Superconvergence for convection-diffusion problems with low regularity

Workshop Dresden-Prague on Numerical Analysis 2012

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Focus on

$$\begin{aligned} \mathcal{L}u := -\varepsilon \Delta u + \mathbf{b} \cdot \nabla u + cu = f \quad & \textit{in } \Omega \subset \mathbb{R}^2 \\ u = 0 \quad & \textit{on } \partial\Omega \end{aligned}$$

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assumptions

$$f \in L^2(\Omega)$$

$$c \in L_\infty(\Omega)$$

$$\mathbf{b} \in (W^1_\infty(\Omega))^2$$

and

$$c - \frac{1}{2}\mathrm{div}\,\mathbf{b} \geq \omega > 0.$$

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Q: What if $u \notin H^3\Omega$?

Typical solution of singularly perturbed problem

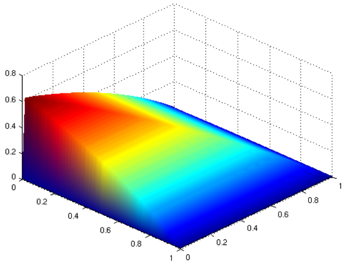


Figure: no compatibility condition

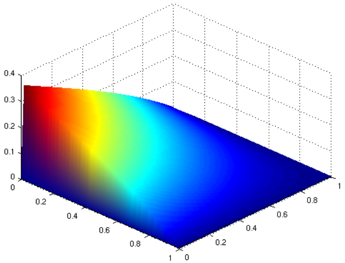
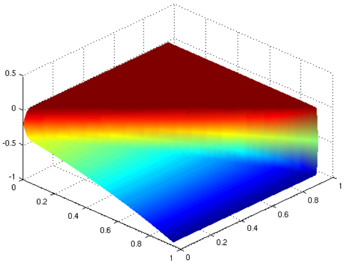


Figure: compatibility condition

Typical solution, ∂_x-derivative (away from layers)



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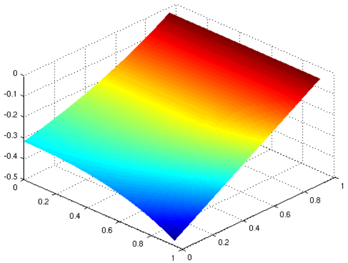

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Galerkin method

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Fractional Sobolev Spaces:

$$\|u^I - u_h\|_\varepsilon \lesssim (\varepsilon^{\frac{1}{2}} h^{1+\sigma} + h^{1+\sigma} + h^2)\|u\|_{2+\sigma}$$

Weighted Sobolev Spaces:

$$\|u^I - u_h\|_\varepsilon \lesssim (\varepsilon^{\frac{1}{2}} h^{2-\alpha} + h^{2-\alpha} + h^2)\|u\|_{3,\alpha}$$

Numerical Experiment 1

Test problem:

$$-\Delta u - 0.5 u_x + u_y + u = 1, \;\; u_{|\partial\Omega} = 0$$

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x_0	1	0.75	0.5	0.25
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- reference solution ... P_2-elements on 2048×2048 grid

Singularly perturbed case

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ε small, Galerkin method + Shishkin mesh at outflow boundary

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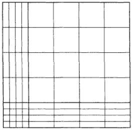

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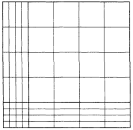

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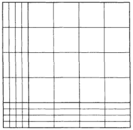

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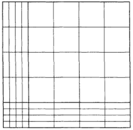

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- approach to analysis: result from B. Kellogg ("Corner Singularities and Singular Perturbations")

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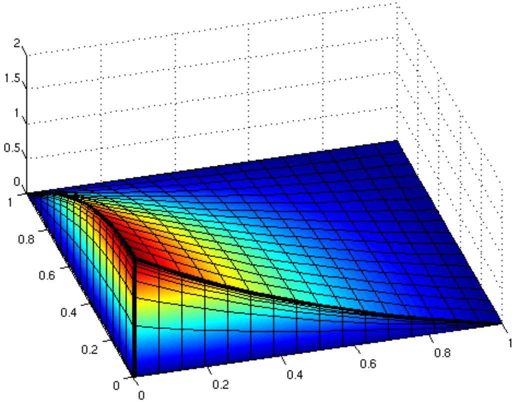
$$u = S + E + Z \in H^3_{2-\frac{\pi}{\omega}+\delta}(\Omega), \forall \delta > 0$$

- u ... finite element solution



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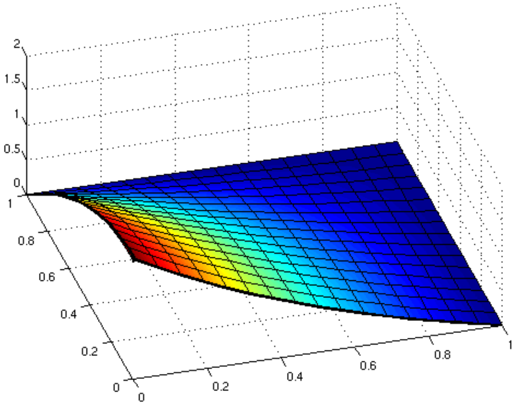

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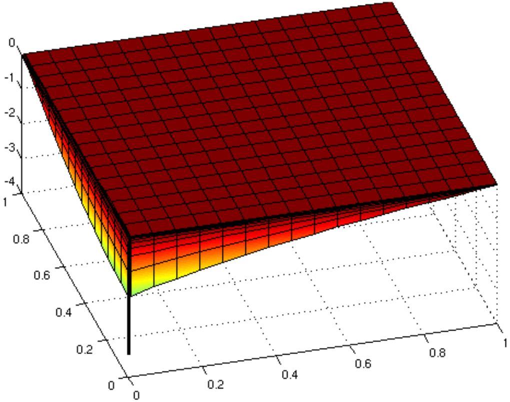

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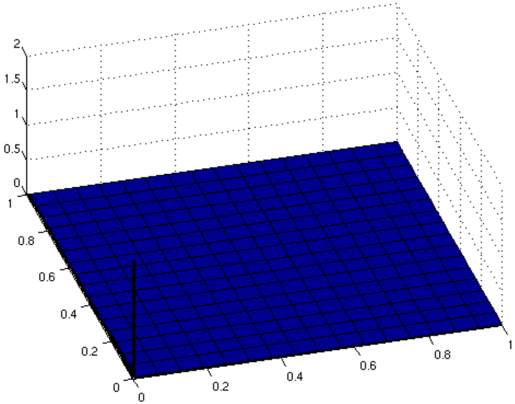

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Numerical Experiment 2

PDE, $\varepsilon = 10^{-4}$

$$\begin{aligned} -\varepsilon \Delta u - u_x - u_y + u &= 1 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

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- careful interpretations of results

Numerical Experiment 2 - results

x_0 / $\frac{\pi}{\omega}$	0.00 / 3.0000	0.2 / 2.7767	0.50 / 2.5442	0.7 / 2.4401	1.00 / 2.3333	1.25 / 2.2735	1.5 / 2.2303
$N = 8$	1.5531	1.6424	1.6273	1.5928	1.4963	1.4615	1.4885
16	1.8064	1.8137	1.8104	1.8092	1.7585	1.7571	1.8247
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Table: rates of convergence against reference solution, $u \in H^{2+\sigma}(\Omega)$

- very weak dependence of convergence on x_0, *i.e.* on regularity
- rate grows and eventually starts to drop in finer discretizazions

Computational analysis of derivatives

$$\varepsilon^3 |u_{xxy}| \sim (1 + \tfrac{1}{\rho}) e^{-a\rho}, \;\; \rho := \tfrac{r}{\varepsilon}$$

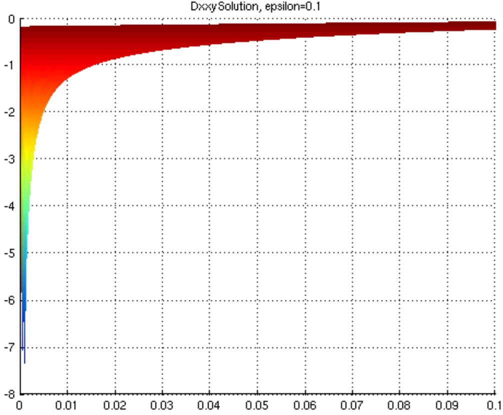

Figure: $\varepsilon = 0.1$

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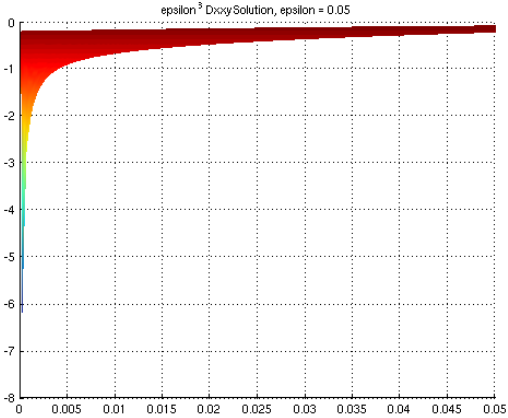

Figure: $\varepsilon = 0.05$

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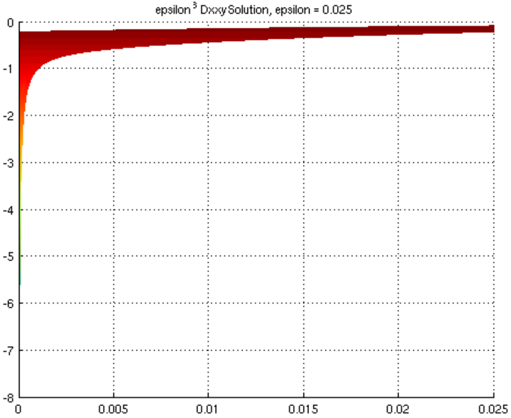

Figure: $\varepsilon = 0.025$

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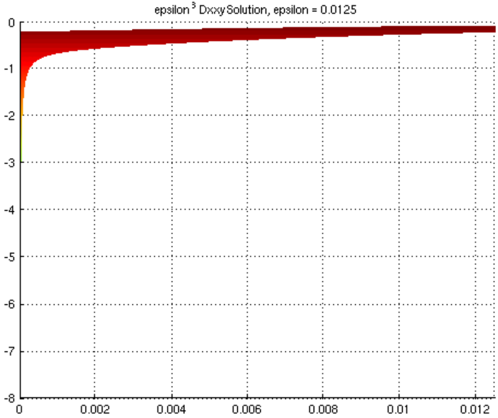

Figure: $\varepsilon = 0.0125$

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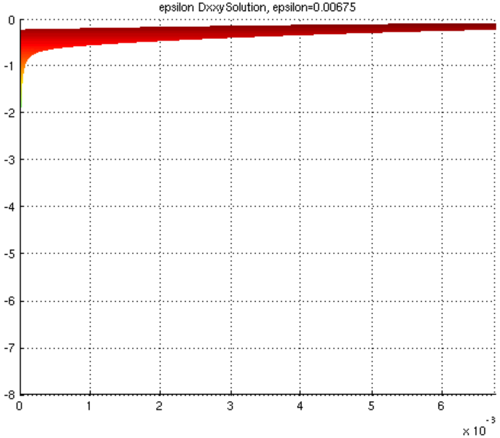

Figure: $\varepsilon = 0.00675$

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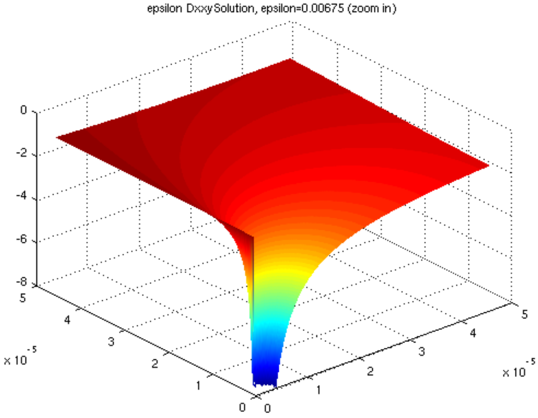

Figure: $\varepsilon = 0.00675$ (zoom in)

Conclusions

- pessimistic bounds for mixed layer/singularity term Z
- singular behaviour only observable very close (order $o(\varepsilon)$) to the origin
- sequence of numerical solutions $\{u_h\}_h$ does not see edge singularity until very fine discretization

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- sequence of numerical solutions $\{u_h\}_h$ does not see edge singularity until very fine discretization
  $\Longrightarrow$ higher order of superconvergence than expected

The End