Numerical simulation of the free fall of a rigid body in a viscous fluid

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Abstract. *We consider the problem of the free fall of a rigid body in a viscous fluid. This problem has many important applications and the present paper is devoted to its numerical solution. We present a model of the problem, discuss various aspects of its finite element discretization and present some first numerical results.*

1 Introduction

This paper is a contribution to the numerical simulation of free fall of rigid bodies in viscous fluids, both of Newtonian and non–Newtonian type. Particularly, we are interested in the orientation of a falling body with respect to the gravity after reaching a steady state. It is a difficult highly coupled problem since the motion of the body affects the flow of the liquid and this, in turn, affects the motion of the body.

There are various applications in which the knowledge of the motion and, particularly, orientation of long bodies in liquids is very important. One example is the production of composite materials based on the addition of short fiber–like particles to a polymer matrix. The properties of the resulting material are strongly influenced by the orientation of the fibers and hence also by the flow in the polymer liquid. Another example is the separation of macromolecules by electrophoresis where the orientation of the molecules is responsible for the loss of separability during steady–field gel electrophoresis. As the last example, let us mention applications of particulate flows where microstructures arise from particle interactions due to inertia and normal stresses. A key to understanding these mechanisms is the knowledge of the stable orientation of long bodies since two particles in momentary contact can be viewed as a rigid, symmetric, long body. It is interesting that the interactions are completely different in Newtonian and viscoelastic fluids.

A first step in modeling the motion and orientation of long bodies in liquids is to investigate their free fall behaviour. Let us consider a homogeneous body $\mathcal{B}$ of revolution around an axis denoted by a. Let us assume that there is also a plane of symmetry of $\mathcal{B}$ which is orthogonal to a. Thus, $\mathcal{B}$ may be, e.g., a cylinder or a round ellipsoid. We assume that the diameter of $\mathcal{B}$ in the direction of a is much prevailing upon the diameter of $\mathcal{B}$ in a direction perpendicular to a (i.e., it is a 'long' body as mentioned above). If we drop the body $\mathcal{B}$ in a quiescent viscous liquid, it will eventually reach a steady state that is purely translatory and with a forming an angle with respect to the gravity $\boldsymbol{g}$, that depends on the weight of the body, on its geometrical properties and on the physical properties of the liquid (cf., e.g., [13]). In particular, in a Newtonian liquid, cylinders will always reach an equilibrium orientation with a orthogonal to the gravity (unless the Reynolds number approaches zero in which case all orientations are admissible). On the contrary, in the purely viscoelastic case, the situation is reversed and the final orientation of the body is with a parallel to the gravity. Depending on the properties of the body and the non–Newtonian liquid, also other stable orientations of the body can be observed.

In this paper we shall concentrate on describing the interaction between an incompressible viscous fluid and a rigid body moving in this fluid, and on discussing various approaches to the numerical solution of this problem. To simplify the presentation, we only consider the Newtonian case and we assume that the gravity is the only outer force acting on the fluid and the body. Thus, the fluid flow will be described by the incompressible Navier–Stokes equations. A suitable, well–established method for their numerical solution is the finite element method, see, e.g., [14, 31, 32]. The motion of the rigid body will be described by ordinary differential equations for its mass center and angular velocity.

The plan of the paper is as follows. In the next section, we describe a mathematical model suitable for the numerical simulation of a rigid body motion in a Navier–Stokes fluid. Then we shall discuss the numerical solution of the model based on the finite element method and we shall present some first numerical results. Finally, we shall close the paper by several concluding remarks.

2 Mathematical model

Let us first consider the case when the whole space $\mathbb{R}^3$ is filled with an incompressible linear viscous fluid, except for the region occupied by a rigid body $\mathcal{B}$. We assume that we are given an inertial Cartesian coordinate system $\mathcal{I}$ in which we shall first formulate the equations describing the considered problem. Since the body can arbitrarily move through the fluid, we shall denote by $\mathcal{B}_t$ the region occupied by the body $\mathcal{B}$ at time t with respect to the coordinate system $\mathcal{I}$.

The fluid is characterized by its velocity $\boldsymbol{v} = \boldsymbol{v}(x,t)$ and pressure $P = P(x,t)$ which obey, for $x \in \mathbb{R}^3 \setminus \mathcal{B}_t$, $t > 0$, the balance laws of linear momentum and mass given by (cf., e.g., [11, 15])

$$\varrho \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} \right) = \operatorname{div} \mathcal{T}(\boldsymbol{v}, P) + \varrho\, \boldsymbol{g}\,, \qquad \operatorname{div} \boldsymbol{v} = 0\,, \tag{1}$$

respectively. Here ϱ is the constant density of the fluid, $\boldsymbol{g}$ is the gravitational acceleration and

$$\mathcal{T}(\boldsymbol{v}, P) = -P\, \boldsymbol{I} + 2\, \mu\, \boldsymbol{D}$$

is the Cauchy stress tensor with $\boldsymbol{I}$ being the identity tensor, $\mu > 0$ the constant dynamic viscosity and

$$\boldsymbol{D} = \frac{1}{2} (\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T)$$

the stretching. The equations (1) can be rewritten into the form

$$\frac{\partial \boldsymbol{v}}{\partial t} - \nu\, \Delta \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \frac{1}{\varrho} \nabla P = \boldsymbol{g}\,, \qquad \operatorname{div} \boldsymbol{v} = 0\,, \tag{2}$$

where $\nu = \mu/\varrho$ is the kinematic viscosity.

Let $x_C(t)$ be the center of mass of $\mathcal{B}$ at time t and let $\boldsymbol{\eta}(t) = \dot{x}_C(t)$ be its velocity. Without loss of generality we may assume that $x_C(0) = 0$. Denoting by $\boldsymbol{\Omega}(t)$ the angular velocity of $\mathcal{B}$, the velocity of any point $x \in \mathcal{B}_t$ at time t is

$$V(x,t) = \boldsymbol{\eta}(t) + \boldsymbol{\Omega}(t) \times (x - x_C(t))\,.$$

To formulate the equations of motion of $\mathcal{B}$ in the coordinate system $\mathcal{I}$, we denote by m the mass of $\mathcal{B}$, by $\varrho_{\mathcal{B}}$ its density, by $\boldsymbol{N}$ the unit normal vector to the boundary of $\mathcal{B}$ pointing into $\mathcal{B}$, by $\boldsymbol{r}_C(x,t) = x - x_C(t)$ the position vector of $x \in \mathcal{B}_t$ with respect to $x_C(t)$ and by

$$\mathbf{J}(t) = \int_{\mathcal{B}_t} (\boldsymbol{r}_C^2\, \boldsymbol{I} - \boldsymbol{r}_C \otimes \boldsymbol{r}_C)\, \varrho_{\mathcal{B}}\, \mathrm{d}x$$

the inertia tensor of $\mathcal{B}$ relative to the center of mass. Then the balance of linear and angular momentum of the body $\mathcal{B}$ can be respectively expressed by the equations

$$m\,\frac{\partial \boldsymbol{\eta}}{\partial t} = m\,\boldsymbol{g} - \int_{\partial\mathcal{B}_t} \mathcal{T}(\boldsymbol{v},P)\,\boldsymbol{N}\,\mathrm{d}\sigma\,, \qquad \frac{\partial(\mathbf{J}\,\boldsymbol{\Omega})}{\partial t} = -\int_{\partial\mathcal{B}_t} \boldsymbol{r}_C \times \mathcal{T}(\boldsymbol{v},P)\,\boldsymbol{N}\,\mathrm{d}\sigma\,. \tag{3}$$

All the above relations can be found, e.g., in [15].

It remains to specify some boundary conditions. We require that

$$\boldsymbol{v}(x,t) = V(x,t) \qquad \forall\, x \in \partial\mathcal{B}_t\,,\ t>0 \tag{4}$$

and assume that the fluid is at rest at infinity, so that we impose

$$\lim_{|x|\to\infty} \boldsymbol{v}(x,t) = 0 \qquad \forall\, t>0\,. \tag{5}$$

Now, the motion of the body and the liquid is determined by the relations (2)–(5) once initial conditions on $\boldsymbol{v}$ and V are prescribed.

However, a drawback of the above formulation is that the region occupied by the fluid is an unknown function of time. Therefore, we reformulate the problem in a Cartesian coordinate system $\mathcal{S}$ attached to the body $\mathcal{B}$ where this region remains the same at all times. Without loss of generality, we may assume that $\mathcal{S} = \mathcal{I}$ at time $t=0$. Denoting by x and y the position vectors in $\mathcal{I}$ and $\mathcal{S}$, respectively, corresponding to the same point in $\mathcal{I}$, we have (cf. [15])

$$x = \boldsymbol{Q}(t)\,y + x_C(t)\,, \qquad \boldsymbol{Q}(0) = \boldsymbol{I}\,, \qquad x_C(0) = 0\,,$$

where $\boldsymbol{Q}$ is a orthogonal tensor, i.e.,

$$\boldsymbol{Q}(t)\,\boldsymbol{Q}^T(t) = \boldsymbol{Q}^T(t)\,\boldsymbol{Q}(t) = \boldsymbol{I}\,.$$

Note that

$$\dot{\boldsymbol{Q}}(t)\,\boldsymbol{Q}^T(t) + \boldsymbol{Q}(t)\,\dot{\boldsymbol{Q}}^T(t) = \dot{\boldsymbol{Q}}^T(t)\,\boldsymbol{Q}(t) + \boldsymbol{Q}^T(t)\,\dot{\boldsymbol{Q}}(t) = 0\,,$$

which shows that the tensors $\dot{\boldsymbol{Q}}(t)\,\boldsymbol{Q}^T(t)$ and $\dot{\boldsymbol{Q}}^T(t)\,\boldsymbol{Q}(t)$ are skew–symmetric. The axial vector of $\dot{\boldsymbol{Q}}(t)\,\boldsymbol{Q}^T(t)$ is the angular velocity $\boldsymbol{\Omega}(t)$, i.e.,

$$\dot{\boldsymbol{Q}}(t)\,\boldsymbol{Q}^T(t)\,\boldsymbol{a} = \boldsymbol{\Omega}(t) \times \boldsymbol{a} \qquad \forall\, \boldsymbol{a} \in \mathbb{R}^3\,.$$

Denoting by

$$\boldsymbol{\xi}(t) = \boldsymbol{Q}^T(t)\,\boldsymbol{\eta}(t)\,, \qquad \boldsymbol{\omega}(t) = \boldsymbol{Q}^T(t)\,\boldsymbol{\Omega}(t)\,, \qquad U(y,t) = \boldsymbol{\xi}(t) + \boldsymbol{\omega}(t) \times y\,,$$

the transformed fields describing the motion of the body, we realize that $\boldsymbol{\omega}(t)$ is the axial vector of $\boldsymbol{Q}^T(t)\,\dot{\boldsymbol{Q}}(t)$, i.e.,

$$\boldsymbol{Q}^T(t)\,\dot{\boldsymbol{Q}}(t)\,\boldsymbol{a} = \boldsymbol{\omega}(t) \times \boldsymbol{a} \qquad \forall\, \boldsymbol{a} \in \mathbb{R}^3\,.$$

The fields $\boldsymbol{v}$ and P transform in the following way:

$$\boldsymbol{u}(y,t) = \boldsymbol{Q}^T(t)\,\boldsymbol{v}(\boldsymbol{Q}(t)\,y + x_C(t),t)\,, \qquad p(y,t) = P(\boldsymbol{Q}(t)\,y + x_C(t),t)\,.$$

It is easy to verify that

$$\Delta_x\,\boldsymbol{v} = \boldsymbol{Q}\,\Delta_y\,\boldsymbol{u}\,, \qquad \nabla_x\,\boldsymbol{v} = \boldsymbol{Q}\,\nabla_y\,\boldsymbol{u}\,\boldsymbol{Q}^T\,, \qquad \nabla_x\,P = \boldsymbol{Q}\,\nabla_y\,p\,, \qquad \mathrm{div}_x\,\boldsymbol{v} = \mathrm{div}_y\,\boldsymbol{u}$$

and

$$\frac{\partial \boldsymbol{v}}{\partial t} = \boldsymbol{Q}\,\frac{\partial \boldsymbol{u}}{\partial t} + \dot{\boldsymbol{Q}}\,\boldsymbol{u} - \boldsymbol{Q}\,(U \cdot \nabla_y\,\boldsymbol{u})\,.$$

Thus, denoting

$$\boldsymbol{G}(t)=\boldsymbol{Q}^T(t)\,\boldsymbol{g}\,,$$

the equations (2) transform to

$$\frac{\partial \boldsymbol{u}}{\partial t}-\nu\,\Delta\boldsymbol{u}+(\boldsymbol{u}-U)\cdot\nabla\,\boldsymbol{u}+\boldsymbol{\omega}\times\boldsymbol{u}+\frac{1}{\varrho}\nabla\,p=\boldsymbol{G}\,,\qquad \operatorname{div}\boldsymbol{u}=0\qquad \text{in }\mathcal{D}\times\mathbb{R}_0^+\,,\tag{6}$$

where $\mathcal{D}=\mathbb{R}^3\setminus\mathcal{B}$ is the region occupied by the fluid which now does not change in time.

Similarly, we obtain from (3)

$$m\,\frac{\partial\boldsymbol{\xi}}{\partial t}+m\,\boldsymbol{\omega}\times\boldsymbol{\xi}\;=\;m\,\boldsymbol{G}-\int_{\partial\mathcal{B}}\mathcal{T}(\boldsymbol{u},p)\,\boldsymbol{n}\,\mathrm{d}\sigma\,,\tag{7}$$

$$\mathbf{I}\,\frac{\partial\boldsymbol{\omega}}{\partial t}+\boldsymbol{\omega}\times(\mathbf{I}\,\boldsymbol{\omega})\;=\;-\int_{\partial\mathcal{B}}y\times\mathcal{T}(\boldsymbol{u},p)\,\boldsymbol{n}\,\mathrm{d}\sigma\,,\tag{8}$$

where $\boldsymbol{n}=\boldsymbol{Q}^T\,\boldsymbol{N}$ is the unit normal vector to the boundary of $\mathcal{B}$ pointing into $\mathcal{B}$, with respect to the coordinate system $\mathcal{S}$. To derive (7) and (8), we used the fact that the tensor $\mathcal{T}(\boldsymbol{v},P)$ transforms to $\boldsymbol{Q}^T\,\mathcal{T}(\boldsymbol{Q}\,\boldsymbol{u},p)\,\boldsymbol{Q}=\mathcal{T}(\boldsymbol{u},p)$ and we applied the relation

$$\mathbf{J}(t)=\boldsymbol{Q}(t)\,\mathbf{I}\,\boldsymbol{Q}^T(t)\qquad\text{with}\qquad\mathbf{I}=\mathbf{J}(0)\,.$$

Thus, the matrix of $\mathbf{J}(t)$ relative to a Cartesian system that rotates with the body is independent of t. Moreover, this coordinate system can be chosen in such a way that the matrix of $\mathbf{J}$ is diagonal; the diagonal entries are called moments of inertia.

Note that, in contrast with the problem in the inertial coordinate system $\mathcal{I}$, the direction of the gravity is an unknown function of time. Nevertheless, it is easy to see from the above relations, that the function $\boldsymbol{G}$ satisfies

$$\frac{\partial\boldsymbol{G}}{\partial t}=\boldsymbol{G}\times\boldsymbol{\omega}\,,\qquad\boldsymbol{G}(0)=\boldsymbol{g}\,.\tag{9}$$

Finally, the boundary conditions (4), (5) become

$$\boldsymbol{u}(y,t)=U(y,t)\qquad\forall\,y\in\partial\mathcal{B}\,,\;t>0\tag{10}$$

and

$$\lim_{|y|\to\infty}\boldsymbol{u}(y,t)=0\qquad\forall\,t>0\,.\tag{11}$$

The equations (6)–(11) completely describe the motion of the body and the fluid in the coordinate system $\mathcal{S}$ once initial conditions have been specified. Theoretical results for this problem can be found, e.g., in [13]. However, if we want to solve this problem numerically, we have to confine ourselves to some bounded domain $\mathcal{D}$ in (6). This bounded domain has to be sufficiently large to allow neglecting the influence of the artificial boundary $\Gamma=\partial\mathcal{D}\setminus\partial\mathcal{B}$ on the motion of the rigid body. It is convenient to prescribe the homogeneous Dirichlet boundary condition corresponding to (11) only on a part Γ^D of Γ and to consider some non–reflecting outflow boundary condition on $\Gamma\setminus\Gamma^D$. The simplest possibility is to use a do–nothing boundary condition on $\Gamma\setminus\Gamma^D$, see [19]. Thus, we replace (11) by

$$\boldsymbol{u}=0\quad\text{on }\Gamma^D\,,\qquad(-p\,\boldsymbol{I}+\mu\,\nabla\,\boldsymbol{u})\,\boldsymbol{n}=0\quad\text{on }\Gamma\setminus\Gamma^D\,,\tag{12}$$

where $\boldsymbol{n}$ is the unit outer normal vector to Γ. The remaining part of this paper will be devoted to the numerical solution of the above problem consisting of the relations (6)–(10) and (12) for the unknown functions $\boldsymbol{u}$, p, $\boldsymbol{\xi}$, $\boldsymbol{\omega}$ and $\boldsymbol{G}$.

3 Numerical solution of the free fall problem

Our aim is to approximate the solution of the problem formulated at the end of the preceding section at some discrete time levels t_k, $k \geq 0$, satisfying $t_k < t_{k+1}$ and $t_0 = 0$. For $k = 0$, the functions $\boldsymbol{u}$, $\boldsymbol{\xi}$, $\boldsymbol{\omega}$ and $\boldsymbol{G}$ are determined by the initial conditions. Let us assume that we have computed the approximations of $\boldsymbol{u}$, p, $\boldsymbol{\xi}$, $\boldsymbol{\omega}$ and $\boldsymbol{G}$ at time t_k. Then we first compute the approximations of $\boldsymbol{u}$ and p at time t_{k+1} by solving the equations (6) and then, using these new approximations, we compute approximations of $\boldsymbol{\xi}$, $\boldsymbol{\omega}$ and $\boldsymbol{G}$ at time t_{k+1} from (7)–(9). In this way the solution of the partial differential equations (6) and the ordinary differential equations (7)–(9) is decoupled. The numerical solution of the ordinary differential equations (7)–(9) can be accomplished using, e.g., a Runge–Kutta method and does not lead to any basic difficulties. Therefore, we shall confine ourselves to a discussion of the numerical solution of the Navier–Stokes equations (6) with boundary conditions (10) and (12) in the following. We shall consider only the finite element method, see, e.g., [14, 31, 32].

First let us consider the discretization of (6) with respect to the time variable. Although space–time finite elements can be used [17], usually the discretizations in space and time are decoupled. In the literature, one can find many various approaches, for example, the implicit Euler method [31], the Crank–Nicolson method [18], the BDF methods [1], the fractional step θ–scheme [24], the projection methods [26] or the Runge–Kutta method [23]. A discussion of various time discretization techniques for the incompressible Navier–Stokes equations can be found in [10]. Many authors prefer the fractional step θ–scheme which is of second order accuracy, strongly A–stable and (nearly) non–dissipative, see also [32] for a numerical study of various time discretization approaches. In our computations, we use a modified version of the Crank–Nicolson method since it is easy to implement and also seems to be sufficiently accurate. The modification consists in treating the pressure only implicitly, see also [32]. We observed that the standard Crank–Nicolson time discretization containing both the new and the old pressure does not converge in certain cases.

At each time step, the nonlinear problem (6) is replaced by a sequence of linearized equations based on simple fixed point iterations. The new iterate is computed from (6) with $(\boldsymbol{u} - U) \cdot \nabla \boldsymbol{u}$ replaced by $(\boldsymbol{u}^{old} - U) \cdot \nabla \boldsymbol{u}$ where $\boldsymbol{u}^{old}$ is the result of the previous fixed point iteration. The iterations are repeated until the nonlinear problem is solved to a prescribed accuracy. Another possibility is to apply Newton's method to linearize (6) but in our numerical tests this always led to increased computational cost.

Thus, after the time discretization and linearization, we solve a problem of the type

$$\alpha\, \boldsymbol{u} - \nu\, \Delta \boldsymbol{u} + \boldsymbol{w} \cdot \nabla \boldsymbol{u} + \boldsymbol{\omega} \times \boldsymbol{u} + \nabla p = \boldsymbol{f}\,, \qquad \operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \mathcal{D} \tag{13}$$

with boundary conditions

$$\boldsymbol{u} = U \quad \text{on } \partial\mathcal{B}\,, \qquad \boldsymbol{u} = 0 \quad \text{on } \Gamma^D\,, \qquad (-p\,\boldsymbol{I} + \nu\, \nabla \boldsymbol{u})\, \boldsymbol{n} = 0 \quad \text{on } \Gamma \setminus \Gamma^D\,, \tag{14}$$

where $\alpha > 0$ ($1/\alpha$ is the time step) and $\boldsymbol{w}$ and $\boldsymbol{f}$ are given functions. The finite element discretization of (13), (14) is based on the standard weak formulation: find $\boldsymbol{u} \in H^1(\mathcal{D})^3$ and $p \in L^2(\mathcal{D})$ such that $\boldsymbol{u} = U$ on $\partial\mathcal{B}$, $\boldsymbol{u} = 0$ on Γ^D and

$$\alpha\, (\boldsymbol{u}, \boldsymbol{v}) + \nu\, (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + (\boldsymbol{w} \cdot \nabla \boldsymbol{u}, \boldsymbol{v}) + (\boldsymbol{\omega} \times \boldsymbol{u}, \boldsymbol{v}) - (p, \operatorname{div} \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in V\,, \tag{15}$$

$$(q, \operatorname{div} \boldsymbol{u}) = 0 \quad \forall\, q \in L^2(\mathcal{D})\,, \tag{16}$$

where the round brackets denote the inner product in $L^2(\mathcal{D})$ or $L^2(\mathcal{D})^3$ and

$$V = \{\boldsymbol{v} \in H^1(\mathcal{D})^3\,;\ \boldsymbol{v} = 0 \ \text{ on } \Gamma^D \cup \partial\mathcal{B}\}\,.$$

The most straightforward way to derive a finite element discretization of the problem (13), (14) is to simply replace the spaces in the weak formulation (15), (16) by some finite element spaces. Unfortunately, this leads to a stable and accurate discretization only if these spaces satisfy an inf–sup condition [5, 14]. Therefore, various stabilizations of the discrete analogue of the incompressibility constraint have been developed to allow the use of arbitrary pairs of finite element spaces for approximating the velocity and the pressure (cf. e.g. [5, 6, 7, 20, 27]). Another approach consists in splitting the problem in several subproblems having a simpler form and thus enabling a more efficient numerical solution [26]. For example, the fulfilment of the incompressibility constraint can be achieved by a simple velocity update based on the solution of a Poisson equation for the pressure. The splitting approach may also be motivated by decoupling the incompressibility constraint and the nonlinearity [3]. In our computations, we always use inf–sup stable pairs of finite element spaces and hence no stabilization of the incompressibility constraint is considered.

If the viscosity ν is small, a stabilization of the convective term $\boldsymbol{w} \cdot \nabla \boldsymbol{u}$ and the Coriolis force $\boldsymbol{\omega} \times \boldsymbol{u}$ is often necessary to suppress spurious oscillations in the discrete solution. One possibility to stabilize the convective term is to use an upwind discretization, which leads to favourable properties of the resulting algebraic systems but is of first order accuracy only [28]. Therefore, for higher order elements, the streamline diffusion stabilization is to be preferred [12]. Since this type of stabilization is residual–based, it is consistent and hence does not decrease the convergence order. To stabilize the Coriolis force, we use the Galerkin/least–squares formulation of [8].

The discretization of (13), (14) corresponds to a system of linear algebraic equations with a large and sparse matrix. Therefore, it is natural to compute the solution of this linear system by means of an iterative method. A large class of these methods consists of Uzawa–type methods [2, 4], another possibility is to use Krylov subspace methods with a suitable preconditioning [9, 25, 30, 34]. A class of very efficient methods is formed by multi–grid methods [16]. Here, let us mention the multi–level pressure Schur complement techniques [33] and the fully coupled approach [22]. In the latter case, which we prefer, efficient solution procedures can be obtained using the Braess–Sarazin smoother or the Vanka smoother, which were analyzed in [35] and [29], respectively. For higher order discretizations, a very efficient approach is the multiple discretization multi–level method where the accurate higher order discretization is combined with fast multi–level solvers based on lower order (nonconforming) finite element discretizations [22, 21]. A multi–grid method can also be used as a preconditioner for a Krylov type method, e.g. for the flexible GMRES method [21]. We refer to [33] for further details on iterative techniques suitable for the solution of incompressible flow problems.

4 Numerical results

In this section we present a few numerical results obtained for the Navier–Stokes equations (6) defined in a two–dimensional computational domain $\mathcal{D} = (-1,1)^2 \setminus B(0,0.2)$ where $B(0,0.2)$ is a circle with radius 0.2 and centre in the origin. The triangulation of this domain obtained after four uniform refinements of a coarse mesh is depicted in Fig. 1. In (6), we consider $\nu = 0.01$, $U = 0$, $\boldsymbol{\omega} = \boldsymbol{e}_3$ (unit vector orthogonal in $\mathbb{R}^3$ to the plain computational domain), $\varrho = 1$ and $\boldsymbol{G} = 0$. The boundary conditions were chosen similarly as in (10) and (12) with Γ^D consisting of the straight parts of $\partial\mathcal{D}$ except for the lower edge of the square $(-1,1)^2$. The only difference to (10), (12) is that we do not consider $\boldsymbol{u} = 0$ on the upper edge of the square $(-1,1)^2$; instead we prescribe

$$\boldsymbol{u}(x,1) = \left(0, \frac{1}{2}(x-1)(x+1)\right) \qquad \forall\, x \in (-1,1)\,.$$

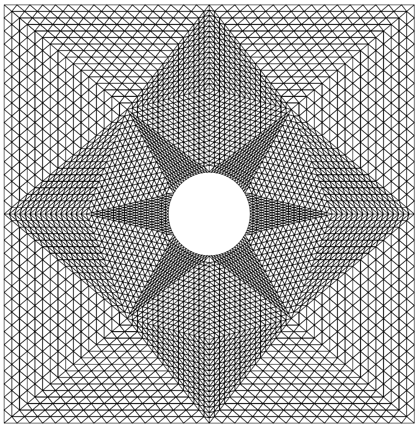

Figure 1: Final triangulation of the computational domain.

The velocity is approximated using continuous piecewise quadratic finite elements and the pressure using using continuous piecewise linear finite elements. We use the above–mentioned stabilization of the Coriolis force but no stabilization of the convective term was needed. The linearized discrete problems were solved by the multiple discretization multi–level method using the Crouzeix–Raviart element with an upwind discretization of the convective term on coarser levels. Fig. 2 and 3 show the velocity vectors and pressure isolines at a time when the steady state was already reached.

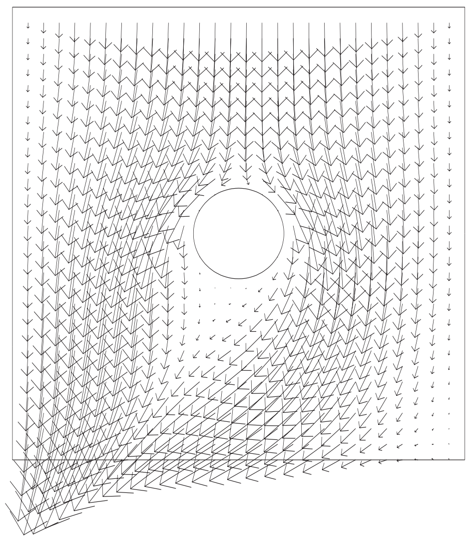

Figure 2: Velocity vectors.

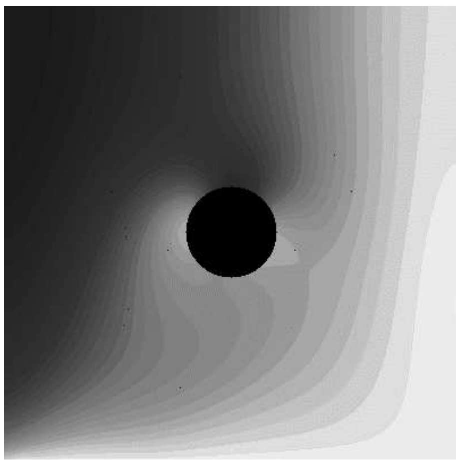

Figure 3: Pressure isolines.

5 Conclusions

In this paper, we formulated a mathematical model describing the free fall of a rigid body in an incompressible viscous fluid and we discussed various approaches to the numerical solution of this problem. In addition, we presented our first numerical results for a simplification of the model. The present work is an important step towards the numerical solution of the complete fluid–body interaction problem described in this paper and of its generalizations to various non–Newtonian fluids. Questions to be studied in the future include testing the robustness and reliability of the method, investigations of its convergence properties and stability, derivation of error estimates and design of other artificial boundary conditions.

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