ON SOME NONCONFORMING FINITE ELEMENTS FOR INCOMPRESSIBLE FLOW PROBLEMS

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1 Introduction

Nonconforming finite elements are attractive for applications from computational fluid dynamics since they usually satisfy an inf–sup condition and lead to an efficient parallel implementation. However, it was realized that they often lead to a rather low accuracy in the convection dominated regime. Therefore, in this paper, we discuss a recently proposed class (cf. [2, 3]) of new nonconforming finite elements, the so called P_1^{mod} element, and we demonstrate that these elements assure a good accuracy in cases of dominant convection and are suitable for approximating the velocity in incompressible flow problems.

2 Assumptions and notation

We assume that we are given a family $\{\mathcal{T}_h\}$ of triangulations of a polygonal domain $\Omega \subset \mathbb{R}^2$ consisting of closed triangular elements K having the usual compatibility properties and satisfying $h_K \equiv \mathrm{diam}(K) \le h$ for any $K \in \mathcal{T}_h$. We assume that the elements K are shape regular, i.e., there exists a constant σ independent of h such that $h_K/\varrho_K \le \sigma$ for any $K \in \mathcal{T}_h$ and $h > 0$, where ϱ_K is the maximum diameter of circles inscribed into K.

We denote by $\mathcal{E}_h$ the set of the edges E of $\mathcal{T}_h$ and by $\mathcal{E}_h^i$ the subset of $\mathcal{E}_h$ consisting of inner edges. Further, we denote by h_E the length of the edge E and by $x_{E,1}$, $x_{E,2}$ the end points of E. For any inner edge $E \in \mathcal{E}_h^i$, we define the jump of a function v across E by

$$[|v|]_E = (v|_K)|_E - (v|_{\widetilde{K}})|_E \,,$$

where K, $\widetilde{K}$ are the two elements adjacent to E (we fix one of the two possible choices of K, $\widetilde{K}$). If an edge $E \in \mathcal{E}_h$ lies on $\partial\Omega$, then we set $[|v|]_E = v|_E$.

3 Definition and properties of the P_1^{mod} element

In [2, 3], a general definition of the P_1^{mod} element was established using a fixed nonconforming bubble function $\widehat{b} \in H^1(\widehat{K})$ defined on the standard reference element $\widehat{K}$. We denote by $\widehat{E}$ one of the edges of $\widehat{K}$ and by $\widehat{x}_1$, $\widehat{x}_2$ the end points of $\widehat{E}$, and we make the following assumptions:

$$\widehat{b}|_{\partial\widehat{K}\setminus\widehat{E}} = 0 \,, \qquad \|\widehat{b}\|_{0,\widehat{E}} \neq 0 \,,$$

$$\widehat{b}|_{\widehat{E}} \quad \text{is odd with respect to the midpoint of } \widehat{E},$$

$$\gamma \equiv \frac{1}{|\widehat{E}|} \int_{\widehat{E}} \widehat{b}\,\widehat{\lambda}_1 \,\mathrm{d}\widehat{\sigma} > 0 \quad \text{for } \widehat{\lambda}_1 \in P_1(\widehat{E}) \text{ with } \widehat{\lambda}_1(\widehat{x}_1) = 1,\ \widehat{\lambda}_1(\widehat{x}_2) = 0.$$

An example of the function $\widehat{b}$ possessing the above properties is

$$\widehat{b} = \widehat{\lambda}_1^2\,\widehat{\lambda}_2 - \widehat{\lambda}_1\,\widehat{\lambda}_2^2\,, \tag{1}$$

where $\widehat{\lambda}_1$, $\widehat{\lambda}_2$ are the barycentric coordinates on $\widehat{K}$ with respect to $\widehat{x}_1$, $\widehat{x}_2$, respectively. For any $K \in \mathcal{T}_h$ and any edge E of K, we introduce a nonconforming bubble function

$$b_{K,E} = \begin{cases} \widehat{b} \circ F_K^{-1} & \text{in } K, \\ 0 & \text{in } \Omega \setminus K, \end{cases}$$

where $F_K : \widehat{K} \to K$ is a unique regular affine mapping satisfying $F_K(\widehat{K}) = K$, $F_K(\widehat{x}_1) = x_{E,1}$ and $F_K(\widehat{x}_2) = x_{E,2}$. Now, on any element K, we define the P_1^{mod} element by the space

$$P_1^{mod}(K) = P_1(K) \oplus \operatorname{span}\{b_{K,E}|_K\}_{E\in\mathcal{E}_h,\,E\subset K}$$

and by the six nodal functionals

$$I_E(v) = \frac{1}{h_E}\int_E v\,\mathrm{d}\sigma\,, \quad J_E(v) = \frac{1}{\gamma\,h_E}\int_E v\,(\lambda_{E,1} - \frac{1}{2})\,\mathrm{d}\sigma\,, \qquad E \in \mathcal{E}_h,\ E \subset K\,,$$

where $\lambda_{E,1}$ is the barycentric coordinate on E with respect to $x_{E,1}$. It is easy to see that the six nodal functionals are unisolvent with the space $P_1^{mod}(K)$. The corresponding finite element space is the space

$$\mathrm{V}_h^{mod} = \{v_h \in \mathrm{V}_h^{nc} \oplus \mathrm{B}_h\,;\ \int_E [|v_h|]_E\,q\,\mathrm{d}\sigma = 0 \ \ \forall\, q \in P_1(E),\ E \in \mathcal{E}_h\}\,,$$

where

$$\mathrm{V}_h^{nc} = \{v_h \in L^2(\Omega)\,;\ v_h|_K \in P_1(K) \ \ \forall\, K \in \mathcal{T}_h\,,\ \ \int_E [|v_h|]_E\,\mathrm{d}\sigma = 0 \ \ \forall\, E \in \mathcal{E}_h\}$$

is the piecewise linear nonconforming space and

$$\mathrm{B}_h = \operatorname{span}\{b_{K,E}\}_{K\in\mathcal{T}_h,\,E\in\mathcal{E}_h,\,E\subset K}\,.$$

We use the notations P_1^{mod} and V_h^{mod} since the new class of elements we just described can be viewed as a *modification* of the nonconforming P_1 element (which leads to the space V_h^{nc}). Note that the space V_h^{mod} contains continuous piecewise linear functions and hence it has first order approximation properties with respect to the discrete H^1 norm. Further, it is very important (cf. the next section) that the space V_h^{mod} satisfies the patch test of order 3, i.e., the property (3) from Section 4 holds for $\mathrm{V}_h = \mathrm{V}_h^{mod}$ with $k = 2$. Another important feature of the space V_h^{mod} is that it contains a basis consisting of functions $\{\chi_E\}_{E\in\mathcal{E}_h^i}$, $\{\psi_E\}_{E\in\mathcal{E}_h^i}$ whose supports are contained always in the two elements K, $\widetilde{K}$ adjacent to the respective edge E. Denoting by E, E_1, E_2 the edges of K and by E, E_3, E_4 the edges of $\widetilde{K}$, the functions χ_E, ψ_E are defined by

$$\begin{aligned} \chi_E &= b_{K,E} + b_{\widetilde{K},E}\,, \\ \psi_E &= \zeta_E + \beta_{E,E_1}\,b_{K,E_1} + \beta_{E,E_2}\,b_{K,E_2} + \beta_{E,E_3}\,b_{\widetilde{K},E_3} + \beta_{E,E_4}\,b_{\widetilde{K},E_4}\,, \end{aligned}$$

where ζ_E is the usual nonconforming piecewise linear basis function assigned to the edge E and $\beta_{E,E_1}, \dots, \beta_{E,E_4}$ are uniquely determined constants. Then $\chi_E \in H_0^1(\Omega)$ whereas ψ_E has jumps across the edges $E_1, \dots, E_4$.

Thus, we can conclude that the P_1^{mod} element leads to an edge–oriented nonconforming first order finite element space satisfying the patch test of order 3. Note that V_h^{mod} can be implemented using the same data structures as the space V_h^{nc} and that if $\widehat{b} \subset C(\widehat{K})$, then V_h^{mod} consists of piecewise continuous functions which are continuous in the midpoints of inner edges and vanish in the midpoints of boundary edges. This is a further feature common with the space V_h^{nc}. The increased number of degrees of freedom ($\dim \mathrm{V}_h^{mod} = 2 \dim \mathrm{V}_h^{nc}$) is worthwhile since the space V_h^{mod} often leads to a substantial improvement of the quality of the discrete solution.

4 Numerical solution of convection dominated problems

To see the properties of the P_1^{mod} element when applied to the numerical solution of convection dominated problems, we consider the convection–diffusion equation

$$-\varepsilon \, \Delta u + \boldsymbol{b} \cdot \nabla u + c \, u = f \quad \text{in } \Omega, \qquad\qquad u = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where $\varepsilon > 0$ is a (small) constant, $\boldsymbol{b} \in W^{1,\infty}(\Omega)^2$, $c \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$. As usual, we assume that

$$c - \frac{1}{2} \operatorname{div} \boldsymbol{b} \geq c_0 \, ,$$

where c_0 is a positive constant. This assumption guarantees that (2) admits a unique weak solution for all positive values of the parameter ε.

To solve the equation (2) numerically, we introduce a nonconforming first order finite element space V_h defined on the triangulation $\mathcal{T}_h$ and look for a solution $u_h \in \mathrm{V}_h$ satisfying an appropriate discrete analogue of the weak formulation including a streamline diffusion term introduced to stabilize the discretization in convection dominated regions (cf. [2, 3]). Let us assume that the space V_h satisfies the patch test of order $k+1$ for some $k \geq 0$, i.e.,

$$\int_E [|v_h|]_E \, q \, \mathrm{d}\sigma = 0 \qquad \forall \; v_h \in \mathrm{V}_h, \; q \in P_k(E), \; E \in \mathcal{E}_h \, , \tag{3}$$

and that $u \in H^m(\Omega)$ with $m = \max\{2, k+1\}$ and $\boldsymbol{b} \in W^{k+1,\infty}(\Omega)^2$. Then one can prove (cf. [2, 3]) that the following error estimate with respect to the streamline diffusion norm holds in case of dominant convection:

$$|||u - u_h||| \leq C \, h^{3/2} \, |u|_{2,\Omega} + C \, h^k \, \min\left\{ \frac{h}{\sqrt{\varepsilon}}, 1 \right\} \|u\|_{m,\Omega} \, , \tag{4}$$

where C is a constant independent of h, ε and u. The second term on the right–hand side of (4) stems from the nonconformity only and it is not present if $\mathrm{V}_h \subset H_0^1(\Omega)$ in which case (4) reduces to the well–known error estimate assuring the convergence order 3/2 which is known to be optimal for first order spaces on general meshes. However, for $\mathrm{V}_h = \mathrm{V}_h^{nc}$, the assumption (3) only holds for $k = 0$ and we only get the convergence order 1. Moreover, we observe that an ε–uniform estimate is only possible with the convergence order 0. Numerical experiments really confirm this pessimistic prediction, which suggests that it is generally a property of the method and not a consequence of an unaccurate estimation. On the other hand, using $\mathrm{V}_h = \mathrm{V}_h^{mod}$, the property (3) holds for $k = 2$ and hence we obtain the optimal ε–uniform convergence order 3/2. The superiority of V_h^{mod} over V_h^{nc} was also confirmed by many numerical tests.

5 Numerical solution of incompressible flow problems

Now let us discuss the application of the P_1^{mod} element to the numerical solution of incompressible flow problems. We already know from the previous section that the P_1^{mod} element is suitable for resolving effects of dominant convection and hence let us focus our attention on the incompressibility. Thus, we consider as a model problem the Stokes equations

$$-\nu\,\Delta\,\boldsymbol{u} + \nabla p = \boldsymbol{f}\,, \qquad \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega\,, \qquad \boldsymbol{u} = \boldsymbol{0} \ \text{ on } \partial\Omega\,, \tag{5}$$

where $\boldsymbol{u}$ is the velocity and p is the pressure in an incompressible viscous fluid contained in Ω. The parameter $\nu > 0$ is the kinematic viscosity and $\boldsymbol{f}$ is an outer volume force. As a discretization we consider just the discrete analogue of the weak formulation of (5), without introducing any stabilization. It is well known that the space $[\mathrm{V}_h^{mod}]^2$ can be used for approximating $\boldsymbol{u}$ only if there is a space Q_h for approximating p such that the two spaces satisfy an inf–sup condition. It was shown in [1] that the inf–sup condition holds for Q_h consisting of discontinuous piecewise linear functions provided that $\int_{\widehat{K}} \widehat{b}\,\mathrm{d}\widehat{x} = 0$ (which is satisfied for $\widehat{b}$ defined by (1)) and that any element $K \in \mathcal{T}_h$ has at least one vertex in Ω. Thus, also for the Stokes equations, one can get much better results using the P_1^{mod} element than using the nonconforming P_1 element which only satisfies the inf–sup condition for Q_h consisting of piecewise constant functions. Therefore, in view of this section and Section 4, it is not surprising that also in case of the incompressible Navier–Stokes equations, the P_1^{mod} element is superior over the nonconforming P_1 element.

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