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On the application of the P_1^{mod} element to incompressible flow problems*

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Abstract. We consider the application of the nonconforming P_1^{mod} element to the approximation of the velocity in the incompressible Stokes and Navier–Stokes equations. We prove the uniform validity of an inf–sup condition if the pressure is approximated by piecewise constant functions. Under additional assumptions, we also prove the inf–sup condition for discontinuous piecewise linear approximations of the pressure. Numerical results show that the P_1^{mod} element allows to obtain significantly better approximations of the velocity than the Crouzeix–Raviart element.

1 Introduction

In computational fluid dynamics, nonconforming finite elements are often used for discretizing incompressible flow problems. One advantage of nonconforming elements in comparison to conforming ones is that they usually satisfy inf–sup conditions with more convenient pressure spaces and that discretely divergence–free bases can often be more easily constructed for this type of elements. Another reason for the application of nonconforming elements may be that they are more suitable for a parallel implementation since their degrees of freedom are associated with edges (or with interior points of the elements of the triangulation), which leads to a cheap local communication between processors. In addition, nonconforming elements often show nice stability properties and lead to very efficient finite element solvers. We refer to [9, 10, 13] and [15] for more details on the properties of nonconforming finite elements applied to incompressible flow problems.

However, it was observed that nonconforming elements sometimes do not lead to the expected accuracy if they are applied to the numerical solution of convection dominated problems. This phenomenon was thoroughly investigated in [12] for a scalar convection–diffusion equation discretized by means of the streamline diffusion method. Numerical experiments in [12] show that, for the simplest nonconforming finite element, which is the linear triangular Crouzeix–Raviart element, it is often not possible to obtain an acceptable accuracy in the convection dominated regime. The reason is that, for the Crouzeix–Raviart element, the interelement continuity is too weak (it is reduced to the continuity at one point on each edge). Therefore, the authors of [12] developed a new nonconforming triangular first order finite element named the P_1^{mod} element for which the interelement continuity is stronger so that the same optimal convergence estimate can be proved as in the conforming case. The finite element space corresponding to the P_1^{mod} element contains *modified* Crouzeix–Raviart functions, which gave rise to the notation P_1^{mod}.

Numerical results for convection–diffusion equations in [12] showed that the discrete solutions obtained using the P_1^{mod} element behave in a very robust way with respect to the perturbation parameter and that their accuracy is significantly better than for the Crouzeix–Raviart element. In addition, the iterative solver used to compute the discrete solutions converged much faster for the P_1^{mod} element than for discretizations using the Crouzeix–Raviart element. Thus, the P_1^{mod} element not only improves the stability of the discrete solution, but also the convergence properties of the solvers. Finally, as a further argument for using the P_1^{mod} element, let us mention that this new element satisfies the discrete Korn inequality (cf. [11]) which does not hold for the most nonconforming first order elements including the Crouzeix–Raviart element.

Since the P_1^{mod} element leads to robust and accurate discretizations of convection dominated convection–diffusion equations, one can expect that the P_1^{mod} element will also be appropriate for approximating the velocity $\boldsymbol{u}$ in an incompressible viscous fluid described by the Navier–Stokes equations

$$-\nu\Delta\boldsymbol{u}+(\nabla\boldsymbol{u})\boldsymbol{u}+\nabla p=\boldsymbol{f} \qquad \text{in } \Omega\,, \tag{1}$$

$$\operatorname{div}\boldsymbol{u}=0 \qquad \text{in } \Omega\,, \tag{2}$$

$$\boldsymbol{u}=\boldsymbol{0} \qquad \text{on } \partial\Omega\,. \tag{3}$$

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Here, $\Omega \subset \mathbb{R}^2$ is a bounded domain with a polygonal boundary $\partial\Omega$, $\nu > 0$ is the kinematic viscosity, $\boldsymbol{f}$ is an outer volume force and p is a second unknown function, the pressure.

In view of the incompressibility constraint, finite element spaces $\boldsymbol{V}_h$ and Q_h for approximating the velocity $\boldsymbol{u}$ and pressure p, respectively, cannot be chosen arbitrarily if one wants to obtain a stable discretization with respect to $h \to 0$ and no additional stabilization of the continuity equation (2) is used (see e.g. [3, 8] for details). A sufficient requirement on the spaces $\boldsymbol{V}_h$, Q_h is the validity of the inf–sup condition

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h \setminus \{\boldsymbol{0}\}} \frac{b_h(\boldsymbol{v}_h, q_h)}{|\boldsymbol{v}_h|_{1,h}} \geq \beta \|q_h\|_{0,\Omega} \qquad \forall\, q_h \in \mathrm{Q}_h\,, \tag{4}$$

where $\beta > 0$ is independent of the discretization parameter h,

$$b_h(\boldsymbol{v}_h, q_h) = -\sum_{K \in \mathcal{T}_h} \int_K q_h \operatorname{div} \boldsymbol{v}_h \, dx\,,$$

$$|\boldsymbol{v}_h|_{1,h} = \left(\sum_{K \in \mathcal{T}_h} |\boldsymbol{v}_h|_{1,K}^2 \right)^{1/2}$$

and $\mathcal{T}_h$ is a triangulation of Ω consisting of elements K used for constructing the spaces $\boldsymbol{V}_h$ and Q_h. The notation $|\boldsymbol{v}_h|_{1,K}$ is used for the seminorm of $\boldsymbol{v}_h|_K$ in the space $H^1(K)^2$. The inf–sup condition (4) makes it possible to establish optimal error estimates for the discrete solution of (1)–(3), see e.g. [8].

The main aim of this paper is to investigate for which spaces Q_h the inf–sup condition (4) is satisfied if the velocity space $\boldsymbol{V}_h$ is defined using the P_1^{mod} element. That means that we consider triangulations $\mathcal{T}_h$ made up of triangles. First, we will show that the inf–sup condition (4) holds if the space Q_h consists of piecewise constant functions. Then, using this result and introducing some additional assumptions, we establish the inf–sup condition for Q_h consisting of discontinuous piecewise linear functions. This particularly implies the validity of the inf–sup condition for Q_h consisting of continuous piecewise linear functions and for Q_h consisting of piecewise linear functions which are continuous in the midpoints of edges of the triangulation. Each of these four pressure spaces Q_h may be appropriate for solving the Navier–Stokes equations as we will see later.

Nowadays, a lot of pairs of finite element spaces are known to satisfy or to fail the inf–sup condition (4) (see e.g. [3] and [8] for overviews) and it is necessary to mention the relation of the present paper to at least some of the known results. The simplest pair of spaces satisfying (4) is the nonconforming P_1/P_0 element of [5], where $\boldsymbol{V}_h$ consists of piecewise linear Crouzeix–Raviart functions and Q_h of piecewise constant functions. Although the P_1^{mod} element was obtained by modifying the Crouzeix–Raviart functions, the inf–sup condition for the P_1^{mod}/P_0 element is not a direct consequence of [5] since all piecewise linear functions which belong to the P_1^{mod} element are continuous. It is well known that $\boldsymbol{V}_h$ consisting of continuous piecewise linear functions does not satisfy (4) for piecewise constant functions q_h. A conforming space $\boldsymbol{V}_h$ which can be used together with a piecewise constant space Q_h was designed in [1]. Here, the space $\boldsymbol{V}_h$ consists of continuous piecewise linear functions enriched by vector functions assigned to edges and the proof of (4) is essentially based on the fact that each additional vector function has a nonzero flux through the edge it is assigned to. The P_1^{mod} element contains a subspace having the structure of $\boldsymbol{V}_h$ from [1], however, the proof of [1] fails in this case since the additional vector functions have zero fluxes through all edges of the triangulation. If Q_h consists of discontinuous piecewise linear functions, the validity of the inf–sup condition is known for spaces $\boldsymbol{V}_h$ consisting of piecewise quadratic functions enriched by conforming or nonconforming bubble functions having their supports always in one element only, see [5] and [7]. The presence of the bubble functions plays a significant role in proving the respective inf–sup conditions. In the case of the P_1^{mod} element, there are no functions available which would have their supports in one element only and hence the present paper also reveals a new structure of a finite element space which can be paired with a space Q_h consisting of discontinuous piecewise linear functions.

The paper is organized in the following way. First, in Sect. 2, we summarize the notation which will be used in the subsequent sections. Then, in Sect. 3, we give the definition of the P_1^{mod} element and mention some of its properties. Section 4 is devoted to the proof of the inf–sup condition in the case when the space Q_h consists of piecewise constant functions and, in Sect. 5, we prove the inf–sup condition for Q_h consisting of discontinuous piecewise linear functions. In Sect. 6, we discuss the choices of various pressure spaces Q_h in the case when the P_1^{mod} element is applied to the solution of the Stokes equations and, finally, in Sect. 7, we present numerical results comparing the P_1^{mod} element with the Crouzeix–Raviart element.

2 Notation

We assume that we are given a family $\{\mathcal{T}_h\}$ of triangulations of the domain Ω consisting of closed triangular elements K having the usual compatibility properties (see e.g. [4]) and satisfying $h_K \equiv \operatorname{diam}(K) \leq h$ for any $K \in \mathcal{T}_h$. We assume that the family of triangulations is regular, i.e., there exists a constant σ independent of h such that

$$\frac{h_K}{\varrho_K} \leq \sigma \qquad \forall\, K \in \mathcal{T}_h,\ h > 0\,, \tag{5}$$

where ϱ_K is the maximum diameter of circles inscribed into K. For any element K, we will denote by $\boldsymbol{n}_{\partial K}$ the unit outer normal vector to the boundary of K. Further, we denote by $\widehat{K}$ the standard reference element and by $F_K : \widehat{K} \to K$ any affine regular mapping which maps $\widehat{K}$ onto K. According to [4, Sect. 15], there exist constants C_1, $C_2 > 0$ depending only on σ such that

$$C_1 |v \circ F_K|_{1,\widehat{K}} \leq |v|_{1,K} \leq C_2 |v \circ F_K|_{1,\widehat{K}} \qquad \forall\, v \in H^1(K),\ K \in \mathcal{T}_h\,. \tag{6}$$

We denote by $\mathcal{E}_h$ the set of edges E of $\mathcal{T}_h$, by $\mathcal{E}_h^i$ the subset of $\mathcal{E}_h$ consisting of inner edges and by $\mathcal{E}_h^b$ the set $\mathcal{E}_h \setminus \mathcal{E}_h^i$, i.e., the set of boundary edges. Further, for any edge E, we denote by h_E the length of E and by $\boldsymbol{n}_E$ a fixed unit normal vector to E. If $E \in \mathcal{E}_h^b$, then $\boldsymbol{n}_E$ coincides with the outer normal vector to $\partial\Omega$. For any inner edge $E \in \mathcal{E}_h^i$, we respectively define the jump and the average of a function v across E by

$$[|v|]_E = \left(v|_K\right)|_E - \left(v|_{\widetilde{K}}\right)|_E\,, \tag{7}$$

$$\langle|v|\rangle_E = \frac{1}{2}\left\{\left(v|_K\right)|_E + \left(v|_{\widetilde{K}}\right)|_E\right\}\,,$$

where K, $\widetilde{K}$ are the two elements adjacent to E denoted in such a way that $\boldsymbol{n}_E$ points into $\widetilde{K}$. If an edge $E \in \mathcal{E}_h$ lies on the boundary of Ω, then we set

$$[|v|]_E = v|_E\,,$$

which is the jump defined by (7) with v extended by zero outside Ω. We will need the nonconforming Crouzeix–Raviart space

$$\mathrm{V}_h^{nc} = \left\{ v_h \in L^2(\Omega)\,;\ v_h|_K \in P_1(K) \quad \forall\, K \in \mathcal{T}_h\,, \right.$$
$$\left. \int_E [|v_h|]_E d\sigma = 0 \quad \forall\, E \in \mathcal{E}_h \right\}$$

and we denote by $\{\zeta_E\}_{E\in\mathcal{E}_h^i}$ the usual basis in V_h^{nc}, i.e., each ζ_E is piecewise linear, equals 1 on E and vanishes in the midpoints of all edges different from E.

Throughout the paper we use standard notation $L^2(\Omega)$, $H^k(\Omega) = W^{k,2}(\Omega)$, $P_k(\Omega)$, $C(\overline{\Omega})$, etc. for the usual function spaces, see e.g. [4]. We only mention that we denote by $L_0^2(\Omega)$ the space of functions from $L^2(\Omega)$ having zero mean value on Ω. The norm and seminorm in the Sobolev space $H^k(\Omega)$ will be denoted by $\|\cdot\|_{k,\Omega}$ and $|\cdot|_{k,\Omega}$, respectively. Finally, we use the notation C to denote a generic constant independent of h and ν.

3 Definition and properties of the P_1^{mod} element

In this section we recall the general definition of the P_1^{mod} element given in [12]. We introduce functions $\widehat{b}_1$, $\widehat{b}_2$ and $\widehat{b}_3$ defined on the reference triangle $\widehat{K}$ and associated respectively with the edges $\widehat{E}_1$, $\widehat{E}_2$ and $\widehat{E}_3$ of $\widehat{K}$. We assume for $i \in \{1,2,3\}$ that

$$\widehat{b}_i \in H^1\left(\widehat{K}\right)\,, \qquad \widehat{b}_i|_{\partial\widehat{K}\setminus\widehat{E}_i} = 0\,, \tag{8}$$

$$\widehat{b}_i|_{\widehat{E}_i} \text{ is odd with respect to the midpoint of } \widehat{E}_i\,, \tag{9}$$

$$\int_{\widehat{E}_i} \left[\left(1 - 2\widehat{\lambda}_{i+1}\right) + \widehat{b}_i\right]\widehat{q}\,d\widehat{\sigma} = 0 \qquad \forall\, \widehat{q} \in P_1\left(\widehat{E}_i\right)\,, \tag{10}$$

where $\widehat{\lambda}_i$ is the barycentric coordinate on $\widehat{K}$ with respect to the vertex of $\widehat{K}$ opposite the edge $\widehat{E}_i$ (we set $\widehat{\lambda}_4 \equiv \widehat{\lambda}_1$). Then the shape functions of the P_1^{mod} element on $\widehat{K}$ form the space

$$P_1^{mod}\left(\widehat{K}\right) = P_1\left(\widehat{K}\right) \oplus \operatorname{span}\left\{\widehat{b}_1, \widehat{b}_2, \widehat{b}_3\right\}\,.$$

For any element $K \in \mathcal{T}_h$, we choose a regular affine mapping $F_K : \widehat{K} \to K$ such that $F_K(\widehat{K}) = K$ and we set

$$b_{K,E} = \begin{cases} \widehat{b}_i \circ F_K^{-1} & \text{in } K, \\ 0 & \text{in } \Omega \setminus K, \end{cases}$$

for $E = F_K(\widehat{E}_i)$, $i = 1,2,3$. Thus, the shape functions on K form the space

$$P_1^{mod}(K) = P_1(K) \oplus \operatorname{span}\{b_{K,E}|_K\}_{E\in\mathcal{E}_h, E\subset\partial K}\,.$$

For each element K, we introduce six local nodal functionals

$$I_{K,E}(v) = \frac{1}{h_E}\int_E v\,d\sigma\,,$$

$$J_{K,E}(v) = \frac{3}{h_E}\int_E v(2\lambda_E - 1)\,d\sigma\,,$$
$$E \in \mathcal{E}_h,\ E \subset \partial K\,, \tag{11}$$

where $\lambda_E \in P_1(E)$ equals 1 at one end point of E and 0 at the other end point of E. Then the finite element space V_h^{mod} approximating the space $H_0^1(\Omega)$ consists of all functions which belong to the space $P_1^{mod}(K)$ on any element $K \in \mathcal{T}_h$, which are continuous on all inner edges in the sense of the equality of nodal functionals and for which all nodal functionals associated with boundary edges vanish. This means that

$$\mathrm{V}_h^{mod} = \left\{ v_h \in L^2(\Omega)\,;\ v_h|_K \in P_1^{mod}(K) \quad \forall\, K \in \mathcal{T}_h\,, \right.$$
$$\left. \int_E [|v_h|]_E\, q\,d\sigma = 0 \quad \forall\, q \in P_1(E),\ E \in \mathcal{E}_h \right\}\,.$$

For any edge $E \in \mathcal{E}_h$, we define global nodal functionals

$$I_E(v) = I_{K,E}(v)\,, \qquad J_E(v) = J_{K,E}(v)\,, \tag{12}$$

where K is any element adjacent to E (for $v \in \mathrm{V}_h^{mod}$, the values of these functionals are independent of the choice of K). We denote by $\{\psi_E, \chi_E\}_{E\in\mathcal{E}_h^i}$ a basis of V_h^{mod} which is dual to the functionals I_E, J_E, i.e., for any $E, E' \in \mathcal{E}_h^i$, we have

$$I_E(\psi_{E'}) = \delta_{E,E'}\,, \qquad I_E(\chi_{E'}) = 0\,,$$
$$J_E(\psi_{E'}) = 0\,, \qquad J_E(\chi_{E'}) = \delta_{E,E'}\,, \tag{13}$$

where $\delta_{E,E'} = 1$ for $E = E'$ and $\delta_{E,E'} = 0$ for $E \neq E'$. To establish formulas for ψ_E and χ_E, we denote by K, $\widetilde{K}$ the two elements adjacent to E, by E, E_1, E_2 the edges of K, by E, E_3, E_4 the edges of $\widetilde{K}$, and by ζ_E the standard basis function of V_h^{nc} associated with the edge E (cf. Sect. 2). Then

$$\psi_E = \zeta_E + \beta_{E,1} b_{K,E_1} + \beta_{E,2} b_{K,E_2} + \beta_{E,3} b_{\widetilde{K},E_3} + \beta_{E,4} b_{\widetilde{K},E_4}\,, \tag{14}$$

$$\chi_E = \beta_{E,5} b_{K,E} + \beta_{E,6} b_{\widetilde{K},E}\,, \tag{15}$$

where the coefficients $\beta_{E,1}, \ldots, \beta_{E,6}$ are uniquely determined and equal 1 or -1. If the functions $\widehat{b}_1, \widehat{b}_2, \widehat{b}_3$ are chosen in a suitable way (e.g. $\widehat{b}_i = \widehat{b}_1 \circ \widehat{F}_i$ where $\widehat{F}_i$ is an affine transformation of $\widehat{K}$ onto $\widehat{K}$) then $\chi_E \in H_0^1(\Omega)$ and hence, in this case, the functions χ_E generate a conforming subspace of V_h^{mod}. The functions ψ_E are always purely nonconforming functions since they have jumps across the edges $E_1, \ldots, E_4$ and they can be viewed as modified basis functions of V_h^{nc}. Note that, in view of (6), we have

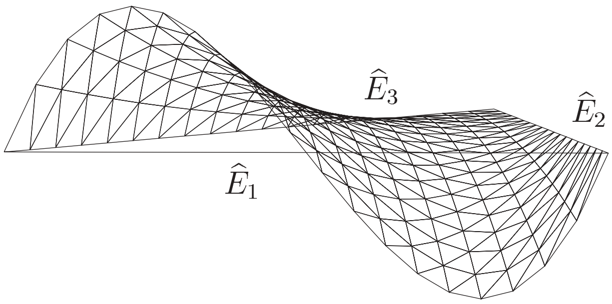

Fig. 1. Function $\widehat{\lambda}_2^2\widehat{\lambda}_3 - \widehat{\lambda}_2\widehat{\lambda}_2^2$

$$|\chi_E|_{1,K} \le C_3\,, \quad |\psi_E|_{1,K} \le C_3 \quad \forall\ E \in \mathcal{E}_h^i,\, K \in \mathcal{T}_h\,, \tag{16}$$

where $C_3 = 2\,C_2[1 + \max\{|\widehat{b}_1|_{1,\widehat{K}}, |\widehat{b}_2|_{1,\widehat{K}}, |\widehat{b}_3|_{1,\widehat{K}}\}]$.

An important property of the space V_h^{mod} is that it satisfies the patch test of order 3, i.e.,

$$\int_E [|v_h|]_E\, q\, d\sigma = 0 \quad \forall\ v_h \in \mathrm{V}_h^{mod},\, q \in P_2(E),\, E \in \mathcal{E}_h\,.$$

This immediately follows from the fact that the space V_h^{mod} satisfies the patch of order 2 and that the basis functions ψ_E and χ_E are odd along all edges of the triangulation. Moreover, if (10) holds for any $\widehat{q} \in P_k(\widehat{E}_i)$ with some $k > 1$, then it is easy to show that the basis functions ψ_E, χ_E satisfy the patch test of order $k+1$. Consequently, in this case, the whole space V_h^{mod} satisfies the patch test of order $k+1$.

We can conclude that the space V_h^{mod} is an edge–oriented nonconforming finite element space possessing first order approximation properties with respect to $|\cdot|_{1,h}$. The supports of the basis functions ψ_E, χ_E are contained in the supports of the basis functions ζ_E of V_h^{nc} and hence the space V_h^{mod} can be implemented using the same data structures as the space V_h^{nc}. However, the higher order of the patch test satisfied by the space V_h^{mod} enables to obtain much more accurate results than for the Crouzeix–Raviart space V_h^{nc}. This was shown for a scalar convection–diffusion equation in [12] and it will also be seen for the Stokes equations in this paper.

A simple example of the P_1^{mod} element can be constructed by setting (cf. Fig. 1)

$$\widehat{b}_i = 10\left(\widehat{\lambda}_{i+1}^2\widehat{\lambda}_{i+2} - \widehat{\lambda}_{i+1}\widehat{\lambda}_{i+2}^2\right)\,, \qquad i = 1, 2, 3, \tag{17}$$

where the indices are to be considered modulo 3. To express the formulas (14), (15) for the basis functions χ_E and ψ_E in terms of the barycentric coordinates, we denote by K and $\widetilde{K}$ the two elements adjacent to an edge $E \in \mathcal{E}_h^i$ and by λ_1, λ_2 and $\widetilde{\lambda}_1$, $\widetilde{\lambda}_2$ the barycentric coordinates on K and $\widetilde{K}$ with respect to the end points of E. Further, we respectively denote by λ_3 and $\widetilde{\lambda}_3$ the remaining barycentric coordinates on K and $\widetilde{K}$. Then

$$\psi_E = \begin{cases} 1 - 2\lambda_3 - 10\left(\lambda_1^2\lambda_3 - \lambda_1\lambda_3^2\right) & \\ \qquad - 10\left(\lambda_2^2\lambda_3 - \lambda_2\lambda_3^2\right) & \text{in } K, \\ 1 - 2\widetilde{\lambda}_3 - 10\left(\widetilde{\lambda}_1^2\widetilde{\lambda}_3 - \widetilde{\lambda}_1\widetilde{\lambda}_3^2\right) & \\ \qquad - 10\left(\widetilde{\lambda}_2^2\widetilde{\lambda}_3 - \widetilde{\lambda}_2\widetilde{\lambda}_3^2\right) & \text{in } \widetilde{K} \setminus E, \\ 0 & \text{in } \Omega \setminus \{K \cup \widetilde{K}\}, \end{cases}$$

and, after dividing by 10,

$$\chi_E = \begin{cases} \lambda_1^2\lambda_2 - \lambda_1\lambda_2^2 & \text{in } K, \\ \widetilde{\lambda}_1^2\widetilde{\lambda}_2 - \widetilde{\lambda}_1\widetilde{\lambda}_2^2 & \text{in } \widetilde{K} \setminus E, \\ 0 & \text{in } \Omega \setminus \{K \cup \widetilde{K}\}. \end{cases}$$

These basis functions were used in the numerical calculations presented both in [12] and in this paper.

4 Inf–sup condition with a piecewise constant pressure space

This section is devoted to the proof of the inf–sup condition

$$\sup_{v_h \in \boldsymbol{V}_h^{mod}\setminus\{\boldsymbol{0}\}} \frac{b_h(\boldsymbol{v}_h, q_h)}{|\boldsymbol{v}_h|_{1,h}} \ge \beta \|q_h\|_{0,\Omega} \qquad \forall\ q_h \in \overline{\mathrm{Q}}_h\,, \tag{18}$$

where $\boldsymbol{V}_h^{mod} = [\mathrm{V}_h^{mod}]^2$ and

$$\overline{\mathrm{Q}}_h = \{q_h \in L_0^2(\Omega)\,;\ q_h|_K \in P_0(K) \ \ \forall\, K \in \mathcal{T}_h\}\,.$$

The proof is based on the validity of an inf–sup condition for the spaces $H_0^1(\Omega)^2$ and $L_0^2(\Omega)$ and on the construction of a suitable operator $r_h : H_0^1(\Omega)^2 \to \boldsymbol{V}_h^{mod}$, which is a classical technique originating from [6]. The operator r_h is constructed analogously as in the proof of the inf–sup condition for the nonconforming P_1/P_0 element in [5]. We will see that the inf–sup condition is assured only by the functions $\{\psi_E\}_{E\in\mathcal{E}_h^i}$.

To simplify the proof, we extend the definition of the functions ψ_E to boundary edges $E \in \mathcal{E}_h^b$. We again require that $\psi_E|_K \in P_1^{mod}(K)$ for any $K \in \mathcal{T}_h$, that it is continuous on inner edges in the sense of the equality of local nodal functionals and that

$$I_E(\psi_{E'}) = \delta_{E,E'}\,, \quad J_E(\psi_{E'}) = 0 \qquad \forall\ E \in \mathcal{E}_h\,,\, E' \in \mathcal{E}_h^b\,. \tag{19}$$

Then, for any $E \in \mathcal{E}_h^b$, the function ψ_E is given by

$$\psi_E = \zeta_E + \beta_{E,1} b_{K,E_1} + \beta_{E,2} b_{K,E_2}\,, \tag{20}$$

where K is the element adjacent to E and E_1, E_2 are the remaining edges of K. The functions ζ_E are defined analogously as for inner edges and the coefficients $\beta_{E,1}$ and $\beta_{E,2}$ are again uniquely determined and equal 1 or -1.

Now we can prove the following auxiliary result.

Lemma 1. *Consider any element $K \in \mathcal{T}_h$ and let E_1, E_2, E_3 be its edges. Then*

$$\sum_{i=1}^{3} \psi_{E_i}|_K = 1\,.$$

Proof. According to (14) and (20), we have

$$\psi_{E_1}|_K = \zeta_{E_1} + \beta_{E_1,E_2} b_{K,E_2} + \beta_{E_1,E_3} b_{K,E_3}\,,$$
$$\psi_{E_2}|_K = \zeta_{E_2} + \beta_{E_2,E_1} b_{K,E_1} + \beta_{E_2,E_3} b_{K,E_3}\,,$$
$$\psi_{E_3}|_K = \zeta_{E_3} + \beta_{E_3,E_1} b_{K,E_1} + \beta_{E_3,E_2} b_{K,E_2}\,.$$

Owing to (13) and (19), the numbers β_{E_i,E_j} are uniquely determined by

$$\beta_{E_i,E_j} = -\frac{J_{K,E_j}(\zeta_{E_i})}{J_{K,E_j}(b_{K,E_j})}, \qquad i,j=1,2,3,\ i\neq j.$$

Since $\zeta_{E_1}+\zeta_{E_2}=0$ on E_3, we obtain

$$\beta_{E_1,E_3}+\beta_{E_2,E_3} = -\frac{J_{K,E_3}(\zeta_{E_1}+\zeta_{E_2})}{J_{K,E_3}(b_{K,E_3})} = 0.$$

Analogously,

$$\beta_{E_1,E_2}+\beta_{E_3,E_2}=0, \qquad \beta_{E_2,E_1}+\beta_{E_3,E_1}=0.$$

Therefore,

$$\sum_{i=1}^{3}\psi_{E_i}|_K = \sum_{i=1}^{3}\zeta_{E_i}|_K = 1.$$

□

The following theorem implies the inf–sup condition (18).

Theorem 1. *Let* $\boldsymbol{Z}_h = [\mathrm{span}\{\psi_E\}_{E\in\mathcal{E}_h^i}]^2$. *Then there exists a constant* $\bar\beta>0$ *depending only on* σ, $\widehat{b}_1$, $\widehat{b}_2$, $\widehat{b}_3$ *and* Ω *such that*

$$\sup_{\boldsymbol{v}_h\in\boldsymbol{Z}_h\setminus\{\boldsymbol{0}\}} \frac{b_h(\boldsymbol{v}_h,q_h)}{|\boldsymbol{v}_h|_{1,h}} \geq \bar\beta\|q_h\|_{0,\Omega} \qquad \forall\, q_h\in\overline{\mathrm{Q}}_h. \tag{21}$$

Proof. For $\boldsymbol{v}\in H^1(\Omega)^2$, let us set

$$r_h\boldsymbol{v} = \sum_{E\in\mathcal{E}_h}\boldsymbol{\alpha}_E\psi_E \qquad \text{with} \qquad \boldsymbol{\alpha}_E = \frac{1}{h_E}\int_E \boldsymbol{v}\,d\sigma.$$

Then, for any element $K\in\mathcal{T}_h$ and any edge $E\subset\partial K$, we have

$$\int_E (r_h\boldsymbol{v})|_K d\sigma = \int_E \boldsymbol{\alpha}_E\psi_E d\sigma = \boldsymbol{\alpha}_E h_E = \int_E \boldsymbol{v}d\sigma$$

and hence we derive by the Gauss integral theorem that

$$\begin{aligned}\int_K \mathrm{div}(r_h\boldsymbol{v})dx &= \int_{\partial K}\boldsymbol{n}_{\partial K}\cdot(r_h\boldsymbol{v})|_K d\sigma\\ &= \int_{\partial K}\boldsymbol{n}_{\partial K}\cdot\boldsymbol{v}d\sigma = \int_K \mathrm{div}\boldsymbol{v}dx.\end{aligned}$$

Therefore,

$$b_h(r_h\boldsymbol{v},q_h) = b_h(\boldsymbol{v},q_h) \qquad \forall\,\boldsymbol{v}\in H^1(\Omega)^2,\, q_h\in\overline{\mathrm{Q}}_h. \tag{22}$$

We will prove that there exists a constant κ, depending only on σ and the functions $\widehat{b}_1$, $\widehat{b}_2$ and $\widehat{b}_3$, such that

$$|r_h\boldsymbol{v}|_{1,h} \leq \kappa|\boldsymbol{v}|_{1,\Omega} \qquad \forall\,\boldsymbol{v}\in H^1(\Omega)^2. \tag{23}$$

Consider any $K\in\mathcal{T}_h$, any regular affine mapping $F_K:\widehat{K}\to K$ with $F_K(\widehat{K})=K$ and any $\boldsymbol{v}\in H^1(\Omega)^2$. We denote $\widehat{\boldsymbol{v}}=\boldsymbol{v}\circ F_K$.

Then, in view of the trace theorems, we deduce that, for any $E\in\mathcal{E}_h$ with $E\subset\partial K$,

$$|\boldsymbol{\alpha}_E| = |F_K^{-1}(E)|^{-1}\left|\int_{F_K^{-1}(E)}\widehat{\boldsymbol{v}}\,d\widehat{\sigma}\right| \leq C\|\widehat{\boldsymbol{v}}\|_{1,\widehat{K}}$$

and hence, by (16),

$$|r_h\boldsymbol{v}|_{1,K} \leq C_3\sum_{E\in\mathcal{E}_h,E\subset\partial K}|\boldsymbol{\alpha}_E| \leq C\|\widehat{\boldsymbol{v}}\|_{1,\widehat{K}}.$$

Using Lemma 1 and this inequality, we derive

$$|r_h\boldsymbol{v}|_{1,K} = \inf_{\boldsymbol{p}\in\mathbb{R}^2}|r_h(\boldsymbol{v}+\boldsymbol{p})|_{1,K} \leq C\inf_{\boldsymbol{p}\in\mathbb{R}^2}\|\widehat{\boldsymbol{v}}+\boldsymbol{p}\|_{1,\widehat{K}}.$$

Applying [4, p. 120, Theorem 14.1], we obtain

$$|r_h\boldsymbol{v}|_{1,K} \leq C|\widehat{\boldsymbol{v}}|_{1,\widehat{K}}$$

and (23) follows using (6).

Now consider any $q_h\in\overline{\mathrm{Q}}_h$. Applying (22) and (23) and using the fact that $r_h\boldsymbol{v}\in\boldsymbol{Z}_h$ for $\boldsymbol{v}\in H_0^1(\Omega)^2$, we deduce that

$$\begin{aligned}\sup_{\boldsymbol{v}_h\in\boldsymbol{Z}_h\setminus\{\boldsymbol{0}\}}\frac{b_h(\boldsymbol{v}_h,q_h)}{|\boldsymbol{v}_h|_{1,h}} &\geq \sup_{\boldsymbol{v}\in H_0^1(\Omega)^2,\, r_h\boldsymbol{v}\neq\boldsymbol{0}}\frac{b_h(r_h\boldsymbol{v},q_h)}{|r_h\boldsymbol{v}|_{1,h}}\\ &\geq \frac{1}{\kappa}\sup_{\boldsymbol{v}\in H_0^1(\Omega)^2\setminus\{\boldsymbol{0}\}}\frac{b_h(\boldsymbol{v},q_h)}{|\boldsymbol{v}|_{1,\Omega}}.\end{aligned}$$

According to [8, p. 81], there exists a constant $\beta>0$ such that

$$\sup_{\boldsymbol{v}\in H_0^1(\Omega)^2\setminus\{\boldsymbol{0}\}}\frac{b_h(\boldsymbol{v},q)}{|\boldsymbol{v}|_{1,\Omega}} \geq \beta\|q\|_{0,\Omega} \qquad \forall\, q\in L_0^2(\Omega)$$

and hence (21) holds with $\bar\beta=\beta/\kappa$. □

5 Inf–sup condition with a piecewise linear pressure space

In this section we will investigate the validity of the inf–sup condition in the case when the pressure is approximated by piecewise linear functions from the space

$$\widehat{\mathrm{Q}}_h = \left\{q_h\in L_0^2(\Omega);\ q_h|_K\in P_1(K)\ \ \forall\,K\in\mathcal{T}_h\right\}.$$

For simplicity, we will assume that

$$\int_{\widehat{K}}\widehat{b}_i\,d\widehat{x} = 0, \qquad i=1,2,3, \tag{24}$$

which is satisfied, e.g., for $\widehat{b}_i$ defined by (17). The property (24) implies that

$$\int_K \chi_E dx = 0 \qquad \forall\ E\in\mathcal{E}_h^i,\, K\in\mathcal{T}_h, \tag{25}$$

$$\int_K \psi_E dx = \frac{1}{3}|K| \qquad \forall\ E\in\mathcal{E}_h^i,\, K\in\mathcal{T}_h,\, E\subset\partial K. \tag{26}$$

The below proof of the inf–sup condition is related to the macroelement technique of [2] and [14].

We denote by $\{x_i\}_{i=1}^{N_h}$ the inner vertices of the triangulation $\mathcal{T}_h$ and, for any vertex x_i, we introduce a macroelement

$$\Delta_i = \bigcup_{K\in\mathcal{T}_h,\, x_i\in K} K$$

consisting of elements grouped around x_i. Further, we denote

$$\boldsymbol{V}_h^i = \{\boldsymbol{v}_h \in \boldsymbol{V}_h^{mod}\,;\quad \boldsymbol{v}_h = 0 \text{ in } \Omega\setminus\Delta_i\}\,,\quad i=1,\dots,N_h\,,$$

$$\widetilde{\mathrm{Q}}_h = \{q_h \in L_0^2(\Omega)\,;\ q_h|_K \in P_1(K)\cap L_0^2(K)\quad \forall\, K\in\mathcal{T}_h\}\,.$$

The following lemma shows that a local inf–sup condition holds on each macroelement.

Lemma 2. *Consider any $i\in\{1,\dots,N_h\}$. For any $\widetilde{q}_h\in\widetilde{\mathrm{Q}}_h$, there exists $\boldsymbol{v}_h^i\in\boldsymbol{V}_h^i$ such that*

$$b_h(\boldsymbol{v}_h^i,\bar{q}_h) = 0 \qquad \forall\ \bar{q}_h\in\overline{\mathrm{Q}}_h\,, \tag{27}$$

$$b_h(\boldsymbol{v}_h^i,\widetilde{q}_h) = \|\widetilde{q}_h\|_{0,\Delta_i}^2\,, \tag{28}$$

$$|\boldsymbol{v}_h^i|_{1,h} \le C_4\|\widetilde{q}_h\|_{0,\Delta_i}\,, \tag{29}$$

where $C_4 = 6\,C_3\,\sigma^{1/2}$.

Proof. Let Δ_i consist of elements $K_1,\dots,K_n$, i.e.,

$$\Delta_i = \bigcup_{j=1}^{n} K_j\,,$$

and let K_{j-1} and K_j have a common edge E_j, $j=1,\dots,n$, see Fig. 2. Here and in the following, the index 0 is considered as the index n and the index $n+1$ is considered as the index 1. We assume that the normal vectors $\boldsymbol{n}_{E_j}$ are pointed into K_j and we introduce tangent vectors $\boldsymbol{t}_{E_j}$ to E_j pointing from x_i to the other vertex of E_j. According to (11), (12) and (13), we have

$$\int_{E_j} \langle|\chi_{E_j}|\rangle_{E_j}(2\lambda_{E_j}-1)\,d\sigma = \frac{1}{3}h_{E_j}J_{E_j}(\chi_{E_j}) = \frac{1}{3}h_{E_j}$$

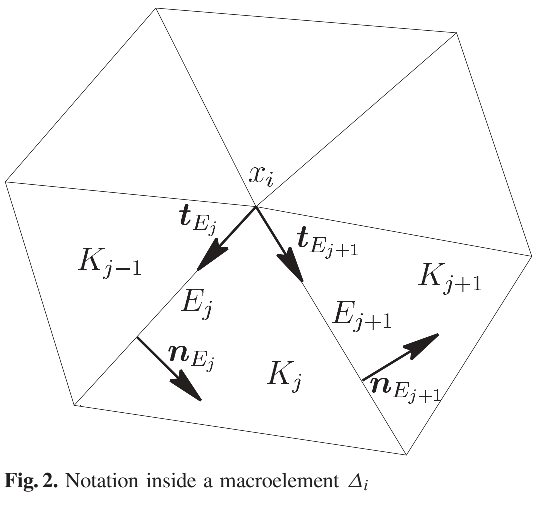

Fig. 2. Notation inside a macroelement Δ_i

and hence

$$\int_{E_j} \langle|\chi_{E_j}|\rangle_{E_j} q\,d\sigma = \gamma_j h_{E_j} \quad \text{for } q\in P_1(E_j)\,,$$

$$q(x_i)=1\,,\ q(C_{E_j})=0\,,\quad j=1,\dots,n\,, \tag{30}$$

where $\gamma_j=\pm 1/3$ and C_{E_j} is the midpoint of E_j. In addition, we will need the relations

$$\int_{E_j} \langle|\chi_{E_j}|\rangle_{E_j}\,d\sigma = 0\,,\qquad j=1,\dots,n\,, \tag{31}$$

$$\int_{K_j} \mathrm{div}(q\psi_{E_k}\boldsymbol{t}_{E_k})\,dx = 0 \qquad \forall\ q\in P_1(K_j)$$

$$j,k=1,\dots,n\,, \tag{32}$$

which easily follow from the definition of the functions ψ_E and χ_E.

Now, let us consider any $\widetilde{q}_h\in\widetilde{\mathrm{Q}}_h$. We will look for numbers α_k, β_k, $k=1,\dots,n$, such that

$$\boldsymbol{v}_h^i = \sum_{k=1}^{n}(\alpha_k\boldsymbol{n}_{E_k}\chi_{E_k} + \beta_k\boldsymbol{t}_{E_k}\psi_{E_k}) \tag{33}$$

satisfies

$$-\int_{K_j} q\,\mathrm{div}\boldsymbol{v}_h^i\,dx = \int_{K_j}\widetilde{q}_h q\,dx \tag{34}$$

for any $q\in P_1(K_j)$, $j=1,\dots,n$. Note that $\boldsymbol{v}_h^i\in\boldsymbol{V}_h^i$. Integrating by parts, applying (25) and (32), and using the fact that any function χ_E vanishes on all edges except E, we derive

$$\begin{aligned}
-&\int_{K_j} q\,\mathrm{div}\boldsymbol{v}_h^i\,dx\\
&= \nabla q\cdot\int_{K_j}\beta_j\boldsymbol{t}_{E_j}\psi_{E_j} + \beta_{j+1}\boldsymbol{t}_{E_{j+1}}\psi_{E_{j+1}}\,dx\\
&\quad + \int_{E_j} q\alpha_j\langle|\chi_{E_j}|\rangle_{E_j}\,d\sigma\\
&\quad - \int_{E_{j+1}} q\alpha_{j+1}\langle|\chi_{E_{j+1}}|\rangle_{E_{j+1}}\,d\sigma\,.
\end{aligned} \tag{35}$$

Owing to (31), this relation implies that

$$\int_{K_j} q\,\mathrm{div}\boldsymbol{v}_h^i\,dx = 0 \quad \forall\ q\in P_0(K_j)\,,\ j=1,\dots,n\,, \tag{36}$$

and hence (34) holds for any $q\in P_0(K_j)$, $j=1,\dots,n$. Therefore, it suffices to look for α_k, β_k, $k=1,\dots,n$, satisfying (34) for $q=\zeta_{E_j}|_{K_j}$ and $q=\zeta_{E_{j+1}}|_{K_j}$, $j=1,\dots,n$. It is easy to see that

$$\nabla\zeta_{E_j}|_{K_j} = -\frac{2}{s_j}\boldsymbol{n}_{E_j}\,,\qquad \nabla\zeta_{E_{j+1}}|_{K_j} = \frac{2}{z_j}\boldsymbol{n}_{E_{j+1}}\,,$$

where s_j is the distance between E_j and the vertex of K_j opposite to E_j, and z_j is the distance between E_{j+1} and the vertex of K_j opposite to E_{j+1}. Since

$$s_j = \boldsymbol{n}_{E_j} \cdot \boldsymbol{t}_{E_{j+1}} h_{E_{j+1}}\,, \qquad z_j = -\boldsymbol{n}_{E_{j+1}} \cdot \boldsymbol{t}_{E_j} h_{E_j}\,,$$

we have

$$\boldsymbol{t}_{E_{j+1}} \cdot \nabla\zeta_{E_j}|_{K_j} = -\frac{2}{h_{E_{j+1}}}\,, \quad \boldsymbol{t}_{E_j} \cdot \nabla\zeta_{E_{j+1}}|_{K_j} = -\frac{2}{h_{E_j}}\,.$$

Thus, substituting the functions $\zeta_{E_j}|_{K_j}$ and $\zeta_{E_{j+1}}|_{K_j}$ into (34) and applying (35), (26), (31) and (30), we derive for $j = 1, \ldots, n$

$$-\frac{2}{3}\frac{|K_j|}{h_{E_{j+1}}}\beta_{j+1} - \gamma_{j+1} h_{E_{j+1}}\alpha_{j+1} = \int_{K_j} \widetilde{q}_h \zeta_{E_j}\, dx\,,$$

$$-\frac{2}{3}\frac{|K_j|}{h_{E_j}}\beta_j \quad + \gamma_j h_{E_j}\alpha_j \qquad = \int_{K_j} \widetilde{q}_h \zeta_{E_{j+1}}\, dx\,.$$

This shows that the coefficients α_k, β_k in (33), for which (34) holds with any $q \in P_1(K_j)$, $j = 1, \ldots, n$, are uniquely determined by

$$\begin{aligned}\gamma_j h_{E_j}(|K_{j-1}| + |K_j|)\alpha_j &= |K_{j-1}| \int_{K_j} \widetilde{q}_h \zeta_{E_{j+1}}\, dx \\ &\quad - |K_j| \int_{K_{j-1}} \widetilde{q}_h \zeta_{E_{j-1}}\, dx\,, \\ 2(|K_{j-1}| + |K_j|)\beta_j &= -3h_{E_j} \int_{K_j} \widetilde{q}_h \zeta_{E_{j+1}}\, dx \\ &\quad - 3h_{E_j} \int_{K_{j-1}} \widetilde{q}_h \zeta_{E_{j-1}}\, dx\,,\end{aligned}$$

where $j = 1, \ldots, n$.

Summing up the relations (34) for $j = 1, \ldots, n$, we get

$$b_h(\boldsymbol{v}_h^i, q_h) = \int_{\Delta_i} \widetilde{q}_h q_h\, dx \qquad \forall\ q_h \in \widetilde{\mathrm{Q}}_h\,,$$

and setting $q_h = \widetilde{q}_h$, we obtain (28). The validity of (27) follows from (36) and hence it remains to prove (29). Using (5), we derive that, for $j = 1, \ldots, n$,

$$|\alpha_j| \le \sigma^{1/2}\|\widetilde{q}_h\|_{0,K_{j-1}\cup K_j}\,, \quad |\beta_j| \le \sigma^{1/2}\|\widetilde{q}_h\|_{0,K_{j-1}\cup K_j}\,.$$

Applying (16), we get for $j = 1, \ldots, n$

$$|\boldsymbol{v}_h^i|_{1,K_j} \le 2C_3\,\sigma^{1/2}(\|\widetilde{q}_h\|_{0,K_{j-1}\cup K_j} + \|\widetilde{q}_h\|_{0,K_j\cup K_{j+1}})$$

and the inequality (29) follows. □

Now we can prove the main result of this section. We will assume that the triangulations $\mathcal{T}_h$ possess the following property:

any element $K \in \mathcal{T}_h$ has at least one vertex in Ω. (37)

This additional assumption guarantees that any element of a triangulation $\mathcal{T}_h$ is contained in at least one macroelement.

Theorem 2. *Let the assumptions (24) and (37) hold. Then*

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h^{mod}\setminus\{\boldsymbol{0}\}} \frac{b_h(\boldsymbol{v}_h, q_h)}{|\boldsymbol{v}_h|_{1,h}} \ge \beta\|q_h\|_{0,\Omega} \qquad \forall\ q_h \in \widehat{\mathrm{Q}}_h\,, \tag{38}$$

where $\beta = \min\{2, \bar{\beta}^2\}/(12\,C_4 + 2\,\bar{\beta})$.

Proof. For completeness we give all details of the proof although the arguments are very similar to those ones used in the conforming case (cf. [2], [14] or [8]). Consider any $q_h \in \widehat{\mathrm{Q}}_h$. Then there exist functions $\widetilde{q}_h \in \widetilde{\mathrm{Q}}_h$ and $\bar{q}_h \in \overline{\mathrm{Q}}_h$ such that

$$q_h = \widetilde{q}_h + \bar{q}_h\,.$$

Since the spaces $\widetilde{\mathrm{Q}}_h$ and $\overline{\mathrm{Q}}_h$ are orthogonal subspaces of $L^2(\Omega)$, we have

$$\|q_h\|_{0,\Omega}^2 = \|\widetilde{q}_h\|_{0,\Omega}^2 + \|\bar{q}_h\|_{0,\Omega}^2\,.$$

According to [8, p. 58, Lemma 4.1], the inf–sup condition (21) implies that there exists a function $\bar{\boldsymbol{v}}_h \in \boldsymbol{V}_h^{mod}$ satisfying

$$\begin{aligned} b_h(\bar{\boldsymbol{v}}_h, \bar{q}_h) &= \|\bar{q}_h\|_{0,\Omega}^2\,, \\ |\bar{\boldsymbol{v}}_h|_{1,h} &\le \frac{1}{\bar{\beta}}\|\bar{q}_h\|_{0,\Omega}\,. \end{aligned}$$

Further, according to Lemma 2, there exist functions $\boldsymbol{v}_h^i \in \boldsymbol{V}_h^i$, $i = 1, \ldots, N_h$, satisfying (27)–(29) with the above function $\widetilde{q}_h$. Setting

$$\widetilde{\boldsymbol{v}}_h = \sum_{i=1}^{N_h} \boldsymbol{v}_h^i\,,$$

we have

$$\begin{aligned} b_h\left(\widetilde{\boldsymbol{v}}_h, \bar{q}_h\right) &= 0\,, \\ b_h\left(\widetilde{\boldsymbol{v}}_h, \widetilde{q}_h\right) &\ge \|\widetilde{q}_h\|_{0,\Omega}^2\,, \\ |\widetilde{\boldsymbol{v}}_h|_{1,h} &\le \sqrt{3\sum_{i=1}^{N_h} |\boldsymbol{v}_h^i|_{1,h}^2} \le 3C_4\|\widetilde{q}_h\|_{0,\Omega}\,. \end{aligned}$$

Denoting

$$\boldsymbol{v}_h = \widetilde{\boldsymbol{v}}_h + \alpha\bar{\boldsymbol{v}}_h\,, \qquad \alpha = \frac{\bar{\beta}^2}{2}\,,$$

we get

$$\begin{aligned} b_h(\boldsymbol{v}_h, q_h) &= b_h\left(\widetilde{\boldsymbol{v}}_h, \widetilde{q}_h\right) + \alpha b_h\left(\bar{\boldsymbol{v}}_h, \widetilde{q}_h\right) + \alpha b_h\left(\bar{\boldsymbol{v}}_h, \bar{q}_h\right) \\ &\ge \|\widetilde{q}_h\|_{0,\Omega}^2 + \alpha\|\bar{q}_h\|_{0,\Omega}^2 - \alpha\sqrt{2}|\bar{\boldsymbol{v}}_h|_{1,h}\|\widetilde{q}_h\|_{0,\Omega} \\ &\ge \frac{1}{2}\|\widetilde{q}_h\|_{0,\Omega}^2 + \alpha\left(1 - \frac{\alpha}{\bar{\beta}^2}\right)\|\bar{q}_h\|_{0,\Omega}^2 \\ &\ge \min\left\{\frac{1}{2}, \frac{\alpha}{2}\right\}\|q_h\|_{0,\Omega}^2\,. \end{aligned}$$

Since

$$|\boldsymbol{v}_h|_{1,h} \le |\widetilde{\boldsymbol{v}}_h|_{1,h} + \alpha|\bar{\boldsymbol{v}}_h|_{1,h} \le 3C_4\|\widetilde{q}_h\|_{0,\Omega} + \frac{\bar{\beta}}{2}\|\bar{q}_h\|_{0,\Omega} \le \left(3C_4 + \frac{\bar{\beta}}{2}\right)\|q_h\|_{0,\Omega}\,,$$

we obtain the theorem. □

Remark 1. We performed various numerical tests for the discretization (42) of the Stokes equations described below. If the pressure was approximated using the space $\widehat{\mathrm{Q}}_h$ and the assumption (37) was not satisfied, we often observed that the discrete pressure contained spurious oscillations. This indicates that the assumption (37) is necessary for the validity of (38).

6 Application of the P_1^{mod} element to the solution of the Stokes equations

An important role in both theoretical investigations and the numerical solution of the Navier–Stokes equations (1)–(3) is played by the linear Stokes equations

$$-\nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega\,, \tag{39}$$

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega\,, \tag{40}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega\,. \tag{41}$$

Here we consider the Stokes equations because they simplify the discussion of various choices of finite element spaces for approximating the velocity and the pressure. It is obvious that pairs of spaces which are detected as not suitable for discretizations of the Stokes equations cannot be expected to give satisfactory results in case of the Navier–Stokes equations.

Assuming that $\boldsymbol{f} \in L^2(\Omega)^2$ and denoting

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,dx\,, \quad b(\boldsymbol{v}, p) = -\int_\Omega p\operatorname{div}\boldsymbol{v}\,dx\,,$$

$$(\boldsymbol{f}, \boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx\,,$$

the usual weak formulation of (39)–(41) reads: Find $\boldsymbol{u} \in H_0^1(\Omega)^2$ and $p \in L_0^2(\Omega)$ such that

$$\nu a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) - b(\boldsymbol{u}, q) = (\boldsymbol{f}, \boldsymbol{v}) \qquad \forall\ \boldsymbol{v} \in H_0^1(\Omega)^2,\ q \in L_0^2(\Omega)\,.$$

It can be shown that this problem has a unique solution (cf. [8, p. 80, Theorem 5.1]).

We will approximate the space $H_0^1(\Omega)^2$ in the weak formulation by the space $\boldsymbol{V}_h^{mod}$ and the space $L_0^2(\Omega)$ by a general finite element space $\mathrm{Q}_h \subset L_0^2(\Omega)$ possessing the approximation property

$$\lim_{h\to 0}\inf_{q_h\in\mathrm{Q}_h}\|q - q_h\|_{0,\Omega} = 0 \qquad \forall\ q \in L_0^2(\Omega)\,.$$

Since the functions from $\boldsymbol{V}_h^{mod}$ are only piecewise in $H^1(\Omega)^2$, we have to replace the bilinear form a by its 'piecewise' counterpart

$$a_h(\boldsymbol{u}, \boldsymbol{v}) = \sum_{K\in\mathcal{T}_h}\int_K \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,dx$$

and the bilinear form b by the bilinear form b_h defined in Sect. 1. Then a finite element discretization of the Stokes equations (39)–(41) reads: Find $\boldsymbol{u}_h \in \boldsymbol{V}_h^{mod}$ and $p_h \in \mathrm{Q}_h$ such that

$$\nu a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) - b_h(\boldsymbol{u}_h, q_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \qquad \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h^{mod},\ q_h \in \mathrm{Q}_h\,. \tag{42}$$

Using the techniques of [5] and [8], we obtain the following result:

Theorem 3. *Let the spaces $\boldsymbol{V}_h^{mod}$ and Q_h satisfy the inf–sup condition (4) with $\beta > 0$ independent of h. Then the problem (42) has a unique solution $\boldsymbol{u}_h$, p_h and we have*

$$\lim_{h\to 0}\{|\boldsymbol{u} - \boldsymbol{u}_h|_{1,h} + \|p - p_h\|_{0,\Omega}\} = 0\,,$$

where $\boldsymbol{u}$, p is the weak solution of (39)–(41). Moreover, if $\boldsymbol{u} \in H^2(\Omega)^2$ and $p \in H^k(\Omega)$, $k \in \{1, 2, 3\}$, then

$$|\boldsymbol{u} - \boldsymbol{u}_h|_{1,h} \le Ch|\boldsymbol{u}|_{2,\Omega} + \frac{C}{\nu}h^k|p|_{k,\Omega} + \frac{\sqrt{2}}{\nu}\inf_{q_h\in\mathrm{Q}_h}\|p - q_h\|_{0,\Omega}\,,$$

$$\|p - p_h\|_{0,\Omega} \le \nu Ch|\boldsymbol{u}|_{2,\Omega} + Ch^k|p|_{k,\Omega} + C\inf_{q_h\in\mathrm{Q}_h}\|p - q_h\|_{0,\Omega}\,.$$

According to Theorem 1, the inf–sup condition mentioned in Theorem 3 holds if $\mathrm{Q}_h = \overline{\mathrm{Q}}_h$. It is well known (cf. e.g. [8, pp. 102 and 126]) that

$$\inf_{q_h\in\overline{\mathrm{Q}}_h}\|p - q_h\|_{0,\Omega} \le Ch|p|_{1,\Omega} \qquad \forall\ p \in H^1(\Omega)\cap L_0^2(\Omega)$$

and hence the choice $\mathrm{Q}_h = \overline{\mathrm{Q}}_h$ is optimal with respect to the convergence order of the discrete solution. However, since the parameter ν is small in typical applications, the pressure parts of the estimates of Theorem 3 will often dominate the velocity parts and the choice $\mathrm{Q}_h = \overline{\mathrm{Q}}_h$ will cause that the relation of these parts will not change significantly for $h \to 0$. Therefore, it may also be reasonable to require that

$$\inf_{q_h\in\mathrm{Q}_h}\|p - q_h\|_{0,\Omega} \le Ch^k|p|_{k,\Omega} \qquad \forall\ p \in H^k(\Omega)\cap L_0^2(\Omega),\ k = 1, 2\,.$$

This is satisfied, for instance, if

$$\mathrm{Q}_h = \mathrm{Q}_h^c \equiv \{q_h \in C(\overline{\Omega})\cap L_0^2(\Omega)\,;\ q_h|_K \in P_1(K)\ \ \forall\, K \in \mathcal{T}_h\}\,,$$

or if $\mathrm{Q}_h = \mathrm{Q}_h^{nc} \equiv \mathrm{V}_h^{nc}\cap L_0^2(\Omega)$, or if $\mathrm{Q}_h = \widehat{\mathrm{Q}}_h$. In all these cases, Theorem 2 assures the validity of the inf–sup condition required in Theorem 3, provided the assumptions of Theorem 2 are satisfied. Hence, if $\boldsymbol{u} \in H^2(\Omega)^2$ and $p \in H^2(\Omega)$, we have

$$|\boldsymbol{u} - \boldsymbol{u}_h|_{1,h} \le Ch|\boldsymbol{u}|_{2,\Omega} + \frac{C}{\nu} h^2 |p|_{2,\Omega} ,$$

$$\|p - p_h\|_{0,\Omega} \le \nu Ch|\boldsymbol{u}|_{2,\Omega} + Ch^2 |p|_{2,\Omega} .$$

Numerical experiments show that, in this way, we can often significantly improve the accuracy of the discrete solution in comparison to the choice $\mathrm{Q}_h = \overline{\mathrm{Q}}_h$.

Since $\dim \widehat{\mathrm{Q}}_h \approx 2 \dim \mathrm{Q}_h^{nc} \approx 6 \dim \mathrm{Q}_h^c$, it may seem at the first sight that the choice $\mathrm{Q}_h = \mathrm{Q}_h^c$ is the best one. However, for implementational reasons, the most attractive choice is the space Q_h^{nc} since this space can be implemented using the same data structures as we use for the space $\boldsymbol{V}_h^{mod}$. Finally, if we want to implement the discrete problem on a parallel computer, we realize that, from the point of view of local communication cost, the best choice is the space $\widehat{\mathrm{Q}}_h$ and the worst choice the space Q_h^c. Thus, each of the above pressure spaces $\overline{\mathrm{Q}}_h$, Q_h^c, Q_h^{nc} and $\widehat{\mathrm{Q}}_h$ may be appropriate in particular situations and the decision to use one of them can be influenced by various factors.

Remark 2. If we use the Crouzeix–Raviart space $[\mathrm{V}_h^{nc}]^2$ instead of the space $\boldsymbol{V}_h^{mod}$, then Theorem 3 holds with $k = 1$ only. The reason is that the consistency error related to the pressure is of order $O(h)$ since the space V_h^{nc} satisfies the patch test of order 1 only. The inf–sup condition is satisfied for $\mathrm{Q}_h = \overline{\mathrm{Q}}_h$ and hence, if $\boldsymbol{u} \in H^2(\Omega)^2$ and $p \in H^1(\Omega)$, we have the estimates

$$|\boldsymbol{u} - \boldsymbol{u}_h|_{1,h} \le Ch|\boldsymbol{u}|_{2,\Omega} + \frac{C}{\nu} h |p|_{1,\Omega} ,$$

$$\|p - p_h\|_{0,\Omega} \le \nu Ch|\boldsymbol{u}|_{2,\Omega} + Ch |p|_{1,\Omega} .$$

7 Numerical results

In this section we present numerical results obtained for the Stokes equations (39)–(41) in $\Omega = (0, 1)^2$ with $\nu = 10^{-4}$ and the exact solution $\boldsymbol{u} = (u_1, u_2)$ and p given by

$$\begin{aligned} u_1(x, y) &= \quad 100x^2(1-x)^2 y(1-y)(1-2y) , \\ u_2(x, y) &= -100y^2(1-y)^2 x(1-x)(1-2x) , \\ p(x, y) &= \quad x^3 + y^3 - \frac{1}{2} . \end{aligned}$$

The function u_1 (cf. Fig. 3) was used as the exact solution of a convection–diffusion equation in some of the numerical experiments in [12] to compare the properties of the P_1^{mod} element and the Crouzeix–Raviart element. The velocity $\boldsymbol{u}$ represents a vortex, see Fig. 5.

We discretized the Stokes equations as described in the preceding section and for approximating the velocity and the pressure we used the following pairs of spaces: P_1^{nc}/P_0, P_1^{mod}/P_0, P_1^{mod}/P_1^{disc}. Here, P_1^{nc} denotes the Crouzeix–Raviart element, P_0 denotes piecewise constant functions and P_1^{disc} denotes discontinuous piecewise linear functions. The triangulations were obtained by uniform refinements of the coarse triangulation depicted in Fig. 4. Thus, after k refinements, we obtain a triangulation $\mathcal{T}_h$ with $h = \sqrt{2}/2^{k+1}$.

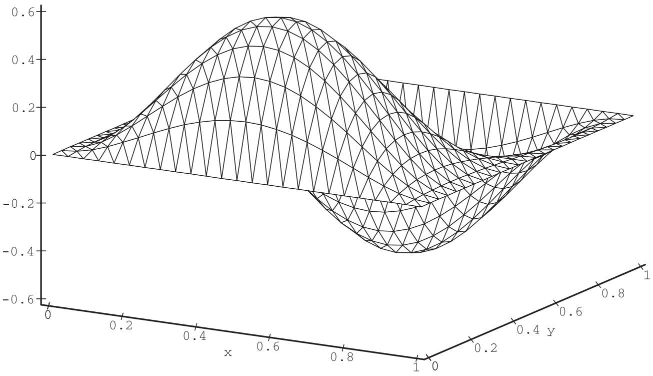

Fig. 3. Component u_1 of the velocity $\boldsymbol{u}$

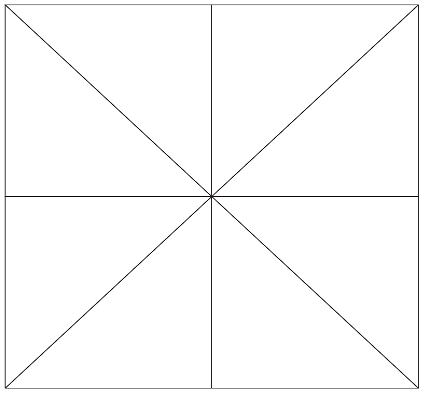

Fig. 4. Coarse triangulation

In Table 1 we present the errors of the discrete solutions computed for a triangulation with $h = \sqrt{2}/64$ (i.e., with 8192 elements). For the P_1^{mod} element, we give the errors of the piecewise linear part of the discrete velocity which is known to converge to $\boldsymbol{u}$ with the same convergence orders as the discrete velocity itself (cf. [12]). In agreement with the results of the preceding section, the discretization with the Crouzeix–Raviart element leads to largest errors and the discretization using the piecewise linear pressure is the most accurate one, in particular, the pressure is approximated very accurately.

In Figs. 6–8, we see the discrete velocity fields corresponding to the three discretizations. The P_1^{nc}/P_0 velocity is far away from the exact solution, the P_1^{mod}/P_0 velocity is significantly better but still with some discrepancies and the P_1^{mod}/P_1^{disc} solution perfectly agrees with the exact solution. In fact, the P_1^{mod}/P_1^{disc} discretization allows to obtain good solutions also on much coarser triangulations. In Fig. 9 we see the discrete velocity field computed for $h = \sqrt{2}/8$. Although the discrete velocity differs a little bit from the exact one, it is still very good from the qualitative point of view. Note that the number of elements corresponding to the used trian-

Table 1. Errors of the discrete solutions for $h = \sqrt{2}/64$

	P_1^{nc}/P_0	P_1^{mod}/P_0	P_1^{mod}/P_1^{disc}
$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}$	7.19×10^{-1}	1.27×10^{-1}	8.88×10^{-4}
$\lvert\boldsymbol{u} - \boldsymbol{u}_h\rvert_{1,h}$	9.36×10^{1}	1.72×10^{1}	1.03×10^{-1}
$\|p - p_h\|_{0,\Omega}$	7.67×10^{-3}	7.53×10^{-3}	4.32×10^{-5}

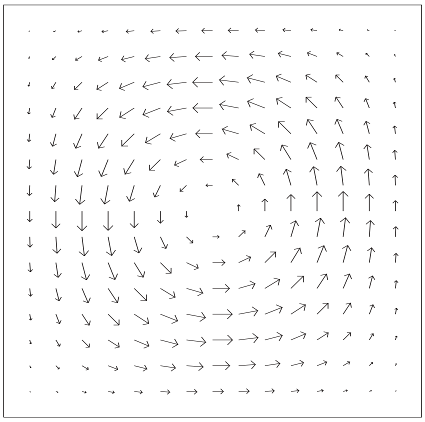

Fig. 5. Exact velocity $\boldsymbol{u}$

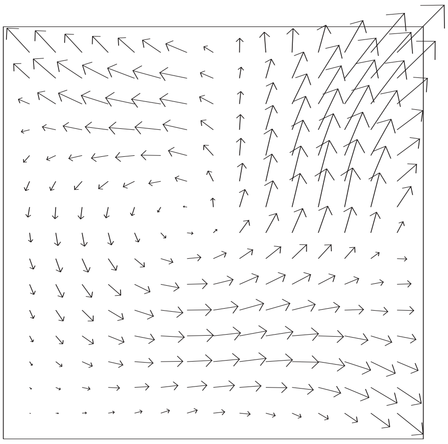

Fig. 6. Discrete velocity $\boldsymbol{u}_h$ for P_1^{nc}/P_0 and $h = \sqrt{2}/64$

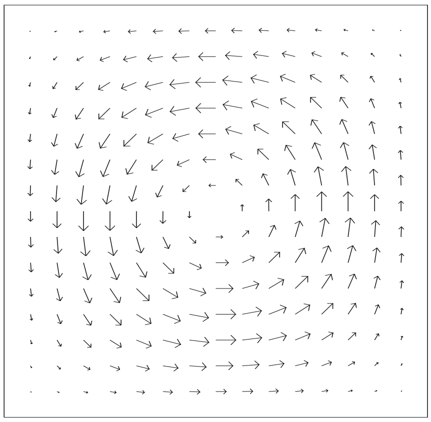

Fig. 8. Discrete velocity $\boldsymbol{u}_h$ for P_1^{mod}/P_1^{disc} and $h = \sqrt{2}/64$

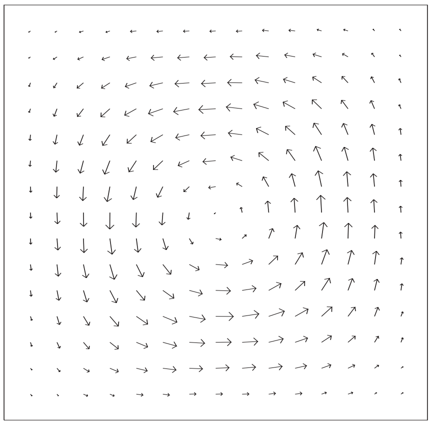

Fig. 9. Discrete velocity $\boldsymbol{u}_h$ for P_1^{mod}/P_1^{disc} and $h = \sqrt{2}/8$

gulation is 128, which is 64 times less than in the preceding computations! The errors of the discrete solution were still smaller than the errors given in Table 1 for the two other discretizations. For these discretizations, the solutions obtained on such a coarse mesh are completly wrong.

At first sight, it is surprising that the discrete velocity from Fig. 6 has a so different character from the exact velocity. However, the explanation is very easy. Let $\boldsymbol{V}_h$ be either the space $\boldsymbol{V}_h^{mod}$ or the space $[\mathrm{V}_h^{nc}]^2$ and let Q_h be one of the above pressure spaces such that the inf–sup condition (4) holds. Let the weak solution of the Stokes equations (39)–(41) satisfy $\boldsymbol{u} \in H^2(\Omega)^2$ and $p \in H^1(\Omega)$. We denote by $\boldsymbol{u}_h^u \in \boldsymbol{V}_h$, $p_h^u \in \mathrm{Q}_h$ the solution of

$$a_h\left(\boldsymbol{u}_h^u, \boldsymbol{v}_h\right) + b_h\left(\boldsymbol{v}_h, p_h^u\right) - b_h\left(\boldsymbol{u}_h^u, q_h\right) = -\left(\Delta \boldsymbol{u}, \boldsymbol{v}_h\right) \qquad \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h,\ q_h \in \mathrm{Q}_h\,,$$

where $(\cdot, \cdot)$ again denotes the $L^2(\Omega)^2$ inner product. Further, we denote by $\boldsymbol{u}_h^p \in \boldsymbol{V}_h$, $p_h^p \in \mathrm{Q}_h$ the solution of

$$a_h\left(\boldsymbol{u}_h^p, \boldsymbol{v}_h\right) + b_h\left(\boldsymbol{v}_h, p_h^p\right) - b_h\left(\boldsymbol{u}_h^p, q_h\right) = \left(\nabla p, \boldsymbol{v}_h\right) \qquad \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h,\ q_h \in \mathrm{Q}_h\,.$$

Then the discrete solution $\boldsymbol{u}_h \in \boldsymbol{V}_h$, $p_h \in \mathrm{Q}_h$ of the Stokes equations (39)–(41) satisfies

$$\boldsymbol{u}_h = \boldsymbol{u}_h^u + \frac{1}{\nu}\boldsymbol{u}_h^p\,, \qquad p_h = p_h^p + \nu p_h^u\,.$$

The functions $\boldsymbol{u}_h^u$, p_h^p tend to the exact soltuion $\boldsymbol{u}$, p and are independent of ν whereas the ν-dependent terms with $\boldsymbol{u}_h^p$ and p_h^u represent pure error terms which tend to zero for $h \to 0$. Note that $\boldsymbol{u}_h^p$ is independent of $\boldsymbol{u}$ but if ν is small, it may happen that the term with $\boldsymbol{u}_h^p$ will dominate the function $\boldsymbol{u}_h^u$ and the discrete velocity will be influenced stronger by p than by $\boldsymbol{u}$. This case can be observed in Fig. 6. On the other hand, in the case of the discretization P_1^{mod}/P_1^{disc}, the above theoretical results guarantee that $|\boldsymbol{u}_h^p|_{1,h} \le C\,h^2\,|p|_{2,\Omega}$ and hence the influence of small ν is supressed.

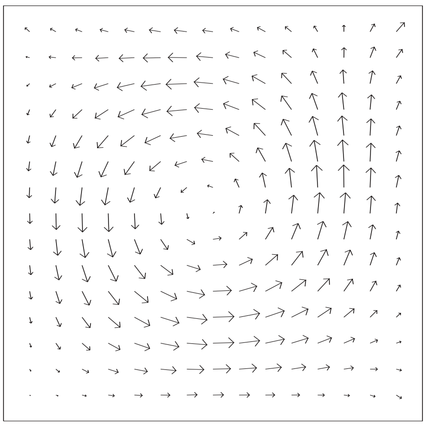

Fig. 7. Discrete velocity $\boldsymbol{u}_h$ for P_1^{mod}/P_0 and $h = \sqrt{2}/64$

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