Computing 76, 41–54 (2006)
Digital Object Identifier (DOI) 10.1007/s00607-005-0125-3

Computing
Printed in Austria

On the inf–sup Condition for the P_3^{mod}/P_2^{disc} Element

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Received July 28, 2004; revised February 27, 2005
Published online: July 15, 2005

Abstract

We consider a recently introduced triangular nonconforming finite element of third-order accuracy in the energy norm called P_3^{mod} element. We show that this finite element is appropriate for approximating the velocity in incompressible flow problems since it satisfies an inf-sup condition for discontinuous piecewise quadratic pressures.

AMS Subject Classifications: 65N30, 65N12, 76D05.

Keywords: Nonconforming finite-element methods, inf-sup condition, incompressible flow problems.

1. Introduction

The P_3^{mod} element considered in this paper is a triangular nonconforming finite element of third-order accuracy in the energy norm which belongs to the family of P_n^{mod} elements introduced by Knobloch and Tobiska [11], and by Knobloch [10]. The distinguished feature of these finite elements is that they satisfy a patch test of a higher order than standard nonconforming finite elements. Therefore, if they are applied to the numerical solution of a scalar convection-diffusion equation-discretized by means of the streamline diffusion method, the same optimal convergence results can be proved as in the conforming case (see [10], [11]) whereas standard nonconforming finite elements lead to a loss of accuracy in the convection-dominated regime.

Nonconforming finite elements are often used for approximating the velocity in incompressible flow problems (see, e.g., [7], [8], [13], [16]) and a natural question is whether the P_n^{mod} elements are also appropriate for such applications. Thus, for any $n \geq 1$, let $\mathbf{V}_h^{mod,n}$ be the velocity space defined using the P_n^{mod} element and approximating the space $H_0^1(\Omega)^2$, where $\Omega \subset \mathbb{R}^2$ is the computational domain under consideration. The pressure is often approximated by discontinuous piecewise polynomial functions since they lead to local mass conservation. As the space $\mathbf{V}_h^{mod,n}$ is of approximation order n in the energy norm, a suitable pressure space is the space

$$\mathrm{Q}_h^{n-1} = \{q_h \in L_0^2(\Omega)\,;\ q_h|_K \in P_{n-1}(K) \ \ \forall\, K \in \mathcal{T}_h\} \tag{1}$$

consisting of discontinuous piecewise polynomial functions of degree $n-1$ having zero mean value on Ω. The notation $\mathcal{T}_h$ denotes a triangulation of Ω consisting of triangular elements K used for constructing the space $\mathbf{V}_h^{mod,n}$. It is well known that finite element spaces for approximating the velocity and pressure in incompressible flow problems cannot be chosen arbitrarily if one wants to obtain a stable discretization with respect to $h \to 0$ and no stabilization of the continuity equation is used (see, e.g., Brezzi and Fortin [2], or Girault and Raviart [6] for details). A sufficient requirement on the spaces $\mathbf{V}_h^{mod,n}$ and Q_h^{n-1} is the validity of the inf-sup condition

$$\sup_{v_h \in \mathbf{V}_h^{mod,n} \setminus \{\mathbf{0}\}} \frac{b_h(v_h, q_h)}{|v_h|_{1,h}} \geq \beta \, \|q_h\|_{0,\Omega} \qquad \forall \, q_h \in \mathrm{Q}_h^{n-1}, \tag{2}$$

where $\beta > 0$ is independent of the discretization parameter h and

$$b_h(v_h, q_h) = -\sum_{K \in \mathcal{T}_h} \int_K q_h \mathrm{div} v_h \, \mathrm{d}x, \qquad |v_h|_{1,h} = \bigg(\sum_{K \in \mathcal{T}_h} |v_h|_{1,K}^2 \bigg)^{1/2}.$$

If the spaces $\mathbf{V}_h^{mod,n}$ and Q_h^{n-1} satisfy (2), then optimal error estimates for the discrete solution of the Stokes or Navier–Stokes equations can be proved.

It is the aim of this paper to fill in the last gap in the stability theory of the pairs of spaces $\mathbf{V}_h^{mod,n}$ and Q_h^{n-1} by proving the inf-sup condition (2) for $n = 3$. The inf-sup condition for $n \leq 2$ follows from the results of Knobloch [9] and, for $n \geq 4$, it follows from the results of Scott and Vogelius [14] provided that some special mesh constructions are avoided.

Let us mention the relation of the P_3^{mod} element to other triangular nonconforming finite elements of third-order accuracy satisfying the inf-sup condition for the space Q_h^2. The first of these finite elements was introduced by Crouzeix and Raviart [4] who elementwise enriched the space of cubic polynomials by two quartic bubbles. Later it was shown by Crouzeix and Falk [5] that these additional bubbles are not necessary to insure stability provided that the triangulation satisfies some geometrical assumptions. Another nonconforming finite element was recently introduced by Matthies and Tobiska [12], again by elementwise enriching the space of cubic polynomials by two quartic polynomials. Unfortunately, all these finite elements lead to the above mentioned loss of accuracy if applied to solving convection dominated convection-diffusion problems. Moreover, also if we do not consider the convection dominated case, the P_3^{mod} element is more advantageous since the space $\mathbf{V}_h^{mod,3}$ has less degrees of freedom than the spaces corresponding to the finite elements from [4] and [12] and no restrictive assumptions on the triangulation are made (in contrast with [5]).

The paper is organized in the following way. First, in Sect. 2, we introduce some assumptions and summarize the notation which will be used in the subsequent sections. Then, in Sect. 3, we recall the definition of the P_3^{mod} element. Finally, in Sect. 4, we prove the validity of the inf-sup condition (2) for the spaces $\mathbf{V}_h^{mod,3}$ and Q_h^2.

2. Assumptions and Notation

We assume that we are given a bounded domain $\Omega \subset \mathbb{R}^2$ with a polygonal boundary $\partial\Omega$ and a family $\{\mathcal{T}_h\}$ of triangulations of Ω. The triangulations are assumed to consist of closed triangular elements K, to possess the usual compatibility properties (see, e.g., Ciarlet [3]) and to satisfy $h_K \equiv \mathrm{diam}(K) \le h$ for any $K \in \mathcal{T}_h$. We assume that the family of triangulations is regular, i.e., there exists a constant σ independent of h such that

$$\frac{h_K}{\varrho_K} \le \sigma \qquad \forall\, K \in \mathcal{T}_h,\ h > 0\,, \tag{3}$$

where ϱ_K is the maximum diameter of circles inscribed into K. Finally, we assume that any element $K \in \mathcal{T}_h$ has at least one vertex in Ω.

We denote by $\mathcal{E}_h$ the set of edges E of $\mathcal{T}_h$ and by $\mathcal{E}_h^i$ the subset of $\mathcal{E}_h$ consisting of inner edges. Further, for any edge E, we denote by h_E the length of E, by $x_{E,1}, x_{E,2}$ the end points of E and by $\lambda_{E,1}, \lambda_{E,2}$ the linear functions on E satisfying $\lambda_{E,i}(x_{E,j}) = \delta_{ij}$, $i, j = 1, 2$, where δ_{ij} denotes the Kronecker symbol. For any edge E, we denote by $\boldsymbol{t}_E = (t_{E1}, t_{E2})$ the unit tangent vector to E which points from $x_{E,1}$ to $x_{E,2}$ and by $\boldsymbol{n}_E \equiv (-t_{E2}, t_{E1})$ a normal vector to E. For any inner edge $E \in \mathcal{E}_h^i$, we define the jump of a function v across E by

$$[|v|]_E = (v|_K)|_E - (v|_{\widetilde{K}})|_E\,,$$

where $K, \widetilde{K}$ are the two elements adjacent to E denoted in such a way that $\boldsymbol{n}_E$ points into $\widetilde{K}$. If an edge $E \in \mathcal{E}_h$ lies on the boundary of Ω, then we set $[|v|]_E = v|_E$.

Throughout the paper we use standard notation $P_k(\Omega)$, $L^2(\Omega)$, $H^k(\Omega) = W^{k,2}(\Omega)$ etc. for the usual function spaces, see, e.g., Ciarlet [3]. We only mention that we denote by $L_0^2(\Omega)$ the space of functions from $L^2(\Omega)$ having zero mean value on Ω. The norm and seminorm in the Sobolev space $H^k(\Omega)$ will be denoted by $\|\cdot\|_{k,\Omega}$ and $|\cdot|_{k,\Omega}$, respectively. Finally, we use the notation $|G|$ to denote the two-dimensional Lebesgue measure of a set $G \subset \mathbb{R}^2$.

3. Definition of the P_3^{mod} Element and the Respective Finite-element Space

The space of P_3^{mod} shape functions on the standard reference triangle $\widehat{K}$ is given by

$$P_3^{mod}(\widehat{K}) = P_3(\widehat{K}) \oplus \mathrm{span}\{\widehat{b}_1, \widehat{b}_2, \widehat{b}_3\}\,,$$

where $\widehat{b}_1$, $\widehat{b}_2$ and $\widehat{b}_3$ are functions on $\widehat{K}$ associated respectively with the edges $\widehat{E}_1$, $\widehat{E}_2$ and $\widehat{E}_3$ of $\widehat{K}$. According to Knobloch [10], we assume that

$$\widehat{b}_1 \in H^1(\widehat{K})\,, \qquad\qquad \widehat{b}_1|_{\partial\widehat{K}\setminus\widehat{E}_1} = 0\,, \tag{4}$$

$$\widehat{b}_1|_{\widehat{E}_1} \text{ is odd with respect to the midpoint of } \widehat{E}_1, \tag{5}$$

$$\int_{\widehat{E}_1} [(1 - 2\,\widehat{\lambda}_2) + \widehat{b}_1]\,\widehat{q}\,\mathrm{d}\widehat{\sigma} = 0 \qquad \forall\,\widehat{q} \in P_k(\widehat{E}_1)\,, \tag{6}$$

where $k \geq 3$ and $\widehat{\lambda}_2$ is the barycentric coordinate on $\widehat{K}$ with respect to the vertex $\widehat{x}_2$ which is the vertex of $\widehat{K}$ opposite the edge $\widehat{E}_2$ (the remaining vertices $\widehat{x}_1$ and $\widehat{x}_3$ and the barycentric coordinates $\widehat{\lambda}_1$ and $\widehat{\lambda}_3$ are defined analogously). The functions $\widehat{b}_2$ and $\widehat{b}_3$ are simply defined by affine transformations of $\widehat{b}_1$, i.e., $\widehat{b}_i = \widehat{b}_1 \circ \widehat{F}_i$, $i = 2, 3$, where $\widehat{F}_2$ and $\widehat{F}_3$ are affine regular mappings on $\mathbb{R}^2$ such that $\widehat{F}_i(\widehat{K}) = \widehat{K}$, $\widehat{F}_i(\widehat{E}_i) = \widehat{E}_1$, $i = 2, 3$.

Because of the proof of the inf-sup condition, we further assume that, for any function $g : \mathbb{R}^2 \to \mathbb{R}$ such that $g(x, y) = g(y, x)$ for all $x, y \in \mathbb{R}$, the function $\widehat{b}_1$ satisfies

$$\int_{\widehat{K}} \widehat{b}_1\, g(\widehat{\lambda}_2, \widehat{\lambda}_3)\, \mathrm{d}\widehat{x} = 0\,. \tag{7}$$

Moreover, denoting

$$A = \frac{1}{|\widehat{K}|} \int_{\widehat{K}} \widehat{b}_1\, \widehat{\lambda}_2\, \mathrm{d}\widehat{x}\,, \tag{8}$$

we assume that

$$A \neq \frac{1}{18}\,, \qquad\qquad A \neq \frac{1}{45}\,. \tag{9}$$

The value of A should not be too close to $1/18$ or $1/45$ since then the inf-sup constant β obtained in this paper is close to zero. Examples of the function $\widehat{b}_1$ satisfying all the above assumptions are

$$\begin{aligned}
\widehat{b}_1 &= (28\,\widehat{\lambda}_2\,\widehat{\lambda}_3 - 126\,\widehat{\lambda}_2^2\,\widehat{\lambda}_3^2)(\widehat{\lambda}_2 - \widehat{\lambda}_3) \qquad (\Rightarrow k = 4,\ \ A = 29/360)\,,\\
\widehat{b}_1 &= (54\,\widehat{\lambda}_2\,\widehat{\lambda}_3 - 594\,\widehat{\lambda}_2^2\,\widehat{\lambda}_3^2 + 1716\,\widehat{\lambda}_2^3\,\widehat{\lambda}_3^3)(\widehat{\lambda}_2 - \widehat{\lambda}_3)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\Rightarrow k = 6,\ \ A = 347/4200)\,.
\end{aligned}$$

For any element $K \in \mathcal{T}_h$, we introduce a regular affine mapping $F_K : \widehat{K} \to K$ such that $F_K(\widehat{K}) = K$ and we define the P_3^{mod} finite-element space by

$$\begin{aligned}
\mathrm{V}_h^{mod,3} = \{v_h \in L^2(\Omega)\,;\ v_h \circ F_K \in P_3^{mod}(\widehat{K}) \ \ \forall\ K \in \mathcal{T}_h\,,\\
\int_E [|v_h|]_E\, q\, \mathrm{d}\sigma = 0 \ \ \forall\ q \in P_3(E),\ E \in \mathcal{E}_h\}\,.
\end{aligned}$$

The above assumptions imply that (see Knobloch [10] for a proof)

$$\int_E [|v_h|]_E\, q\, \mathrm{d}\sigma = 0 \qquad\quad \forall\ v_h \in \mathrm{V}_h^{mod,3},\ q \in P_k(E),\ E \in \mathcal{E}_h\,, \tag{10}$$

where k is the integer introduced in (6). Thus, choosing the function $\widehat{b}_1$ in a suitable way, we can enforce the validity of (10) with an arbitrarily high k.

To describe a basis of the space $V_h^{mod,3}$, let us first introduce some notation. Consider any $K \in \mathcal{T}_h$ and any $E \in \mathcal{E}_h$ such that $E \subset \partial K$. Let $i \in \{1, 2, 3\}$ be such that $E = F_K(\widehat{E}_i)$. Then we set

$$b_{K,E} = \begin{cases} \pm \widehat{b}_i \circ F_K^{-1} & \text{in } K, \\ 0 & \text{in } \Omega \setminus K, \end{cases}$$

where the sign is chosen in such a way that

$$\int_E (b_{K,E}|_K)\, \lambda_{E,1}\, \mathrm{d}\sigma > 0\,. \tag{11}$$

Further, for any edge $E \in \mathcal{E}_h$, we denote by ζ_E the standard nonconforming piecewise linear basis function associated with E, i.e., ζ_E is piecewise linear, equals 1 on E and vanishes at the midpoints of all edges different from E.

Now consider any inner edge $E \in \mathcal{E}_h^i$ and let us denote by K, $\widetilde{K}$ the two elements adjacent to E, by E, E_1, E_2 the edges of K, and by E, E_3, E_4 the edges of $\widetilde{K}$. Then we define functions ψ_E, χ_E by

$$\psi_E = \zeta_E + \beta_{E,1}\, b_{K,E_1} + \beta_{E,2}\, b_{K,E_2} + \beta_{E,3}\, b_{\widetilde{K},E_3} + \beta_{E,4}\, b_{\widetilde{K},E_4}\,, \tag{12}$$

$$\chi_E = \begin{cases} b_{K,E} & \text{in } K, \\ b_{\widetilde{K},E} & \text{in } \Omega \setminus K, \end{cases} \tag{13}$$

where $\beta_{E,i} = -1$ if $x_{E_i,1} \in E$ and $\beta_{E,i} = 1$ if $x_{E_i,1} \notin E$, $i = 1, \ldots, 4$. It is easy to verify that $\psi_E \in V_h^{mod,3}$ and $\chi_E \in V_h^{mod,3} \cap H_0^1(\Omega)$. Further, for any inner edge $E \in \mathcal{E}_h^i$, we introduce functions $\varrho_E, \varphi_E \in V_h^{mod,3}$ vanishing outside the two elements adjacent to E and defined by

$$\varrho_E|_K = \lambda_1\, \lambda_2\,, \qquad \varphi_E|_K = \lambda_1\, \lambda_2\, (\lambda_1 - \lambda_2)$$

for any element K adjacent to E, where λ_1, λ_2 are the barycentric coordinates on K with respect to $x_{E,1}$, $x_{E,2}$, respectively. Finally, for any element $K \in \mathcal{T}_h$, we define the bubble function π_K by

$$\pi_K|_K = \lambda_1\, \lambda_2\, \lambda_3\,, \qquad \pi_K|_{\Omega\setminus K} = 0\,,$$

where $\lambda_1, \lambda_2, \lambda_3$ are the barycentric coordinates on K. Then

$$V_h^{mod,3} = \operatorname{span}\Big\{\{\psi_E, \chi_E, \varrho_E, \varphi_E\}_{E\in\mathcal{E}_h^i} \cup \{\pi_K\}_{K\in\mathcal{T}_h}\Big\}\,.$$

Consequently, the functions from $V_h^{mod,3}$ are determined by four degrees of freedom on each inner edge E, e.g., by the moments

$$\frac{1}{h_E} \int_E v\, \lambda_{E,1}^j\, \mathrm{d}\sigma\,, \qquad j = 0, 1, 2, 3\,,$$

and by one degree of freedom on each element $K \in \mathcal{T}_h$, e.g., by the mean value over K. This is illustrated by Fig. 1. Combining these degrees of freedom, we can also

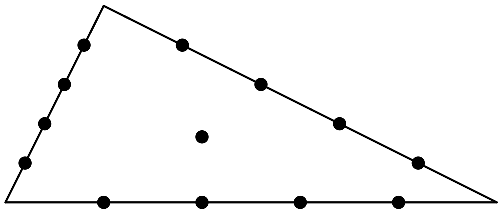

Fig. 1. Degrees of freedom of the P_3^{mod} element (see the text for details)

define degrees of freedom which are dual to the basis functions of $\mathrm{V}_h^{mod,3}$ introduced above. Such degrees of freedom are convenient for implementing the P_3^{mod} element. Since all the basis functions can be obtained by affine transformations of functions defined on the reference element $\widehat{K}$, we have (cf. Ciarlet [3, Sect. 15])

$$|\psi_E|_{1,h} + |\chi_E|_{1,\Omega} + |\varrho_E|_{1,\Omega} + |\varphi_E|_{1,\Omega} \leq C\,, \qquad |\pi_K|_{1,\Omega} \leq C \tag{14}$$

for any $E \in \mathcal{E}_h^i$ and $K \in \mathcal{T}_h$, where the constant C depends only on σ and $\widehat{b}_1$.

4. Inf-sup Condition for the P_3^{mod}/P_2^{disc} Element

In this section, we prove that the spaces $\mathbf{V}_h^{mod,3} \equiv [\mathrm{V}_h^{mod,3}]^2$ and Q_h^2 (cf. (1)) satisfy the inf-sup condition (2). The proof of the inf–sup condition will be related to the macroelement technique of Boland and Nicolaides [1], and Stenberg [15] which was extended to the nonconforming case by Crouzeix and Falk [5], and Knobloch [9].

We denote by $\{x_i\}_{i=1}^{N_h}$ the inner vertices of the triangulation $\mathcal{T}_h$ and, for any vertex x_i, we introduce a macroelement

$$\Delta_i = \bigcup_{K \in \mathcal{T}_h,\, x_i \in K} K$$

consisting of elements grouped around x_i. Further, we denote

$$\begin{aligned}
\mathbf{V}_h^{i,3} &= \{\boldsymbol{v}_h \in \mathbf{V}_h^{mod,3}\,;\ \boldsymbol{v}_h = \mathbf{0} \text{ in } \Omega \setminus \Delta_i\}\,, \qquad i = 1, \dots, N_h\,,\\
\widetilde{\mathrm{Q}}_h^2 &= \{q_h \in L_0^2(\Omega)\,;\ q_h|_K \in P_2(K) \cap L_0^2(K) \ \ \forall\, K \in \mathcal{T}_h\}\,,\\
\overline{\mathrm{Q}}_h &= \{q_h \in L_0^2(\Omega)\,;\ q_h|_K \in P_0(K) \ \ \forall\, K \in \mathcal{T}_h\}\,.
\end{aligned}$$

The following theorem shows that, for proving the inf-sup condition for the spaces $\mathbf{V}_h^{mod,3}$ and Q_h^2, it suffices to verify that they satisfy certain local conditions on the macroelements Δ_i.

Theorem 1: *For any $\widetilde{q}_h \in \widetilde{\mathrm{Q}}_h^2$ and any $i \in \{1, \dots, N_h\}$, let there exists $\boldsymbol{v}_h^i \in \mathbf{V}_h^{i,3}$ satisfying*

$$b_h(\boldsymbol{v}_h^i, \bar{q}_h) = 0 \qquad \forall\, \bar{q}_h \in \overline{\mathrm{Q}}_h\,, \tag{15}$$

$$b_h(\boldsymbol{v}_h^i, \widetilde{q}_h) = \|\widetilde{q}_h\|_{0,\Delta_i}^2\,, \tag{16}$$

$$|\boldsymbol{v}_h^i|_{1,h} \leq \overline{C}\, \|\widetilde{q}_h\|_{0,\Delta_i}\,, \tag{17}$$

where $\overline{C}$ is a constant independent of h. Then there exists a constant $\beta > 0$ depending only on $\overline{C}$, σ, $\widehat{b}_1$ and Ω such that

$$\sup_{v_h \in \mathbf{V}_h^{mod,3} \setminus \{\mathbf{0}\}} \frac{b_h(\boldsymbol{v}_h, q_h)}{|\boldsymbol{v}_h|_{1,h}} \geq \beta \, \|q_h\|_{0,\Omega} \qquad \forall \, q_h \in \mathrm{Q}_h^2 \, . \tag{18}$$

Proof: See the proof of Theorem 2 in Knobloch [9]. □

Remark 1: The assumption that any element $K \in \mathcal{T}_h$ has at least one vertex in Ω (cf. Sect. 2) assures that any element $K \in \mathcal{T}_h$ is contained in at least one macroelement, which is crucial for the proof of Theorem 1.

To verify the validity of the assumptions of Theorem 1, it is convenient to replace the basis functions χ_E, ϱ_E, φ_E and π_K introduced in Sect. 3 by functions having some special properties. Consider any $E \in \mathcal{E}_h^i$ and let K, $\widetilde{K}$ be the two elements adjacent to E denoted in such a way that $\boldsymbol{n}_E$ points into $\widetilde{K}$. Then we set

$$\overline{\chi}_E = \chi_E - 180\, A\, \varphi_E \, ,$$

$$\overline{\varrho}_E = 6\, \varrho_E - \psi_E - \begin{cases} 10\, \pi_K & \text{in } K, \\ 10\, \pi_{\widetilde{K}} & \text{in } \widetilde{K}, \\ 0 & \text{in } \Omega \setminus \{K \cup \widetilde{K}\}, \end{cases}$$

$$\overline{\varphi}_E = \chi_E - 10\, \varphi_E \, ,$$

$$\pi_E = (180\, A - 10)\, h_E^2 \left(\frac{\pi_K}{|K|} - \frac{\pi_{\widetilde{K}}}{|\widetilde{K}|} \right).$$

Some properties of these functions are summarized in the following two lemmas.

Lemma 1: *For any $E \in \mathcal{E}_h^i$, the functions $\overline{\chi}_E$, $\overline{\varrho}_E$, $\overline{\varphi}_E$ and π_E satisfy*

$$\int_E \overline{\chi}_E \, \mathrm{d}\sigma = \int_E \overline{\varrho}_E \, q \, \mathrm{d}\sigma = \int_E \overline{\varphi}_E \, q \, \mathrm{d}\sigma = 0 \qquad \forall \, q \in P_1(E) \, , \tag{19}$$

$$\int_K \overline{\chi}_E \, q \, \mathrm{d}x = \int_K \overline{\varrho}_E \, \mathrm{d}x = \int_K \overline{\varphi}_E \, \mathrm{d}x = 0 \qquad \forall \, q \in P_1(K), \; K \in \mathcal{T}_h \, , \tag{20}$$

$$b_h(\boldsymbol{\alpha}\, \overline{\varrho}_E, q_h) = b_h(\boldsymbol{\alpha}\, \overline{\varphi}_E, q_h) = 0 \qquad \forall \, \boldsymbol{\alpha} \in \mathbb{R}^2, \; q_h \in \mathrm{Q}_h^1 \, , \tag{21}$$

$$b_h(\overline{\chi}_E \, \boldsymbol{n}_E + \pi_E \, \boldsymbol{t}_E, q_h) = 0 \qquad \forall \, q_h \in \mathrm{Q}_h^1 \, . \tag{22}$$

Proof: Consider any $E \in \mathcal{E}_h^i$. Then

$$\int_E \chi_E \, \lambda_{E,1} \, \mathrm{d}\sigma = \frac{h_E}{6} \, , \qquad \int_E \varphi_E \, \lambda_{E,1} \, \mathrm{d}\sigma = \frac{h_E}{60} \, , \tag{23}$$

where the first equality follows from (6), (11) and (13). Further, we know that $\overline{\varrho}_E|_E = 6\, \lambda_{E,1}\, \lambda_{E,2} - 1$ and $\overline{\chi}_E|_E$, $\overline{\varphi}_E|_E$ are odd. Thus, it is easy to see that (19) holds.

Let us consider any $K \in \mathcal{T}_h$ adjacent to E. In view of (7), it is easy to verify that (20) holds for $\overline{\varrho}_E$, $\overline{\varphi}_E$ and $\overline{\chi}_E$ with $q = 1$ and $q = \zeta_E|_K$, where ζ_E is the nonconforming piecewise linear basis function defined in Sect. 3. Let λ_1 be the barycentric coordinate on K with respect to $x_{E,1}$. Then, according to (11), (7) and (8), we have

$$\int_K \chi_E \, \lambda_1 \, \mathrm{d}x = A \, |K| \tag{24}$$

and since $\int_K \varphi_E \, \lambda_1 \, \mathrm{d}x = |K|/180$, we deduce that $\int_K \overline{\chi}_E \, \lambda_1 \, \mathrm{d}x = 0$. This completes the proof of (20).

The proof of (21) and (22) is based on the fact that, due to the Gauss integral theorem,

$$b_h(\boldsymbol{v}, q_h) = \sum_{K \in \mathcal{T}_h} \left(\nabla q_h|_K \cdot \int_K \boldsymbol{v} \, \mathrm{d}x - \int_{\partial K} (q_h \, \boldsymbol{v})|_K \cdot \boldsymbol{n}_{\partial K} \, \mathrm{d}\sigma \right) \quad \forall \, q_h \in \mathrm{Q}_h^1$$

for any piecewise H^1 vector field $\boldsymbol{v}$ ($\boldsymbol{n}_{\partial K}$ denotes the unit outer normal vector to the boundary of K). Thus, the validity of (21) immediately follows from (10), (19) and (20). To prove (22), we have to show that, for any $K \in \mathcal{T}_h$ adjacent to E and for any $q \in P_1(K)$,

$$\begin{aligned} &\nabla q \cdot \int_K \pi_E \, \boldsymbol{t}_E \, \mathrm{d}x - \int_E q \, \overline{\chi}_E \, \boldsymbol{n}_E \cdot \boldsymbol{n}_{\partial K} \, \mathrm{d}\sigma \\ &\quad = \left[\frac{1}{6} \, (18A - 1) \, h_E^2 \, \boldsymbol{t}_E \cdot \nabla q - \int_E q \, \overline{\chi}_E \, \mathrm{d}\sigma \right] \boldsymbol{n}_E \cdot \boldsymbol{n}_{\partial K}|_E = 0 \, . \end{aligned} \tag{25}$$

Since $\overline{\chi}_E|_E$ is odd, this is obvious for $q = 1$ and $q = \zeta_E|_K$. Denoting by E_1 the edge of K opposite $x_{E,1}$ and setting $q = \zeta_{E_1}|_K$, it follows from (23) that $\int_E \chi_E \, q \, \mathrm{d}\sigma = -h_E/3$ and $\int_E \varphi_E \, q \, \mathrm{d}\sigma = -h_E/30$. Further, $\boldsymbol{t}_E \cdot \nabla q = 2/h_E$ and hence we deduce that the term in the square brackets in (25) vanishes. Consequently, (25) holds for any $q \in P_1(K)$. □

Lemma 2: *Consider any $E \in \mathcal{E}_h^i$ and let $K \in \mathcal{T}_h$ be an element adjacent to E. Let E_1 and E_2 be the remaining two edges of K opposite $x_{E,1}$ and $x_{E,2}$, respectively. Then*

$$\int_K \overline{\varrho}_E \, \zeta_E \, \mathrm{d}x = \frac{4}{45} \, (45 \, A - 1) \, |K| \, , \tag{26}$$

$$\int_K \overline{\varrho}_E \, \zeta_{E_1} \, \mathrm{d}x = \int_K \overline{\varrho}_E \, \zeta_{E_2} \, \mathrm{d}x = \frac{2}{45} \, (1 - 45 \, A) \, |K| \, , \tag{27}$$

$$\int_E \overline{\varrho}_E \, \zeta_{E_1}^2 \, \mathrm{d}\sigma = \int_E \overline{\varrho}_E \, \zeta_{E_2}^2 \, \mathrm{d}\sigma = -\frac{2}{15} \, h_E \, , \tag{28}$$

$$\int_K \overline{\varphi}_E \, (\zeta_{E_1} - \zeta_{E_2}) \, \mathrm{d}x = \frac{2}{9} \, (1 - 18 \, A) \, |K| \, , \tag{29}$$

$$\int_K \pi_K \, \zeta_E \, \mathrm{d}x = \int_K \pi_K \, \zeta_{E_1} \, \mathrm{d}x = \int_K \pi_K \, \zeta_{E_2} \, \mathrm{d}x = \frac{|K|}{180} \, , \tag{30}$$

where ζ_E, ζ_{E_1} and ζ_{E_2} are the nonconforming piecewise linear basis functions associated with the edges E, E_1 and E_2, respectively, introduced in Sect. 3.

Proof: A straightforward computation gives (28), (30) and, using (7) and (24),

$$\int_K \psi_E\,\zeta_E\,\mathrm{d}x = \frac{1}{3}\,(1-12\,A)\,|K|\,,$$
$$\int_K \psi_E\,\zeta_{E_1}\,\mathrm{d}x = \int_K \psi_E\,\zeta_{E_2}\,\mathrm{d}x = 2\,A\,|K|\,,$$
$$\int_K \varrho_E\,\zeta_E\,\mathrm{d}x = \frac{|K|}{20}\,, \qquad \int_K \varrho_E\,\zeta_{E_1}\,\mathrm{d}x = \int_K \varrho_E\,\zeta_{E_2}\,\mathrm{d}x = \frac{|K|}{60}\,,$$
$$\int_K \chi_E\,\zeta_{E_2}\,\mathrm{d}x = -\int_K \chi_E\,\zeta_{E_1}\,\mathrm{d}x = 2\,A\,|K|\,,$$
$$\int_K \varphi_E\,\zeta_{E_2}\,\mathrm{d}x = -\int_K \varphi_E\,\zeta_{E_1}\,\mathrm{d}x = \frac{|K|}{90}\,.$$

Combining these relations, we obtain (26), (27) and (29). □

Now we can prove the main result of this paper.

Theorem 2: *There exists a constant $\beta > 0$ depending only on σ, $\widehat{b}_1$ and Ω such that the inf-sup condition (18) holds.*

Proof: Consider any $i \in \{1,\dots,N_h\}$ and let the macroelement Δ_i consist of elements $K_1,\dots,K_n$, i.e.,

$$\Delta_i = \bigcup_{j=1}^{n} K_j\,,$$

and let K_{j-1} and K_j have a common edge E_j, $j = 1,\dots,n$, see Fig. 2. Here and in the following, the index 0 is considered as the index n and the index $n+1$ is considered as the index 1. Without loss of generality, we may assume that $x_{E_j,1} = x_i$

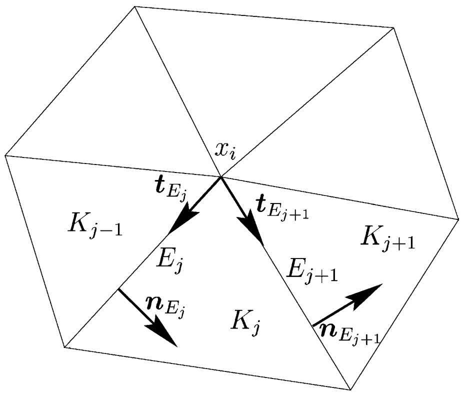

Fig. 2. Notation inside a macroelement Δ_i

for $j = 1, \ldots, n$. Then, for any $j \in \{1, \ldots, n\}$, the normal vector $\boldsymbol{n}_{E_j}$ points into K_j and the tangent vector $\boldsymbol{t}_{E_j}$ points from x_i to the other vertex of E_j (cf. Fig. 2).

Now, let us consider any function $\widetilde{q}_h \in \widetilde{\mathrm{Q}}_h^2$. We shall show that there exist a function $\boldsymbol{v}_h^i \in \mathbf{V}_h^{i,3}$ and a constant $\overline{C}$ (depending only on σ and $\widehat{b}_1$) which satisfy (15), (17) and

$$b_j(\boldsymbol{v}_h^i, q) = (\widetilde{q}_h, q)_{K_j} \qquad \forall\, q \in P_2(K_j),\ j = 1, \ldots, n\,, \tag{31}$$

where

$$b_j(\boldsymbol{v}_h^i, q) = -\int_{K_j} q \operatorname{div} \boldsymbol{v}_h^i \,\mathrm{d}x\,, \qquad (\widetilde{q}_h, q)_{K_j} = \int_{K_j} \widetilde{q}_h\, q \,\mathrm{d}x\,.$$

Setting $q = \widetilde{q}_h|_{K_j}$ in (31) and summing up over $j = 1, \ldots, n$, we then obtain (16). According to Theorem 1, this will prove the inf-sup condition (18).

It was shown in the proof of Lemma 2 in Knobloch [9] that there exists a uniquely determined function $\boldsymbol{v}_h^{i,1} \in \operatorname{span}\{\chi_{E_j}\, \boldsymbol{n}_{E_j}, \psi_{E_j}\, \boldsymbol{t}_{E_j}\}_{j=1}^n$ such that

$$b_j(\boldsymbol{v}_h^{i,1}, q) = (\widetilde{q}_h, q)_{K_j} \qquad \forall\, q \in P_1(K_j),\ j = 1, \ldots, n\,, \tag{32}$$

$$|\boldsymbol{v}_h^{i,1}|_{1,h} \le C\, \|\widetilde{q}_h\|_{0,\Delta_i}\,, \tag{33}$$

with C depending only on σ and $\widehat{b}_1$. Obviously,

$$b_h(\boldsymbol{v}_h^{i,1}, \bar{q}_h) = 0 \qquad \forall\, \bar{q}_h \in \overline{\mathrm{Q}}_h\,. \tag{34}$$

Let us denote

$$\boldsymbol{v}_h^{i,2} = \sum_{k=1}^{n} \{\alpha_k\, \overline{\varrho}_{E_k}\, \boldsymbol{n}_{E_k} + \beta_k\, \overline{\varphi}_{E_k}\, \boldsymbol{t}_{E_k}\}\,,$$

$$\boldsymbol{v}_h^{i,3} = \sum_{k=1}^{n} \{\gamma_k\, (\overline{\chi}_{E_k}\, \boldsymbol{n}_{E_k} + \pi_{E_k}\, \boldsymbol{t}_{E_k}) + \delta_k\, \overline{\varrho}_{E_k}\, \boldsymbol{t}_{E_k}\}$$

with constants $\alpha_k, \beta_k, \gamma_k, \delta_k$, $k = 1, \ldots, n$, to be determined later. According to (21) and (22), we have

$$b_h(\boldsymbol{v}_h^{i,2}, q_h) = b_h(\boldsymbol{v}_h^{i,3}, q_h) = 0 \qquad \forall\, q_h \in \mathrm{Q}_h^1 \tag{35}$$

and hence, setting

$$\boldsymbol{v}_h^i = \boldsymbol{v}_h^{i,1} + \boldsymbol{v}_h^{i,2} + \boldsymbol{v}_h^{i,3}\,,$$

we obtain a function from $\mathbf{V}_h^{i,3}$ satisfying

$$b_j(\boldsymbol{v}_h^i, q) = (\widetilde{q}_h, q)_{K_j} \qquad \forall\, q \in P_1(K_j),\ j = 1, \ldots, n\,.$$

Our aim is to choose the constants α_k, β_k, γ_k, δ_k, $k = 1, \ldots, n$, in such a way that $\boldsymbol{v}_h^i$ also satisfies (31). For this, it is sufficient to fulfil

$$b_j(\boldsymbol{v}_h^i, \zeta_{E_j}^2) = (\widetilde{q}_h, \zeta_{E_j}^2)_{K_j}\,, \qquad j = 1, \ldots, n\,, \tag{36}$$

$$b_j(\boldsymbol{v}_h^i, \zeta_{E_{j+1}}^2) = (\widetilde{q}_h, \zeta_{E_{j+1}}^2)_{K_j}\,, \qquad j = 1, \ldots, n\,, \tag{37}$$

$$b_j(\boldsymbol{v}_h^i, \bar{q}_j) = (\widetilde{q}_h, \bar{q}_j)_{K_j}\,, \qquad j = 1, \ldots, n\,, \tag{38}$$

where

$$\bar{q}_j = (\zeta_{E_j}^2 + \zeta_{E_{j+1}}^2 + \zeta_{E_{i,j}}^2)|_{K_j}$$

and $E_{i,j}$ is the edge of K_j opposite x_i.

Consider any $j \in \{1, \ldots, n\}$. Integrating by parts and applying (10), we derive for any $\boldsymbol{v} \in \mathbf{V}_h^{i,3}$ and $q \in P^2(K_j)$

$$b_j(\boldsymbol{v}, q) = \int_{K_j} \boldsymbol{v} \cdot \nabla q \,\mathrm{d}x + \int_{E_j} q\, \boldsymbol{v} \cdot \boldsymbol{n}_{E_j} \,\mathrm{d}\sigma - \int_{E_{j+1}} q\, \boldsymbol{v} \cdot \boldsymbol{n}_{E_{j+1}} \,\mathrm{d}\sigma\,. \tag{39}$$

Particularly, we get

$$\begin{aligned} b_j(\boldsymbol{v}_h^{i,2}, q) = \int_{K_j} \nabla q \cdot (\alpha_j\, \overline{\varrho}_{E_j}\, \boldsymbol{n}_{E_j} + \beta_j\, \overline{\varphi}_{E_j}\, \boldsymbol{t}_{E_j} + \alpha_{j+1}\, \overline{\varrho}_{E_{j+1}}\, \boldsymbol{n}_{E_{j+1}} \\ + \beta_{j+1}\, \overline{\varphi}_{E_{j+1}}\, \boldsymbol{t}_{E_{j+1}}) \,\mathrm{d}x + \alpha_j \int_{E_j} q\, \overline{\varrho}_{E_j} \,\mathrm{d}\sigma - \alpha_{j+1} \int_{E_{j+1}} q\, \overline{\varrho}_{E_{j+1}} \,\mathrm{d}\sigma\,. \end{aligned}$$

Is is easy to see that

$$\boldsymbol{t}_{E_j} \cdot \nabla \zeta_{E_{j+1}}|_{K_j} = -\frac{2}{h_{E_j}}\,, \qquad \boldsymbol{t}_{E_{j+1}} \cdot \nabla \zeta_{E_j}|_{K_j} = -\frac{2}{h_{E_{j+1}}}\,. \tag{40}$$

In addition, in view of the Gauss integral theorem, we have

$$|K_j|\,(\nabla \zeta_{E_j}|_{K_j}) = \int_{K_j} \nabla \zeta_{E_j} \,\mathrm{d}x = \int_{\partial K_j} (\zeta_{E_j}|_{K_j})\, \boldsymbol{n}_{\partial K_j} \,\mathrm{d}\sigma = h_{E_j}\, \boldsymbol{n}_{\partial K_j}|_{E_j}\,,$$

where $\boldsymbol{n}_{\partial K_j}$ is the unit outer normal vector to the boundary of K_j. An analogous relation also holds for $\zeta_{E_{j+1}}$. This implies that

$$\begin{aligned} \boldsymbol{t}_{E_j} \cdot \nabla \bar{q}_j &= \frac{4}{h_{E_j}}\, (\zeta_{E_{i,j}} - \zeta_{E_{j+1}})\,, \\ \boldsymbol{t}_{E_{j+1}} \cdot \nabla \bar{q}_j &= \frac{4}{h_{E_{j+1}}}\, (\zeta_{E_{i,j}} - \zeta_{E_j})\,, \\ \boldsymbol{n}_{E_j} \cdot \nabla \bar{q}_j &= \frac{2\, h_{E_j}}{|K_j|}\, (\zeta_{E_{i,j}} - \zeta_{E_j}) - \frac{2\, h_{E_{j+1}}}{|K_j|}\, \boldsymbol{n}_{E_j} \cdot \boldsymbol{n}_{E_{j+1}}\, (\zeta_{E_{i,j}} - \zeta_{E_{j+1}})\,, \\ \boldsymbol{n}_{E_{j+1}} \cdot \nabla \bar{q}_j &= -\frac{2\, h_{E_{j+1}}}{|K_j|}\, (\zeta_{E_{i,j}} - \zeta_{E_{j+1}}) \\ &\quad + \frac{2\, h_{E_j}}{|K_j|}\, \boldsymbol{n}_{E_j} \cdot \boldsymbol{n}_{E_{j+1}}\, (\zeta_{E_{i,j}} - \zeta_{E_j})\,. \end{aligned}$$

Thus, we derive using (19) and (26)–(29) that

$$b_j(\boldsymbol{v}_h^{i,2}, \bar{q}_j) = 12\,A\,(\alpha_{j+1}\,h_{E_{j+1}} - \alpha_j\,h_{E_j}) + \frac{8}{9}\,(1 - 18\,A)\,|K_j|\,\left(\frac{\beta_j}{h_{E_j}} + \frac{\beta_{j+1}}{h_{E_{j+1}}}\right).$$

Now, we set

$$\beta_j = \frac{27\,A\,h_{E_j}^2}{2\,(18\,A - 1)\,|K_{j-1}|}\,\alpha_j\,. \tag{41}$$

Then we obtain

$$b_j(\boldsymbol{v}_h^{i,2}, \bar{q}_j) = -12\,A\,h_{E_j}\left(1 + \frac{|K_j|}{|K_{j-1}|}\right)\alpha_j\,. \tag{42}$$

Further, it follows from (39) and (20) that, for any $q \in P_2(K_j)$ which is even along all edges of K_j, we have

$$b_j(\boldsymbol{v}_h^{i,3}, q) = \int_{K_j} \nabla q \cdot [(\gamma_j\,\pi_{E_j} + \delta_j\,\overline{\varrho}_{E_j})\,\boldsymbol{t}_{E_j} + (\gamma_{j+1}\,\pi_{E_{j+1}} + \delta_{j+1}\,\overline{\varrho}_{E_{j+1}})\,\boldsymbol{t}_{E_{j+1}}]\,\mathrm{d}x\,.$$

Particularly, using (27) and (30), we obtain

$$b_j(\boldsymbol{v}_h^{i,3}, \bar{q}_j) = 0\,. \tag{43}$$

Applying (40) and again (27) and (30), we derive

$$b_j(\boldsymbol{v}_h^{i,3}, \zeta_{E_j}^2) = -\frac{2}{9}\,(18\,A - 1)\,h_{E_{j+1}}\,\gamma_{j+1} + \frac{8}{45}\,(45\,A - 1)\,|K_j|\,\frac{\delta_{j+1}}{h_{E_{j+1}}}\,, \tag{44}$$

$$b_j(\boldsymbol{v}_h^{i,3}, \zeta_{E_{j+1}}^2) = \frac{2}{9}\,(18\,A - 1)\,h_{E_j}\,\gamma_j + \frac{8}{45}\,(45\,A - 1)\,|K_j|\,\frac{\delta_j}{h_{E_j}}\,. \tag{45}$$

For any $j \in \{1, \ldots, N_h\}$ and any $\boldsymbol{v} \in H^1(K_j)^2$ and $q \in P_2(K_j)$, let us denote

$$r_j(\boldsymbol{v}, q) = (\widetilde{q}_h, q)_{K_j} - b_j(\boldsymbol{v}, q)\,.$$

In view of (41)–(43), we see that (38) is satisfied if we set

$$\alpha_j = -\frac{|K_{j-1}|}{12\,A\,h_{E_j}\,(|K_{j-1}| + |K_j|)}\,r_j(\boldsymbol{v}_h^{i,1}, \bar{q}_j)\,, \qquad j = 1, \ldots, n\,.$$

Further, owing to (44) and (45), we deduce that (36) and (37) hold, if we set, for $j = 1, \ldots, n$,

$$\gamma_j = 9\,\frac{|K_{j-1}|\,r_j(\boldsymbol{v}_h^{i,1} + \boldsymbol{v}_h^{i,2}, \zeta_{E_{j+1}}^2) - |K_j|\,r_{j-1}(\boldsymbol{v}_h^{i,1} + \boldsymbol{v}_h^{i,2}, \zeta_{E_{j-1}}^2)}{2\,(18\,A - 1)\,h_{E_j}\,(|K_{j-1}| + |K_j|)}\,,$$

$$\delta_j = 45\,h_{E_j}\,\frac{r_{j-1}(\boldsymbol{v}_h^{i,1} + \boldsymbol{v}_h^{i,2}, \zeta_{E_{j-1}}^2) + r_j(\boldsymbol{v}_h^{i,1} + \boldsymbol{v}_h^{i,2}, \zeta_{E_{j+1}}^2)}{8\,(45\,A - 1)\,(|K_{j-1}| + |K_j|)}\,.$$

The validity of (15) follows from (34) and (35) and hence it remains to prove (17). First, note that

$$r_j(\boldsymbol{v},q) \le \sqrt{|K_j|}\,(\|\widetilde{q}_h\|_{0,K_j} + \sqrt{2}\,|\boldsymbol{v}|_{1,K_j})\,\|q\|_{0,\infty,K_j} \qquad \forall\, \boldsymbol{v} \in H^1(K_j)^2,\ q \in P_2(K_j)\,.$$

Using (3), we deduce that

$$h_{E_j}^2 \le \sigma\, h_{E_j}\, \varrho_{K_{j-1}} \le 2\,\sigma\,|K_{j-1}|\,, \qquad |K_{j-1}| \le \frac{1}{2}\,h_{E_j}\,h_{K_{j-1}} \le \frac{\sigma}{2}\,h_{E_j}^2\,.$$

Similarly, we get

$$h_{E_j}^2 \le 2\,\sigma\,|K_j|\,, \qquad |K_j| \le \frac{\sigma}{2}\,h_{E_j}^2\,.$$

Thus, by virtue of (9), (33) and the above relations, we derive that $|\alpha_j| + |\beta_j| \le C\,\|\widetilde{q}_h\|_{0,\Delta_i}$, $j = 1,\dots,N_h$, with C depending only on σ and $\widehat{b}_1$. Since all angles in the elements of $\mathcal{T}_h$ are bounded from below by $\arcsin(1/\sigma)$, the number n of elements in Δ_i satisfies $n \le 2\,\pi/\arcsin(1/\sigma)$. Therefore, according to (14), we have $|\boldsymbol{v}_h^{i,2}|_{1,h} \le C\,\|\widetilde{q}_h\|_{0,\Delta_i}$, again with C depending only on σ and $\widehat{b}_1$. Consequently, we deduce that also $|\gamma_j| + |\delta_j| \le C\,\|\widetilde{q}_h\|_{0,\Delta_i}$, $j = 1,\dots,N_h$, and $|\boldsymbol{v}_h^{i,3}|_{1,h} \le C\,\|\widetilde{q}_h\|_{0,\Delta_i}$ with C depending only on σ and $\widehat{b}_1$. □

Acknowledgements

This research has been supported by the Grant Agency of the Academy of Sciences of the Czech Republic under the Grant No. IAA100190505 and by the Grant No. MSM 0021620839.

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