{B}_h = \mathrm{span}\{b_{K,E}\}_{K\in\mathcal{T}_h,\, E\in\mathcal{E}_h,\, E\subset\partial K}\,. \tag{32}$$

Then one can show in the same way as in[13] that the space V_h^{mod} defined by (28) and (32) contains a basis consisting of functions $\{\chi_E\}_{E\in\mathcal{E}_h^i}$, $\{\psi_E\}_{E\in\mathcal{E}_h^i}$ whose supports are contained always in the two elements K, $\widetilde{K}$ adjacent to the respective edge E. Denoting by E, E_1, E_2 the edges of K and by E, E_3, E_4 the edges of $\widetilde{K}$, the functions χ_E, ψ_E are defined by

$$\chi_E \;=\; b_{K,E} + b_{\widetilde{K},E}\,, \tag{33}$$

$$\psi_E \;=\; \zeta_E + \beta_{E,E_1}\, b_{K,E_1} + \beta_{E,E_2}\, b_{K,E_2} + \beta_{E,E_3}\, b_{\widetilde{K},E_3} + \beta_{E,E_4}\, b_{\widetilde{K},E_4}\,, \tag{34}$$

where ζ_E are the basis functions of V_h^{nc} defined in Section 2 and $\beta_{E,E_1},\dots,\beta_{E,E_4}$ are uniquely determined coefficients. Note that $\chi_E \in H_0^1(\Omega)$ whereas ψ_E has jumps across the edges $E_1,\dots,E_4$.

Again, the space V_h^{mod} defined by (28) and (32) is an edge–oriented nonconforming first order finite element space satisfying the patch test of order 2. The stiffness matrix corresponding to the basis functions χ_E, ψ_E is now easy to implement since it consists of four matrices having the same structure as the stiffness matrix corresponding to the space V_h^{nc}. The enlargement of the stiffness matrix and of the number of unknowns ($\dim \mathrm{V}_h^{mod} = 2\dim \mathrm{V}_h^{nc}$) is worthwhile since the space V_h^{mod} often leads to a *substantial* improvement of the quality of the discrete solution as we shall see in Section 7.

Remark 3 Let us consider the discrete problem (5) with $\mathrm{V}_h = \mathrm{V}_h^{mod}$. Then the discrete solution u_h can be uniquely decomposed into its piecewise linear part u_h^{lin} and its bubble part $u_h^{bub} \in \mathrm{B}_h$, i.e., $u_h = u_h^{lin} + u_h^{bub}$. It can be shown in the same way as in[13] that, for $u \in H^2(\Omega)$, we have

$$\begin{aligned}
|u - u_h^{lin}|_{1,h} &\le |u - u_h|_{1,h} + 2\,|u - i_h u|_{1,\Omega}\,,\\
\|u - u_h^{lin}\|_{0,\Omega} &\le C\,\|u - u_h\|_{0,\Omega} + C\,\|u - i_h u\|_{0,\Omega}\,,\\
|||u - u_h^{lin}||| &\le C\Big(1 + \max_{K\in\mathcal{T}_h} \delta_K^{1/2}\Big)(|||u - u_h||| + |||u - i_h u|||)\,.
\end{aligned}$$

Thus, u_h^{lin} converges to the weak solution with the same convergence orders as u_h and the estimates of Theorems 2 and 3 remain valid for u_h^{lin}. Therefore, it is possible and for practical reasons sensible to consider the linear part of u_h as a discrete solution of (1).

Remark 4 The definitions of the functions ζ_E and ψ_E can be extended to boundary edges in an obvious way. Setting

$$\alpha_E = (i_h\widetilde{u}_b)(C_E)\,, \qquad \beta_E = \frac{1}{\gamma\, h_E}\int_E (i_h\widetilde{u}_b - \alpha_E\,\zeta_E)\,\lambda_{E,1}\,\mathrm{d}\sigma\,, \qquad E\in\mathcal{E}_h^b\,,$$

the function

$$\widetilde{u}_{bh} = \sum_{E\in\mathcal{E}_h^b} (\alpha_E\,\psi_E + \beta_E\,\chi_E)$$

satisfies $\widetilde{u}_{bh} - i_h\widetilde{u}_b \in \mathrm{V}_h^{mod}$. Thus, we can use $\widetilde{u}_{bh}$ as the boundary condition in the discrete problem (5), which is more convenient for implementational reason than the use of $i_h\widetilde{u}_b$. Details can be found in[13].

6 PATCH TEST OF ORDER 3

In the preceding section, we have constructed a general nonconforming first order finite element space V_h^{mod} satisfying the patch test of order 2. Further properties of this space depend on the definition of the function $\widehat{b}$. Particularly, one can ask whether a suitable choice of $\widehat{b}$ can assure the patch test of order 3, i.e., the validity of

$$\int_E [|v_h|]_E\, q\,\mathrm{d}\sigma = 0 \qquad \forall\ v_h \in \mathrm{V}_h^{mod},\ q\in P_2(E),\ E\in\mathcal{E}_h\,. \tag{35}$$

Since V_h^{mod} satisfies the patch test of order 2, the property (35) is equivalent to

$$\int_E [|v_h|]_E\, \lambda_{E,1}\,\lambda_{E,2}\,\mathrm{d}\sigma = 0 \qquad \forall\ v_h \in \mathrm{V}_h^{mod},\ E\in\mathcal{E}_h\,.$$

The function $\lambda_{E,1}\,\lambda_{E,2}$ is even with respect to C_E and therefore, it suffices to assure that $[|v_h|]_E$ is odd with respect to C_E for any $v_h\in\mathrm{V}_h^{mod}$ and any $E\in\mathcal{E}_h$. This is satisfied for functions from V_h^{nc} and hence (35) holds under the additional assumption that

$$\widehat{b}|_{\widehat{E}} \quad \text{is odd with respect to the midpoint of } \widehat{E}. \tag{36}$$

If the function $\widehat{b}$ is continuous, then the assumption (36) guarantees that functions from B_h vanish in the midpoints of all edges of the triangulation. In this case, the space V_h^{mod} consists of piecewise continuous functions which are continuous in the midpoints of inner edges and vanish in the midpoints of boundary edges. This is a further feature common with V_h^{nc}.

However, there is a much more important consequence of the property (36). In view of (35), we can replace $\mathcal{M}_E^0$ and $\mathcal{M}_E^1$ in (17) and (20), respectively, by $\mathcal{M}_E^2$. Assuming that $u\in H^3(\Omega)$ and $\boldsymbol{b}\in W^{3,\infty}(\Omega)^2$, we then derive

$$|r_h^s(u,v_h)| + \left|\sum_{E\in\mathcal{E}_h}\int_E (\boldsymbol{b}\cdot\boldsymbol{n}_E)\,w\,[|v_h|]_E\,\mathrm{d}\sigma\right| \le C\,h^2\left(\sum_{E\in\mathcal{E}_h}\gamma_E\,\|u\|_{3,S_E}^2\right)^{1/2} |||v_h|||$$

so that

$$|||u-u_h||| \le C\,h\left(\sum_{K\in\mathcal{T}_h}\gamma_K\,|u|_{2,K}^2\right)^{1/2} + C\,h^2\left(\sum_{E\in\mathcal{E}_h}\gamma_E\,\|u\|_{3,S_E}^2\right)^{1/2}.$$

Let us consider the convection dominated case in which $\varepsilon \le h$. Then, defining δ_K by (8) with bounded κ_K satisfying (9), we have $\gamma_K \le C\,h$ and hence it follows from the above estimate that

$$|||u - u_h||| \le C\,h^{3/2}\,\|u\|_{3,\Omega}\,,$$

where the constant C is independent of ε. It is known from the conforming finite element method that on general meshes the convergence order 3/2 is optimal[16].

As we see, the assumption (36) guarantees that the convective consistency error is of order $O(h^2)$ (ε–uniformly) and that, for a fixed ε, it is of order $O(h^3)$. Let us remark that for the space V_h^{nc}, we only have $O(1)$ and $O(h)$, respectively. The diffusive consistency error r_h^d can be even estimated by

$$r_h^d(u, v_h) \le C\,h^3\,\varepsilon^{1/2}\,|u|_{4,\Omega}\,|||v_h|||\,,$$

provided that $u \in H^4(\Omega)$. This estimate is again better by the factor h^2 compared with V_h^{nc}. The improvement of the estimate of r_h^d does not influence the asymptotic behaviour of the discrete solution but it certainly improves the accuracy.

The space V_h^{mod} defined using a function $\widehat{b}$ satisfying (30), (31) and (36) was already introduced in[13] and the corresponding finite element was named the P_1^{mod} element. A particular example of the P_1^{mod} element can be constructed by setting

$$\widehat{b} = \widehat{\lambda}_1^2\,\widehat{\lambda}_2 - \widehat{\lambda}_1\,\widehat{\lambda}_2^2\,,$$

where $\widehat{\lambda}_1$, $\widehat{\lambda}_2$ are the barycentric coordinates on the reference element $\widehat{K}$ with respect to $\widehat{x}_1$, $\widehat{x}_2$, respectively. To express the formulas (33), (34) for the basis functions χ_E and ψ_E in terms of the barycentric coordinates, we denote by K and $\widetilde{K}$ the two elements adjacent to an edge $E \in \mathcal{E}_h^i$ and by λ_1, λ_2 and $\widetilde{\lambda}_1$, $\widetilde{\lambda}_2$ the barycentric coordinates on K and $\widetilde{K}$ with respect to $x_{E,1}$, $x_{E,2}$, respectively. Further, we respectively denote by λ_3 and $\widetilde{\lambda}_3$ the remaining barycentric coordinates on K and $\widetilde{K}$. Then

$$\chi_E = \begin{cases} \lambda_1^2\,\lambda_2 - \lambda_1\,\lambda_2^2 & \text{in } K, \\ \widetilde{\lambda}_1^2\,\widetilde{\lambda}_2 - \widetilde{\lambda}_1\,\widetilde{\lambda}_2^2 & \text{in } \widetilde{K} \setminus E, \\ 0 & \text{in } \Omega \setminus \{K \cup \widetilde{K}\} \end{cases}$$

and

$$\psi_E = \begin{cases} 1 - 2\,\lambda_3 - 10\,(\lambda_1^2\,\lambda_3 - \lambda_1\,\lambda_3^2) - 10\,(\lambda_2^2\,\lambda_3 - \lambda_2\,\lambda_3^2) & \text{in } K, \\ 1 - 2\,\widetilde{\lambda}_3 - 10\,(\widetilde{\lambda}_1^2\,\widetilde{\lambda}_3 - \widetilde{\lambda}_1\,\widetilde{\lambda}_3^2) - 10\,(\widetilde{\lambda}_2^2\,\widetilde{\lambda}_3 - \widetilde{\lambda}_2\,\widetilde{\lambda}_3^2) & \text{in } \widetilde{K} \setminus E, \\ 0 & \text{in } \Omega \setminus \{K \cup \widetilde{K}\}. \end{cases}$$

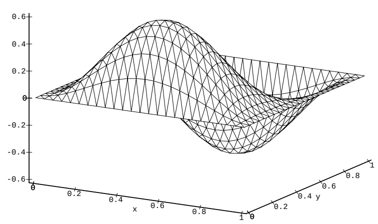

Figure 1: Exact solution of Example 1

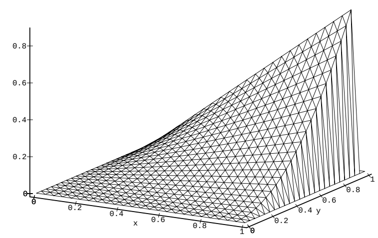

Figure 2: Exact solution of Example 2

7 NUMERICAL RESULTS

The aim of this section is to compare streamline diffusion finite element discretizations of the equation (1) employing the P_1^{nc} element (i.e., the space V_h^{nc}) with discretizations in which the modified space V_h^{mod} is used and to demonstrate the robustness of the latter discretizations. We only consider the piecewise cubic example of the space V_h^{mod} described at the end of the preceding section which we denote as the P_1^{mod} element in the following. We shall investigate both the discretization (5) and the discretization obtained from (5) by replacing a_h^{skew} by a_h^{conv} defined in (11). Thus, we have two types of discretizations and two types of spaces, which gives four combinations. However, since the combination a_h^{conv}/P_1^{mod} mostly gave very similar results as a_h^{skew}/P_1^{mod}, we mainly consider the following three methods: a_h^{conv}/P_1^{nc}, a_h^{skew}/P_1^{nc} and a_h^{skew}/P_1^{mod}. We recall that no theoretical results are available for the problems with a_h^{conv} because of the missing coercivity proof for a_h^{conv}.

The bilinear forms a_h^{skew} and a_h^{conv} were computed exactly whereas the right–hand side l_h was evaluated using a quadrature formula which is exact for piecewise cubic f. The arising linear systems were solved applying the GMRES method with ILU preconditioning. The computations were terminated if the ratio of the norms of the residuum and the right–hand side was smaller than 10^{-8}. The errors of the discrete solutions were measured in the norms $|||\cdot|||$ and $|\cdot|_{1,h}$. The evaluation of $|||\cdot|||$ (resp. $|\cdot|_{1,h}$) was exact for piecewise quadratic (resp. cubic) functions. For the P_1^{mod} element, we give the errors of the piecewise linear part u_h^{lin} of u_h (see Remark 3).

Example 1 Smooth polynomial solution.
Let $\Omega = (0,1)^2$, $\boldsymbol{b} = (3,2)$, $c = 2$ and $u_b = 0$. For a given $\varepsilon > 0$, the right–hand side f is chosen such that

$$u(x,y) = 100\,x^2\,(1-x)^2\,y\,(1-y)\,(1-2\,y)$$

is the exact solution of (1), see Fig. 1.

It is not surprising that, for all three methods and for any fixed ε, numerical experiments confirm the linear convergence of the discrete solution to the above solution u in both

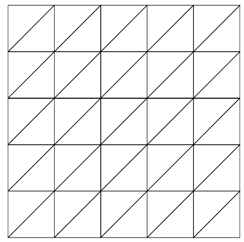

Figure 3: Type of the triangulations used for Examples 1–3

the norms $|||\cdot|||$ and $|\cdot|_{1,h}$. However, the convergence behaviour with respect to ε is more interesting. Therefore, in Tables 1 and 2, we present errors of the discrete solutions for various values of h and for $\varepsilon = h^4$. The results were obtained for Friedrichs–Keller triangulations of Ω of the type depicted in Fig. 3 and for δ_K defined by (8) with $\kappa_K = 1$. The convergence orders were always computed using values from triangulations with $h = 1.77 \cdot 10^{-2}$ and $h = 8.84 \cdot 10^{-3}$. Table 1 shows that the solutions of the discretization a_h^{skew}/P_1^{mod} converge with the optimal order 3/2 in the streamline diffusion norm $|||\cdot|||$, as predicted by our theory. The same convergence order is also observed for the discretization a_h^{conv}/P_1^{nc}. Further, we observe that the solutions of (5) with the

h	ε	a_h^{conv}/P_1^{nc}	a_h^{skew}/P_1^{nc}	a_h^{skew}/P_1^{mod}
7.07−2	2.50−5	1.43−1	7.79−1	1.48−1
3.54−2	1.56−6	5.10−2	7.43−1	5.24−2
1.77−2	9.77−8	1.80−2	7.09−1	1.85−2
8.84−3	6.10−9	6.36−3	6.86−1	6.56−3
conv. order		1.50	0.05	1.50

Table 1: Example 1, errors $|||u - u_h|||$

h	ε	a_h^{conv}/P_1^{nc}	a_h^{skew}/P_1^{nc}	a_h^{skew}/P_1^{mod}
7.07−2	2.50−5	1.40+0	4.29+1	2.14−1
3.54−2	1.56−6	1.09+0	8.66+1	1.07−1
1.77−2	9.77−8	7.57−1	1.78+2	5.37−2
8.84−3	6.10−9	4.98−1	3.72+2	2.69−2
conv. order		0.60	−1.06	1.00

Table 2: Example 1, errors $|u - u_h|_{1,h}$

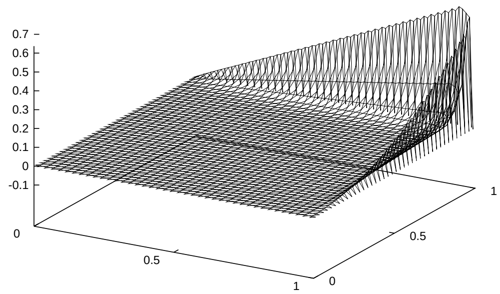

Figure 4: Example 2, errors of solution of a_h^{skew}/P_1^{mod} for $h = 0.0354$

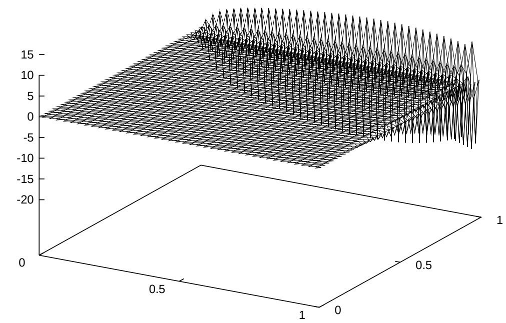

Figure 5: Example 2, errors of solution of a_h^{conv}/P_1^{nc} for $h = 0.0354$

P_1^{nc} element do not converge in $|||\cdot|||$ which is in agreement with Theorem 2. Thus, the discretization a_h^{skew}/P_1^{nc} cannot compete with the other two methods. According to Table 1, the discretizations a_h^{skew}/P_1^{mod} and a_h^{conv}/P_1^{nc} seem to be comparable, however, Table 2 indicates that the discretization a_h^{skew}/P_1^{mod} is much more accurate than both a_h^{conv}/P_1^{nc} and a_h^{skew}/P_1^{nc}. In addition, we observe an optimal ε–uniform convergence of solutions of a_h^{skew}/P_1^{mod} with respect to $|\cdot|_{1,h}$. The superiority of a_h^{skew}/P_1^{mod} can also be seen from the following example.

Example 2 Layers at the outflow part of the boundary.
Let $\Omega = (0,1)^2$, $\varepsilon = 10^{-8}$, $\boldsymbol{b} = (2,3)$ and $c = 1$. The right–hand side f and the boundary condition u_b are chosen such that

$$u(x,y) = x\,y^2 - y^2\,\exp\left(\frac{2\,(x-1)}{\varepsilon}\right) - x\,\exp\left(\frac{3\,(y-1)}{\varepsilon}\right) + \exp\left(\frac{2\,(x-1)+3\,(y-1)}{\varepsilon}\right)$$

is the exact solution of (1). This function has boundary layers at $x = 1$ and $y = 1$, see Fig. 2.

The domain Ω was again discretized using a triangulation of the type depicted in Fig. 3 and δ_K was defined by (8) with $\kappa_K = 0.25$. Fig. 4–6 show errors $u_h - u$ for all three discretizations obtained for $h = 0.0354$. We observe that, for a_h^{skew}/P_1^{mod}, the errors are located in a region near the boundary layers. For a_h^{conv}/P_1^{nc}, the errors are located in nearly the same region but they oscillate and are more than ten times larger. Finally, for a_h^{skew}/P_1^{nc}, the errors are smaller than for a_h^{conv}/P_1^{nc} but they are distributed over a large part of Ω and they again oscillate. Again, we can conclude, that the discretization employing the P_1^{mod} element is much more better than discretizations using the P_1^{nc} element.

To demonstrate the robustness of discretizations employing the P_1^{mod} element, we also consider the following two examples which do not fit into the theory presented in this paper.

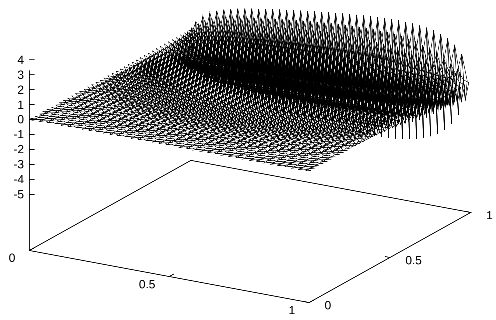

Figure 6: Example 2, errors of solution of a_h^{skew}/P_1^{nc} for $h = 0.0354$

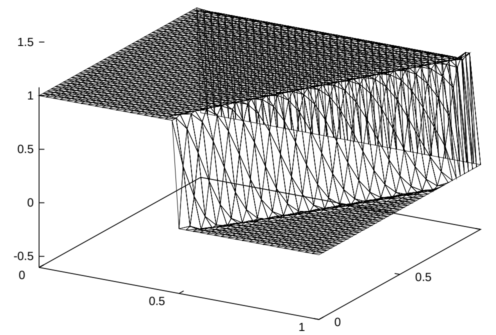

Figure 7: Example 3, solution of a_h^{skew}/P_1^{mod} for $h = 0.0354$

Example 3 Inner and boundary layers.
We consider $\Omega = (0,1)^2$, $\varepsilon = 10^{-6}$, $\boldsymbol{b} = (1/2, \sqrt{3}/2)$, $c = 0$, $f = 0$ and

$$u_b(x,y) = \begin{cases} 0 & \text{for } x \geq 1/2 \text{ or } y = 1, \\ 1 & \text{else.} \end{cases}$$

The solution u has an inner layer along the line $y = \sqrt{3}\,(x - 1/2)$ and boundary layers along $y = 1$ and $x = 1$, $y > \sqrt{3}/2$.

Example 4 Inner and boundary layers.
Let $\Omega = (-3,9) \times (-3,3) \setminus \{(x,y) \in \mathbb{R}^2\,;\ x^2 + y^2 \leq 1\}$, $\varepsilon = 10^{-6}$, $\boldsymbol{b} = (1,0)$, $c = 0$, $f = 0$ and

$$u_b(x,y) = \begin{cases} 0 & \text{for } x = -3 \text{ or } y = \pm 3, \\ 1 & \text{else.} \end{cases}$$

In addition, instead of the Dirichlet boundary condition along the line $x = 9$, we prescribe $\partial u/\partial x = 0$. The solution u has two inner layers along $(0,9) \times \{\pm 1\}$ and a boundary layer along the curve $x \leq 0$, $x^2 + y^2 = 1$.

For solving Example 3, we used the same triangulation as for Example 2. The triangulation used for Example 4 is depicted in Fig. 8. In both cases we used δ_K defined by (8) with $\kappa_K = 0.2$. The computed solutions are shown in Fig. 7 and 9. Instead of showing the discontinuous solutions u_h directly, we present corresponding conforming functions $\widetilde{u}_h \in \widetilde{V}_h^{conf}$ such that the value of $\widetilde{u}_h$ at any inner vertex is equal to the arithmetic mean value of the values of u_h at the midpoints of edges connected with this vertex. We see that inner and boundary layers are detected very well and that the methods behave in a robust way although the assumptions made in Section 1 are not satisfied.

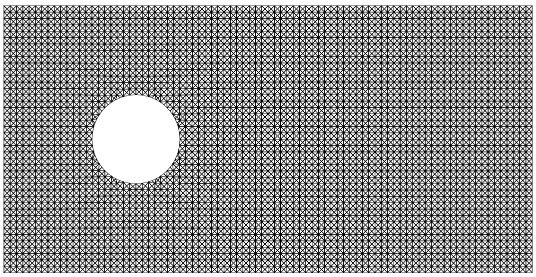

Figure 8: Triangulation for Example 4 (13488 elements)

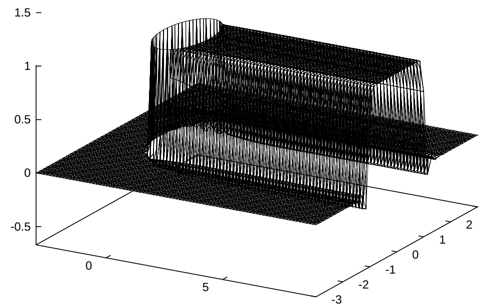

Figure 9: Example 4, solution of a_h^{conv}/P_1^{mod} for the triangulation from Fig. 8

The above numerical results show that the discretizations employing the P_1^{mod} element are substantially better than discretizations using the P_1^{nc} element. The P_1^{mod} element always gave optimal convergence orders and behaved very robust with respect to ε. In addition, the iterative solver used to compute the discrete solutions converged much faster for the P_1^{mod} element than for discretizations using the P_1^{nc} element. Thus, the P_1^{mod} element not only improves the stability of the discrete solution, but also the convergence properties of the solver.

ACKNOWLEDGMENT

The research of P. Knobloch has been supported by the Czech Grant Agency under the grant No. 201/99/P029, by the Grant Agency of the Charles University under the grant No. 3/1998/B MAT/MFF and by the grant CEZ J13/98:113200007.

REFERENCES

[1] T.J.R. Hughes and A.N. Brooks, A multidimensional upwind scheme with no crosswind diffusion. In: T.J.R. Hughes (ed.): *Finite Element Methods for Convection Dominated Flows*, Vol. 34 of AMD, ASME (1979).

[2] H.–G. Roos, M. Stynes, and L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Equations. Convection–Diffusion and Flow problems*, Springer–Verlag (1996).

[3] K. Eriksson and C. Johnson, Adaptive streamline diffusion finite element methods for stationary convection–diffusion problems, *Math. Comput.*, **60**, 167–188 (1993).

[4] C. Johnson and J. Saranen, Streamline diffusion methods for the incompressible Euler and Navier–Stokes equations, *Math. Comput.*, **47**, 1–18 (1986).

[5] K. Nijima, Pointwise error estimates for a streamline diffusion finite element scheme, *Numer. Math.*, **56**, 707–719 (1990).

[6] L. Tobiska and R. Verfürth, Analysis of a streamline diffusion finite element method for the Stokes and Navier–Stokes equations, *SIAM J. Numer. Anal.*, **33**, 107–127 (1996).

[7] O. Dorok, V. John, U. Risch, F. Schieweck, and L. Tobiska, Parallel finite element methods for the incompressible Navier–Stokes equations. In: E.H. Hirschel (ed.): *Flow Simulation with High–Performance Computers II*, Notes on Numerical Fluid Mechanics Vol. 52, 20–33, Vieweg–Verlag (1996).

[8] V. John, *Parallele Lösung der inkompressiblen Navier–Stokes Gleichungen auf adaptiv verfeinerten Gittern*, PhD Thesis, Otto–von–Guericke–Universität Magdeburg (1997).

[9] F. Schieweck, *Parallele Lösung der stationären inkompressiblen Navier–Stokes Gleichungen*, Habilitationsschrift, Otto–von–Guericke–Universität Magdeburg (1997).

[10] V. John, G. Matthies, F. Schieweck, and L. Tobiska, A streamline–diffusion method for nonconforming finite element approximations applied to convection–diffusion problems, *Comput. Methods Appl. Mech. Eng.*, **166**, 85–97 (1998).

[11] V. John, J.M. Maubach, and L. Tobiska, Nonconforming streamline–diffusion–finite–element–methods for convection–diffusion problems, *Numer. Math.*, **78**, 165–188 (1997).

[12] M. Stynes and L. Tobiska, The streamline–diffusion method for nonconforming Q_1^{rot} elements on rectangular tensor–product meshes, to appear in *IMA J. Numer. Anal.*

[13] P. Knobloch and L. Tobiska, *The P_1^{mod} element: a new nonconforming finite element for convection–diffusion problems*, Preprint Nr. 28, Fakultät für Mathematik, Otto–von–Guericke–Universität Magdeburg (1999).

[14] P.G. Ciarlet, Basic Error Estimates for Elliptic Problems. In: P.G. Ciarlet and J.L. Lions (eds.): *Handbook of Numerical Analysis*, Vol. 2 – *Finite Element Methods* (pt. 1), 17–351, North–Holland (1991).

[15] M. Crouzeix and P.–A. Raviart, Conforming and nonconforming finite element methods for solving the stationary Stokes equations I, *RAIRO*, **7**, 33–76 (1973).

[16] G. Zhou, How accurate is the streamline diffusion finite element method?, *Math. Comput.*, **66**, 31–44 (1997).