Convergence Behavior of Barrier-Penalty Methods Applied to Optimal Control with PDEs

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Scope of the Talk

- Problem Formulation, Basic Properties
- Penalties for Control Bounds, Approximate Reduction
- Discretization of Reduced Problems
- Numerical Examples
- Outlook

1. Problem Formulation, Basic Properties

1.1. Distributed Controls (see G/Kunz/Meischner[3], G/Winkler[4])

$\Omega \subset \mathbb{R}^n,\ n = 2, 3$ bounded convex domain,
$q, b, d \in L_\infty(\Omega)$ given.

Considered optimal control problem

$$
\begin{aligned}
\hat{J}(y,u) &:= \frac{1}{2}\int_\Omega (y-q)^2 + \frac{\alpha}{2}\int_\Omega u^2 \to \min! \\
\text{s.t.} -\Delta y &= u \qquad \text{in } \Omega, \\
y + \frac{\partial y}{\partial n} &= 0 \qquad \text{on } \Gamma := \partial\Omega, \\
u &\in U_{ad},
\end{aligned}
\tag{1}
$$

$\alpha > 0$ regularization parameter
$U_{ad} \subset L^2(\Omega)$ set of admissible controls (convex, closed).

$$U_{ad} := \left\{ u \in L^2(\Omega) : u \le b \quad \text{a.e.in } \Omega \right\} \tag{2}$$

in case of only control constraints - or

$$U_{ad} := \left\{ u \in L^2(\Omega) : u \le b,\ y \le d \quad \text{a.e.in } \Omega \right\} \tag{3}$$

in case of state and control constraints.
One-sided bounds only to simplify the presentation.

General assumption: $U_{ad} \neq \emptyset.$

State equations of (1) in the weak sense in $V := H^1(\Omega)$,
$a(\cdot,\cdot) : V \times V \to \mathbb{R}$ defined by

$$a(y,v) := \int_\Omega \nabla y \cdot \nabla v + \int_\Gamma yv \quad \forall y, v \in V. \tag{4}$$

Lemma

For any $u \in U_{ad}$ *there is a unique* $y \in V$ *such that*

$$a(y,v) = (u,v) \quad \forall v \in V. \tag{5}$$

$Su := y$ *and* $V \hookrightarrow L^2(\Omega)$ *defines a linear, continuous mapping* $S : L^2(\Omega) \to L^2(\Omega)$.

Thus, problem (1) can be reduced to

$$J(u) := \hat{J}(Su, u) \to \text{ min! } \quad \text{s.t. } u \in U_{ad}. \tag{6}$$

U_{ad} is nonempty, closed and convex and J is continuous and strongly convex.

Remark

Since Ω is convex, the solution $y \in V$ of (5) has the additional regularity $y \in H^2(\Omega)$. Hence, S is a linear, continuous mapping from $L^2(\Omega)$ to $H^2(\Omega)$.

Theorem

Problem (6) possesses a unique optimal solution $\bar{u}$ and $(S\bar{u}, \bar{u}) \in V \times U_{ad}$ is the unique optimal solution of (1). Further, system

$$\begin{aligned} (\bar{y} - q, y) - a(y, \bar{v}) &= 0 \quad \forall y \in V, \\ a(\bar{y}, v) - (\bar{u}, v) &= 0 \quad \forall v \in V, \\ \alpha(\bar{u}, u - \bar{u}) + (u - \bar{u}, \bar{v}) &\geq 0 \quad \forall u \in U_{ad} \end{aligned} \tag{7}$$

is necessary and sufficient for $\bar{u} \in U_{ad}$ to be optimal.

Remark

The inequality in system (7) is equivalent to

$$\bar{u} = P(\bar{u} - \sigma(\bar{v} + \alpha \bar{u})) \tag{8}$$

for any $\sigma > 0$*, where* P *is the* $L^2(\Omega)$*-ortho-projection onto* U_{ad}*. For sufficiently small* $\sigma > 0$ *the mapping*

$$Tu := P(u - \sigma((S')^*(Su - q) + \alpha u)) \quad \forall u \in U \tag{9}$$

is a contraction. (Here $S' = S$*)*

Especially for $\sigma = 1/\alpha$ formula (8) yields $\bar{u} = P(-\frac{1}{\alpha}\bar{v})$ i.e., $\bar{u}$ is expressed by the remaining variables (see Hinze[6]).

If only control constraints occur then P is easy to evaluate, not so in case of state constraints!

1.2. Case of Boundary Controls (see G/Winkler[5])

Consider the semi-linear boundary control problem

$$
\begin{array}{rcll}
\tilde{J}(y,u) := \frac{1}{2}\int\limits_{\Omega}(y-q)^2 & + & \frac{\alpha}{2}\int\limits_{\Gamma} u^2 \to \ \min! & \\
\text{subject to} \qquad -\Delta y + f(\cdot, y) & = & 0 \quad \text{in } \Omega, & (10)\\
\dfrac{\partial y}{\partial n} + y & = & u \quad \text{on } \Gamma,\ u \in U_{ad}\,. &
\end{array}
$$

with

$$
U_{ad} := \{ u \in L^2(\Gamma) \ : \ a \le u \le b \quad \text{a.e. on } \Gamma \}
$$

$q \in L^2(\Omega)$, a, $b \in \mathbb{R}$, $a < b$ given. $f \in C^2(\Omega \times \mathbb{R}) \to \mathbb{R}$, twice L-continuously differentiable and Minty monotone, i. e.

$$
(f(x,s) - f(x,t))(s-t) \ge 0 \quad \forall x \in \bar{\Omega},\ s,\ t \in \mathbb{R} \qquad (11)
$$

The state equation (10) is understood as weak formulation

$$a(y,v) + (f(\cdot,y),v)_\Omega = (u,v)_\Gamma \qquad \forall v \in V. \tag{12}$$

Lemma

For any $u \in U$ problem (12) possesses a unique solution $y \in V$. The operator $S : U \to V$ defined by $Su := y$ is Lipschitz continuous, i.e. there is some $c > 0$ such that

$$\|Su - S\tilde{u}\|_V \le c\,\|u - \tilde{u}\|_U \qquad \forall u, \tilde{u} \in U, \tag{13}$$

and weakly sequentially continuous, i.e.

$$u_k \rightharpoonup u \text{ in } U \quad \Longrightarrow \quad y_k \rightharpoonup y \text{ in } V \quad y_k := Su_k,\ y := Su. \tag{14}$$

Furthermore holds $\quad y_k \to y$ *in* $L^2(\Omega)$.

The control-to-state mapping leads to the reduced problem

$$J(u) := \frac{1}{2}\|Su - q\|^2_{0,\Omega} + \frac{\alpha}{2}\|u\|^2_{0,\Gamma} \rightarrow \ \text{min!} \quad \text{s.t. } u \in U_{ad} \quad (15)$$

Theorem

Problem (15) possesses an optimal solution $\bar{u} \in U_{ad}$. Any optimal solution $\bar{u} \in U_{ad}$ of (15) satisfies

$$\langle J'(\bar{u}), u - \bar{u}\rangle \geq 0 \qquad \forall\, u \in U_{ad}, \qquad (16)$$

which is equivalent to

$$(S\bar{u} - q, S'(u - \bar{u}))_\Omega + \alpha\,(\bar{u}, u - \bar{u})_\Gamma \geq 0 \qquad \forall\, u \in U_{ad}. \qquad (17)$$

2. Penalties for Control Bounds

Let

$$U_{ad} = \left\{ u \in L^2(\Omega) : u \le b \quad \text{a.e.in } \Omega \right\}$$

Barrier-penalty modification

$$\tilde{J}(u,s) := J(u) + \int_\Omega \phi(u(x) - b(x), s)dx \tag{18}$$

of the objective. Here $s > 0$ denotes the penalty parameter with $s \to 0+$ and $\phi : \mathbb{R} \to \overline{\mathbb{R}}$ some barrier-penalty function that satisfies

$$\frac{\partial}{\partial t}\phi(t,s) \;=\; \psi\left(\frac{t}{s}\right) \; \forall t \in \mathsf{dom}\,\phi(\cdot, s) \tag{19}$$

with an appropriate function $\psi : \mathbb{R} \to \overline{\mathbb{R}}$.

Results for the finite dimensional case

Problem

$J : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$, sufficiently smooth and consider

$$J(u) \to \ min! \qquad s.t. \quad u \in \mathbb{R}^n,\ g(u) \leq 0. \tag{20}$$

Auxiliary problem via general barrier-penalty approach

Problem

$$\tilde{J}(u,s) := J(u) + \sum_{i=1}^{m} \phi_i(g_i(u), s) \qquad s.t. \quad u \in \mathbb{R}^n \tag{21}$$

Theorem

Let $\bar{u}$ a local solution of (20) and $\bar{u}(s)$ a related local solution of (21). Further, assume that LICQ and strict complementarity locally hold. Then

$$\lim_{s \to 0+} \bar{u}(s) = \bar{u}$$

and with some $c > 0$ and $s_0 > 0$ we have

$$\|\bar{u}(s) - \bar{u}\| \le c\, s \quad \forall s \in (0, s_0].$$

Difficulties: $c = c(n)$ with $c(n) \to \infty$ for $n \to \infty$,
local application of implicit function theorem (using second order sufficiency conditions).
In optimal control: *no 'uniform' strict complementarity.*

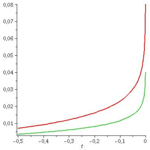
For optimal control consider either the quadratic loss

$$\psi(t) := \max\{0, t\}, \qquad \phi(t,s) = \frac{1}{2s}\max^2\{0, t\}$$

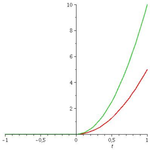
or the smoothed exact penalty

$$\psi(t) := \delta\left(1 + \frac{t}{\sqrt{1+t^2}}\right) \qquad \phi(t,s) = \delta\left(t + \sqrt{s^2 + t^2}\right).$$

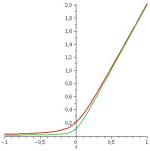
$\delta \geq \|\bar{v}\|_\infty + \varepsilon$ with some $\varepsilon > 0$, optimal multiplier $\bar{v}$.



log-barrier



quadratic loss



smoothed exact

Log-barrier analyzed by Schiela [8].

Back to optimal control

For the smoothed exact as well as for the quadratic loss functional under mild additional assumptions holds

Theorem (G/Kunz/Meischner[3])

For any $s > 0$ the penalty problem

$$\tilde{J}(u, s) \to \ min! \quad s.t. \ u \in L^2(\Omega) \tag{22}$$

possesses a unique solution $\bar{u}(s)$ and there holds $\lim\limits_{s \to 0+} \bar{u}(s) = \bar{u}$. Further, if only control constraints occur then

$$\|\bar{u} - \bar{u}(s)\| = O(\sqrt{s}).$$

Theorem (Improved estimate, Winkler[Dipl.thesis 2011])

Assume that only control constraints are considered. Then in case of the quadratic loss penalty some some $c > 0$ exists such that

$$\|u(s) - \bar{u}\| \leq \; c\, s$$

holds.

Proof (main idea)
The unconstrained auxiliary problem yields

$$J'(u(s)) + \psi\left(\frac{u(s) - b}{s}\right) = 0.$$

Direct calculations lead to

$$Tu(s) = Pu(s) \quad \text{with} \quad Tu := P(u - \sigma J'(u)).$$

Taking $\sigma > 0$ such that T contracts. With $\bar{u} = T\bar{u}$ and with $u^0 := u(s)$ we obtain

$$\begin{aligned} \|u(s) - \bar{u}\| &\leq \frac{1}{1-\kappa}\, \|Tu(s) - u(s)\| \\ &= \frac{1}{1-\kappa}\, \|Pu(s) - u(s)\|. \end{aligned}$$

Now,

$$\|u(s) - Pu(s)\| \leq s\,\|\bar{v}\| \qquad \forall\, s > 0 \qquad \text{(shown in C/K/M[3])}$$

completes the proof. □

Remark

The penalty multiplier $\bar{v}(s) := \psi((\bar{u}(s)-b)/s)$ *approximates the optimal multiplier* $\bar{v}$ *by*

$$\|\bar{v}(s)-\bar{v}\| \leq c\,\|\bar{u}(s)-\bar{u}\|.$$

Problem (22) by (18) leads to the necessary and sufficient optimality conditions

$$\begin{aligned} (\bar{y}(s)-q, y) - a(y, \bar{v}(s)) &= 0 \quad \forall y \in V, \\ -a(\bar{y}(s), v) + (\bar{u}(s), v) &= 0 \quad \forall v \in V, \\ \alpha\,\bar{u}(s) + \bar{v}(s) + \psi((\bar{u}(s)-b)/s) &= 0 \quad \text{a.e. in } \Omega. \end{aligned} \tag{23}$$

Theorem

For any $s > 0$ *system (23) possesses a unique solution* $(\bar{y}(s), \bar{v}(s), \bar{u}(s))$.

The structure of ψ allows to find $\bar{u}(s)$ in dependence of $\bar{v}(s)$. Due to $\bar{v} \in H^2(\Omega)$ and $H^2(\Omega) \hookrightarrow C(\bar{\Omega})$ this can be done by pointwise.

Let $\bar{u}(s) := g(\bar{v}(s), s)$. Then (23) leads to the reduced optimality system

$$\begin{array}{rcll} (\bar{y}(s) - q, y) - a(y, \bar{v}(s)) & = & 0 & \forall y \in V, \\ -a(\bar{y}(s), v) + (g(\bar{v}(s), s), v) & = & 0 & \forall v \in V. \end{array} \qquad (24)$$

Theorem

For any $s > 0$ the system (24) possesses a unique solution $(\bar{u}(s), \bar{v}(s)) \in V \times V$ and $\bar{u}(s) := g(\bar{v}(s), s)$ forms the optimal solution of the problem (22).

Consider the boundary control case (10).
Assumption: There exist $\delta, \varepsilon > 0$ such that

$$J''(\bar{u})[h,h] \geq \delta \|h\|_{0,\Gamma}^2 \qquad \forall\, \|h\|_{0,\Gamma} \leq \varepsilon.$$

Theorem

Let $\{s_k\} \subset \mathbb{R}_+$ with $s_k \to 0$ for $k \to \infty$. Then any related sequence $u_k := u(s_k)$ is bounded in U and therefore $\{u_k\}$ weakly compact. Any weakly convergent $\{u_k\}_{\mathcal{K}} \subset \{u_k\}$ converges also strongly in U to $\bar{u}$, i.e. $\lim_{k\in\mathcal{K},k\to\infty} \|u(s_k) - \bar{u}\|_{0,\Gamma} = 0$. Further, some $\sigma \in (0,1)$, $s_0 > 0$ exist such that

$$\|u(s_k) - \bar{u}\|_{0,\Gamma} \leq \frac{2\sigma}{1-\sigma} \|\bar{v}\|_{0,\Gamma}\, s_k \quad \forall k \in \mathcal{K},\ s_k \in (0, s_0],$$

where $\bar{v}$ denotes the optimal Lagrange multiplier at $\bar{u}$ related to the constraint $u \leq b$.

3. Discretization of Reduced Problems

Consider conforming finite element discretizations

$$V_h \subset V$$

applied to the control reduced systems (24).
This leads to the finite dimensional nonlinear systems

$$\begin{aligned} (\bar{y}_h - q, y_h) - a(y_h, \bar{v}_h) &= 0 \quad \forall y_h \in V_h, \\ -a(\bar{y}_h, v_h) + (g(\bar{v}_h, s), v_h) &= 0 \quad \forall v_h \in V_h. \end{aligned} \tag{25}$$

Like in the continuous case system (25) defines uniquely the solution $(\bar{y}_h(s), \bar{v}_h(s)) \in V_h \times V_h$.

Remark

Unlike in full discretization no a-priori discretization of the control space U is used. The penalty yields an approximate projection.

Convergence analysis

follows widely standard arguments of conforming FEM discretization.

Galerkin orthogonality

$$
\begin{aligned}
(\bar{y}_h - \bar{y}, y_h) - a(y_h, \bar{v}_h - \bar{v}) &= 0 \quad \forall y_h \in V_h, \\
-a(\bar{y}_h - \bar{y}, v_h) + (g(\bar{v}_h, s) - g(\bar{v}, s), v_h) &= 0 \quad \forall v_h \in V_h.
\end{aligned}
\tag{26}
$$

Define

Galerkin projections

$\tilde{y}_h$, $\tilde{v}_h \in V_h$ of $\bar{y}$, $\bar{v} \in V$ by

$$a(\bar{y} - \tilde{y}_h, v_h) = 0 \quad \forall v_h \in V_h$$

and

$$a(y_h, \bar{v} - \tilde{v}_h) = 0 \quad \forall y_h \in V_h.$$

Lemma (Schiela[8])

From (26) with the Galerkin projections follows

$$(\bar{y}_h - \bar{y}, \tilde{y}_h - \bar{y}_h) - (g(\bar{v}_h, s) - g(\bar{v}, s), \tilde{v}_h - \bar{v}_h) = 0.$$

Theorem

There exist some constant $c > 0$ such that

$$\|\bar{y} - \bar{y}_h\|^2 + \|\bar{v} - \bar{v}_h\|^2 \le c\left(\|\bar{y} - \tilde{y}_h\|^2 + \|\bar{v} - \tilde{v}_h\|^2\right)$$

with the Galerkin projections $\tilde{y}_h$, $\tilde{v}_h \in V_h$.

Remark

If no state constraints are given then this constant is independent of the embedding parameter $s > 0$.

4. Numerical Example

Consider piecewise linear conforming finite elements

$$V_h \subset V$$

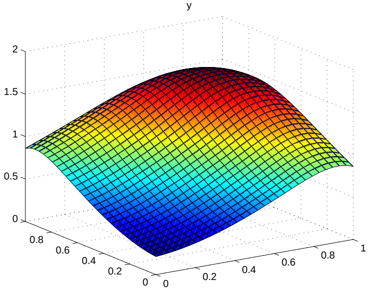
with a criss-cross triangulation applied to the control reduced systems (24).

Example 1

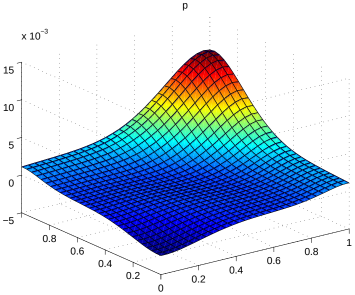
$$\begin{aligned} J(y,u) := \tfrac{1}{2}\|y-q\|_0^2 \quad &+ \quad \tfrac{\alpha}{2}\|u\|_0^2 \to \min! \\ \text{s.t.} \qquad -\Delta y \quad &= \quad u \quad \text{in } \Omega = [0,1]^2 \\ \frac{\partial y}{\partial n} + y \quad &= \quad 0 \quad \text{on } \Gamma, \\ u \in U_{ad} \quad &:= \quad \{u \in U : -4 \le u \le 12 \quad \text{a.e. in } \Omega\} \end{aligned}$$

with $q(x_1,x_2) = x_1 + x_2$.

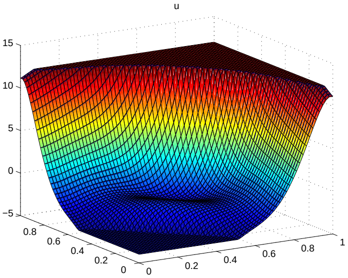
Solution obtained with loss penalty for $s = 10^{-10}$ and overall $N = 900$.



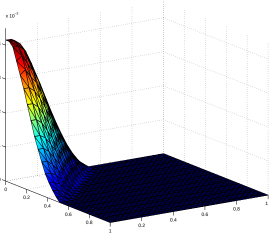
optimal state $\bar{y}$



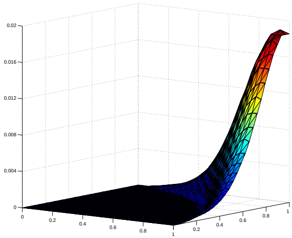
adjoint state $\bar{v}$



optimal control $\bar{u}$



multiplier for a

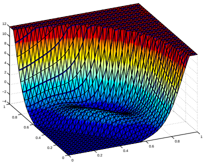


multiplier for b

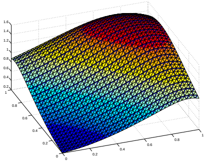
Similar results for the smoothed exact penalty

$$\psi(t) = \delta\,(1 + \frac{t}{\sqrt{1+t^2}}).$$

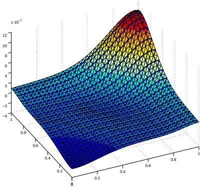
Unlike in the quadratic loss penalty case here feasibility is obtained for sufficiently small $s > 0$.



optimal control $\bar{u}$



optimal state $\bar{y}$



adjoint state $\bar{v}$

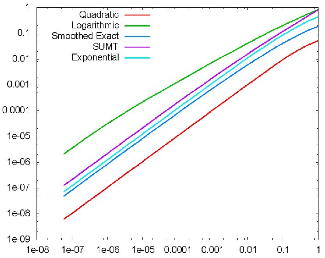
Experimental order of convergence in long-step path following with one Newton step per iteration and reduction

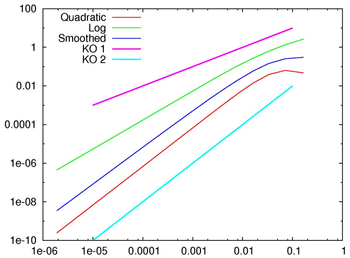
$$s_{k+1} = 0.5\, s_k$$

s	$EOC_0(y)$	$EOC_1(y)$	$EOC_0(u)$
1	1.0097	0.9419	1.3334
2^{-4}	1.00	0.9955	1.0279
2^{-8}	1.00	1.00	1.0026
2^{-12}	1.00	1.00	1.00
2^{-16}	1.00	1.00	1.00
2^{-20}	1.00	1.00	1.00

Accuracies for a 1D-example with known exact solution

	quadratic loss		log-barrier		smoothed exact	
s	$\|\bar{u}(s) - \bar{u}\|$	EOC	$\|\bar{u}(s) - \bar{u}\|$	EOC	$\|\bar{u}(s) - \bar{u}\|$	EOC
2^{-5}	0.1869	0.9171	4.4882	0.6552	0.8625	0.77546
2^{-10}	0.0061	0.9972	0.3717	0.7499	0.0408	0.9272
2^{-15}	0.0002	0.9999	0.0257	0.7864	0.0014	0.9869
2^{-20}	6.0315 E-06	1.0000	0.0014	0.9115	4.4722 E-05	0.9991
2^{-25}	1.8848 E-07	1.0000	4.5904 E-05	0.9954	1.3985 E-06	0.9999



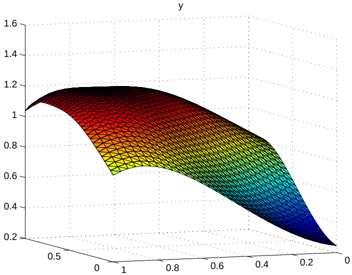


Modification of Example 1 by the additional state constraint

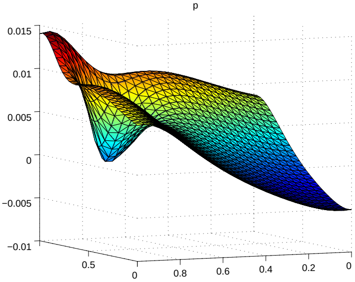
$$y(x) \leq 1.2 \qquad \text{in} \quad \Omega$$

and with no control bounds.

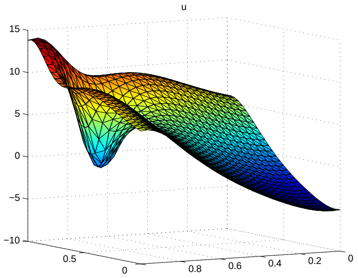
Numerical results for $N = 256$ and $s = 10^{-3}$

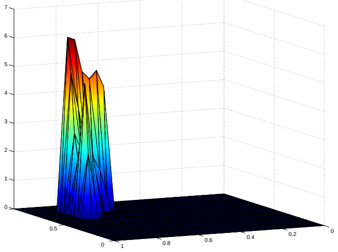


optimal state y



adjoint state v



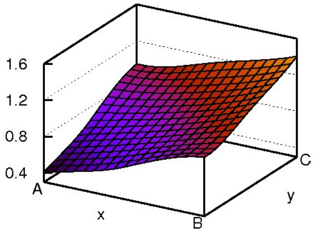


optimal control u **approximate multiplier**

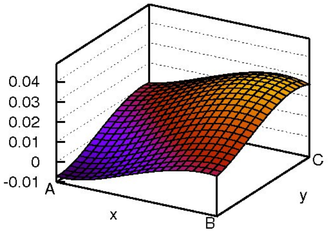
Example 2

$$
\begin{aligned}
J(y,u) := \tfrac{1}{2}\|y-q\|_0^2 \quad &+ \quad \tfrac{\alpha}{2}\|u\|_0^2 \to \min!\\
\text{s.t.} \quad -\Delta y + y^3 \quad &= \quad 0 \quad \text{in } \Omega = [0,1]^2,\\
\frac{\partial y}{\partial n} + y \quad &= \quad u \quad \text{on } \Gamma,\\
u \in U_{ad} \quad &:= \quad \{u \in L^2(\Gamma) : \; 0 \le u \le 2.2 \quad \text{a.e. on } \Gamma\}
\end{aligned}
$$

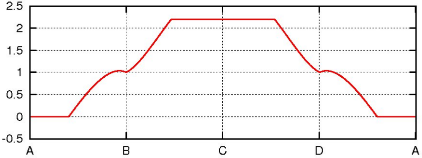
with $q(x_1,x_2) = x_1 + x_2\,.$



optimal state



adjoint state



optimal control

Solution obtained for $N = 200$ (each direction), $s = 10^{-40}$.

Obtained experimental order of convergence

s	$\|u_h(s) - \bar{u}_h\|$	EOC
2^{-05}	$1.93e-02$	0.84
2^{-10}	$6.79e-04$	0.99
2^{-15}	$2.13e-05$	1.00
2^{-20}	$6.66e-07$	1.00
2^{-25}	$2.08e-08$	1.00
2^{-30}	$6.49e-10$	1.01

Error $e(s) := \|u_h(s) - \bar{u}_h\|$

Outlook

- extend the analytical proof for the numerically observed convergence rate $O(s)$ to more general types of barrier-penalty functions;
- deriving specific solution techniques for the elimination of u_h for reduced FEM discretizations;
- finding sharp bounds for the radius of convergence of Newton's method;
- studying long-step path-following Newton methods;
- study of stability and convergence properties in the case of state constraints.

J.P. Dussault, P. - Shen, H. - Bandrauk, A.: *Penalty algorithms in Hilbert spaces.*, Acta Math. Sin., Engl. Ser. 23(2007), 229-236.

Gugat, M. - Herty, M.: *The smoothed–penalty algorithm for state constrained optimal control problems for partial differential equations*, Optim. Methods Softw. 25(2010), 573-599.

Grossmann, C. - Kunz, H. - Meischner, R.: *Elliptic control by penalty techniques with control reduction*, System modeling and optimization, 251-267, IFIP Adv. Inf. Commun. Technol. 312, Springer, Berlin, 2009.

Grossmann, C. - Winkler, M.: *Mesh-independent convergence of penalty methods applied to optimal control with partial differential equations*, (to appear in Optimization (2012), DOI: 10.1080/02331934.2012.655693).

Grossmann, C. - Winkler, M.: *The quadratic penalty method applied to boundary control of semilinear elliptic equations*, 2012, (submitted)

Hinze, M.: *A variational discretization concept in control constrained optimization: The linear-quadratic case*, Comput. Optim. Appl. 30(2005), 45-61.

Krumbiegel K. - Neitzel I. - Rösch A.; *Regularization for semilinear elliptic optimal control problems with pointwise state and control constraints*, Comput. Optim. Appl. (2010). 1-27.

Schiela, A.: *The control reduced interior point method*, PhD-thesis, FU Berlin, 2006.

Tröltzsch, F.: *Optimal Control of Partial Differential Equations. Theory, Methods and Applications.* Amer. Math. Soc. (AMS), Providence, RI, 2010.

Winkler, M.: *Strafmethoden für steuerbeschränkte Kontrollprobleme*, Diploma thesis, TU Dresden, May 2011.