On a new approach of using exponentially fitted splines for singularly perturbed convection-diffusion problems

Martin Schopf

TECHNISCHE UNIVERSITÄT DRESDEN
Fachrichtung Mathematik
Institut für Numerische Mathematik

14th April 2012

Convection-diffusion problem

$$\mathcal{L}u := -\varepsilon u'' - bu' + cu = f \quad \text{in } \Omega = (0,1),$$
$$u(0) = u(1) = 0,$$

$0 < \varepsilon \ll 1$, b, c, f smooth and for $x \in [0,1]$

$$0 < \beta \leq b(x),$$
$$c(x) \geq C_0 > 0,$$
$$c(x) + \frac{1}{2}b'(x) \geq C_1 > 0.$$

Standard methods **fail**!
Two techniques of **layer-adapted** discretization:

- Fine mesh to resolve layer
- Adequate base functions

Convection-diffusion problem

$$\begin{aligned} \mathcal{L}u := -\varepsilon u'' - bu' + cu = f \quad \text{in } \Omega = (0,1), \\ u(0) = u(1) = 0, \end{aligned}$$

$0 < \varepsilon \ll 1$, b, c, f smooth and for $x \in [0,1]$

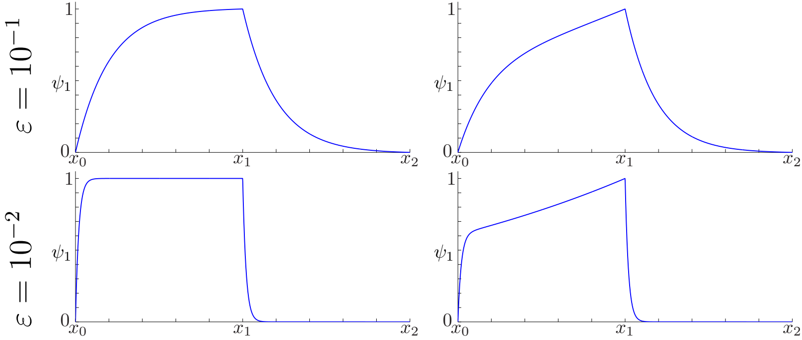
$$\begin{aligned} 0 < \beta \le b(x), \\ c(x) \ge C_0 > 0, \\ c(x) + \frac{1}{2}b'(x) \ge C_1 > 0. \end{aligned}$$

Standard methods **fail**!
Two techniques of **layer-adapted** discretization:

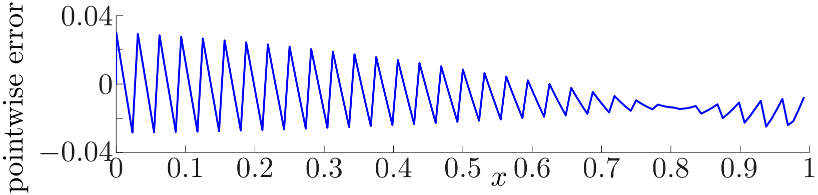
- Fine mesh to resolve layer
- Adequate base functions

Hemker, 1977: exponentially fitted spline ψ_i:
Let $\Omega_N = \{x_i\}_{i=0,\dots,N}$ equidistant grid; $\bar{b}$ and $\bar{c}$ piecewise constant approximations of b and c

$$\begin{aligned} \overline{\mathcal{L}}\psi_i := -\varepsilon\psi_i'' - \bar{b}\psi_i' + \bar{c}\psi_i = 0 \quad &\text{in } \Omega \setminus \Omega_N, \\ \psi_i(x_j) = \delta_{ij}, & \end{aligned} \qquad i = 0, \dots, N-1.$$



$\bar{b} = 1$, $\bar{c} = 0$ $\qquad$ $\bar{b} = 1$, $\bar{c} = 1$

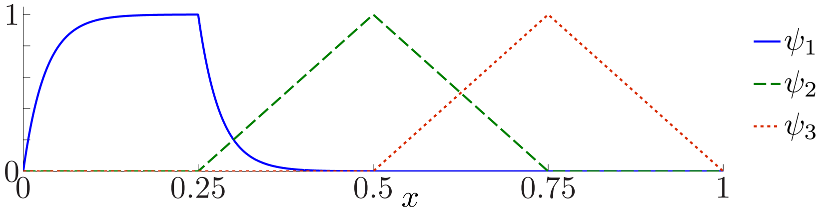


Error of the Ritz-Galerkin $\overline{\mathcal{L}}$-spline FEM for a constant coefficient problem, $\varepsilon = 10^{-8}$, $N = 32$

N	$\sqrt{\varepsilon}\left\vert u - u_N^{\overline{\mathcal{L}}\text{-spline}}\right\vert_1$		$\left\Vert u - u_N^{\overline{\mathcal{L}}\text{-spline}}\right\Vert_0$		$\left\Vert u - u_N^{\overline{\mathcal{L}}\text{-spline}}\right\Vert_\infty$	
	error	rate	error	rate	error	rate
32	1.88e-1		1.81e-2		2.93e-2	
		0.49		0.98		0.99
64	1.34e-1		9.17e-3		1.47e-2	
		0.49		0.99		0.99
128	9.53e-2		4.61e-3		7.37e-3	
		⋮		⋮		⋮
⋮	⋮		⋮		⋮	
		0.50		1.00		1.00
32768	5.98e-3		1.81e-5		2.89e-5	

$$\left\| u - u_N^{\overline{\mathcal{L}}\text{-spline}} \right\|_\varepsilon \le C N^{-1/2} \quad \text{is sharp!} \qquad (*)$$

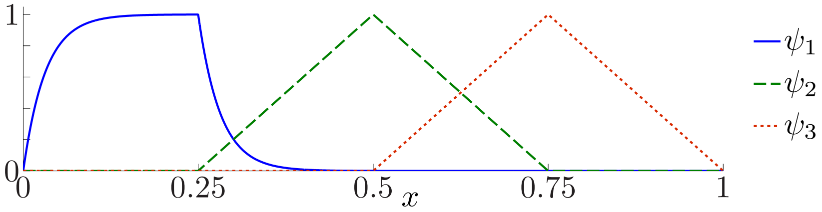
Stynes and O'Riordan 1991: use $\overline{\mathcal{L}}$-splines only in the layer region: in all elements that cover $[0, 2\varepsilon/\beta \ln(1/\varepsilon)]$.



- First order convergence in L_2 and for interpolation error
- No improvement over (*) uniformly in ε

$$\left\| u - u_N^{\overline{\mathcal{L}}\text{-spline}} \right\|_\varepsilon \le C N^{-1/2} \quad \text{is sharp!} \qquad (*)$$

Stynes and O'Riordan 1991: use $\overline{\mathcal{L}}$-splines only in the layer region: in all elements that cover $[0, 2\varepsilon/\beta \ln(1/\varepsilon)]$.



- First order convergence in L_2 and for interpolation error
- No improvement over (*) uniformly in ε

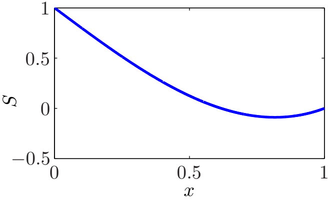
Solution Decomposition: Let b, c, f be sufficiently smooth, then

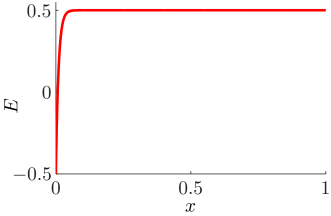
$$u = S + E$$

with

$$\begin{aligned} \mathcal{L}S &= f \text{ in } (0,1), \\ -bS'(0) + cS(0) &= f(0), \\ S(1) &= 0, \\ \|S\|_2 &\le C, \end{aligned}$$

$$\begin{aligned} \mathcal{L}E &= 0 \text{ in } (0,1), \\ E(0) &= -S(0), \\ E(1) &= 0, \\ \left|E^{(i)}(x)\right| &\le C\varepsilon^{-i}\mathrm{e}^{-\beta x/\varepsilon}. \end{aligned}$$





Derivation of the method: weak formulations:

$$V := \{v \in H^1(\Omega) : \; v(1) = 0\}$$
$$a(w,v) := \varepsilon(w',v') + (-bw' + cw, v)$$

$S, E \in V$ s.t.

$$\underbrace{a^S(S,v)}_{:= a(S,v) + \varepsilon \frac{c(0)}{b(0)} S(0)v(0)} = f^S(v) \quad \text{for all } v \in V,$$
$$f^S(v) := (f,v) + \varepsilon \frac{f(0)}{b(0)} v(0),$$

$$a(E,v) = 0 \quad \text{for all } v \in H_0^1(\Omega),$$
$$E(0) = -S(0),$$

Bilinear forms coercive in V with respect to:

$$\|\cdot\|_{\varepsilon,0}, \qquad \|v\|_{\varepsilon,0}^2 := \|v\|_\varepsilon^2 + |v(0)|^2.$$

$$\|\cdot\|_\varepsilon, \qquad \|v\|_\varepsilon^2 := \varepsilon|v|_1^2 + \|v\|_0^2.$$

Derivation of the method: weak formulations:

$$V := \{v \in H^1(\Omega) :\ v(1) = 0\}$$
$$a(w, v) := \varepsilon(w', v') + (-bw' + cw, v)$$

$S, E \in V$ s.t.

$$\underbrace{a^S(S, v)} = f^S(v) \quad \text{for all } v \in V,$$
$$:= a(S, v) + \varepsilon\frac{c(0)}{b(0)}S(0)v(0)$$
$$f^S(v) := (f, v) + \varepsilon\frac{f(0)}{b(0)}v(0),$$

$$a(E, v) = 0 \quad \text{for all } v \in H_0^1(\Omega),$$
$$E(0) = -S(0),$$

Bilinear forms coercive in V with respect to:

$$\|\cdot\|_{\varepsilon,0},$$
$$\|v\|_{\varepsilon,0}^2 := \|v\|_\varepsilon^2 + |v(0)|^2.$$

$$\|\cdot\|_\varepsilon,$$
$$\|v\|_\varepsilon^2 := \varepsilon|v|_1^2 + \|v\|_0^2.$$

Derivation of the method: Discretization:

$$u_N = S_N + E_N$$

with

$$S_N \in V_N^S (\subset V) \quad \text{span of standard hat functions over } \{x_i\}_{i=0}^{N-1}$$

$$a^S(S_N, v) = f^S(v) \quad \text{for all } v \in V_N^S,$$

$$E_N \in V_N^E (\subset V) \quad \text{span of } \overline{\mathcal{L}} \text{ splines } \psi_i \text{ over } \{x_i\}_{i=0}^{N-1}$$

$$a(E_N, v) = 0 \quad \text{for all } v \in V_N^E \cap H_0^1(\Omega),$$

$$E_N(0) = -S_N(0).$$

Use $\overline{\mathcal{L}}$ splines only for the approximation of the layer component!

Derivation of the method: Discretization:

$$u_N = S_N + E_N$$

with

$$S_N \in V_N^S (\subset V) \quad \text{span of standard hat functions over } \{x_i\}_{i=0}^{N-1}$$
$$a^S(S_N, v) = f^S(v) \quad \text{for all } v \in V_N^S,$$

$$E_N \in V_N^E (\subset V) \quad \text{span of } \overline{\mathcal{L}} \text{ splines } \psi_i \text{ over } \{x_i\}_{i=0}^{N-1}$$
$$a(E_N, v) = 0 \quad \text{for all } v \in V_N^E \cap H_0^1(\Omega),$$
$$E_N(0) = -S_N(0).$$

Use $\overline{\mathcal{L}}$ splines only for the approximation of the layer component!

Error analysis: Smooth part S:

Standard arguments give

$$\|S - S_N\|_\varepsilon \le CN^{-1},$$
$$|S(0) - S_N(0)| \le CN^{-1}.$$

Layer part E:

$$E^I(x) := \sum_{i=0}^{N-1} E(x_i)\psi_i(x)$$

Using stability estimates (estimate of the Green's function):

$$\begin{aligned}
\|(E^I)'\|_{L_1(0,1)} &\le \|E'\|_{L_1(0,1)} + \|(E^I - E)'\|_{L_1(0,1)} \\
&\le \|E'\|_{L_1(0,1)} + \frac{2}{\beta}\|\overline{\mathcal{L}}(E^I - E)\|_{L_1(0,1)} \\
&\le C\big(\varepsilon\|E''\|_{L_1(0,1)} + \|E'\|_{L_1(0,1)} + \|E\|_{L_1(0,1)}\big) \le C.
\end{aligned}$$

Error analysis: Layer part E (2):

Interpolation error:

$$\begin{aligned}\|E - E^I\|_\infty &\le C\|\mathcal{L}(E - E^I)\|_{L_1(0,1)} = C\|\mathcal{L}E^I - \overline{\mathcal{L}}E^I\|_{L_1(0,1)} \\ &\le C\int_0^1 \left|\bar{b}(x) - b(x)\right|\left|(E^I)'(x)\right| + \left|\bar{c}(x) - c(x)\right|\left|E^I(x)\right|\mathrm{d}x. \\ &\le C\left(\|\bar{b} - b\|_\infty + \|\bar{c} - c\|_\infty\right) \le CN^{-1}.\end{aligned}$$

In energy norm: Use $\mathcal{L}E = \overline{\mathcal{L}}E^I = 0$ in a weak sense (tested with $E - E^I$) and the L_∞ estimate to obtain

$$\|E - E^I\|_\varepsilon \le CN^{-1}.$$

Error analysis: Layer part E (3):

Discrete error:
Let $\omega \in V_N^E$ with $\omega(0) = 0$ and $\|\bar{b} - b\|_\infty \le Ch_N$, then

$$\|\omega\|_\varepsilon \le CN^{-1/2}\left(\sum_{j=1}^{N-1}\omega^2(x_j)\right)^{1/2} \le C \max_{j=1,\dots,N-1}|\omega(x_j)|.$$

Proof: use $0 = (\overline{\mathcal{L}}\omega, \omega)_{(x_i,x_{i+1})}$ and integration by parts.

- Consider $\omega(x) := \big(E^I - E_N\big)(x) - \big(E^I - E_N\big)(0)\psi_0(x)$
- Estimate $\big(E^I - E_N\big)(0)\psi_0(x)$ (for $\epsilon \le CN^{-1}$)
- Discrete L_∞ estimates available
- Use comparison principle to control the influence of $E_N(0) = -S_N(0) \approx -S(0)$.

Error analysis: Layer part E (3):

Discrete error:
Let $\omega \in V_N^E$ with $\omega(0) = 0$ and $\|\bar{b} - b\|_\infty \le C h_N$, then

$$\|\omega\|_\varepsilon \le C N^{-1/2} \left(\sum_{j=1}^{N-1} \omega^2(x_j) \right)^{1/2} \le C \max_{j=1,\dots,N-1} |\omega(x_j)|.$$

Proof: use $0 = (\overline{\mathcal{L}}\omega, \omega)_{(x_i, x_{i+1})}$ and integration by parts.

- Consider $\omega(x) := \big(E^I - E_N\big)(x) - \big(E^I - E_N\big)(0)\psi_0(x)$
- Estimate $\big(E^I - E_N\big)(0)\psi_0(x)$ (for $\epsilon \le C N^{-1}$)
- Discrete L_∞ estimates available
- Use comparison principle to control the influence of $E_N(0) = -S_N(0) \approx -S(0)$.

Result:

Assume $\|S\|_2 \le C$ and the bounds on E from the solution decomposition. Suppose $\varepsilon \le C_2 h_N$ and let $\bar{b}$ and $\bar{c}$ be $\mathcal{O}(N^{-1})$ piecewise constant approximations of b and c, respectively. Then for $u_N := S_N + E_N$ it holds

$$\|u - u_N\|_\varepsilon \le C N^{-1}.$$

Problem: amount of calculations is **doubled**!

Result:

Assume $\|S\|_2 \leq C$ and the bounds on E from the solution decomposition. Suppose $\varepsilon \leq C_2 h_N$ and let $\bar{b}$ and $\bar{c}$ be $\mathcal{O}(N^{-1})$ piecewise constant approximations of b and c, respectively. Then for $u_N := S_N + E_N$ it holds

$$\|u - u_N\|_\varepsilon \leq C N^{-1}.$$

Problem: amount of calculations is **doubled**!

Work reduction(1):

$\ell := \min\{i : x_i \geq \varepsilon/\beta \ln N\} \geq 1 \quad \Rightarrow \quad |E(x)| \leq CN^{-1}$ for $x \geq x_\ell$.

New interpolant:

$$\tilde{E}^I(x) := \sum_{i=0}^{\ell-1} E(x_i)\psi_i(x).$$

Clearly $\tilde{E}^I(x) = 0$ for $x \geq x_\ell$, $E^I - \tilde{E}^I \in V_N^E$ vanishes at the end points of $(0,1)$, hence

$$\begin{aligned} \|E^I - \tilde{E}^I\|_\varepsilon &\leq C \max_{j=\ell,\dots,N-1} \left|E^I(x_j) - \tilde{E}^I(x_j)\right| \\ &\leq C \max_{j=\ell,\dots,N-1} \left|E(x_j)\right| \leq CN^{-1}. \end{aligned}$$

$$\|E - \tilde{E}^I\|_\varepsilon \leq \|E - E^I\|_\varepsilon + \|E^I - \tilde{E}^I\|_\varepsilon \leq CN^{-1}.$$

Work reduction(1):

$\ell := \min\{i : x_i \geq \varepsilon/\beta \ln N\} \geq 1 \quad \Rightarrow \quad |E(x)| \leq CN^{-1}$ for $x \geq x_\ell$.

New interpolant:

$$\tilde{E}^I(x) := \sum_{i=0}^{\ell-1} E(x_i)\psi_i(x).$$

Clearly $\tilde{E}^I(x) = 0$ for $x \geq x_\ell$, $E^I - \tilde{E}^I \in V_N^E$ vanishes at the end points of $(0,1)$, hence

$$\begin{aligned} \|E^I - \tilde{E}^I\|_\varepsilon &\leq C \max_{j=\ell,\dots,N-1} \left|E^I(x_j) - \tilde{E}^I(x_j)\right| \\ &\leq C \max_{j=\ell,\dots,N-1} \left|E(x_j)\right| \leq CN^{-1}. \end{aligned}$$

$$\|E - \tilde{E}^I\|_\varepsilon \leq \|E - E^I\|_\varepsilon + \|E^I - \tilde{E}^I\|_\varepsilon \leq CN^{-1}.$$

Work reduction(2):

Assume $\|S\|_2 \leq C$ and the bounds on E from the solution decomposition. Suppose $\varepsilon \leq C_2 h_N$ and let $\bar{b}$ and $\bar{c}$ be $\mathcal{O}(N^{-1})$ piecewise constant approximations of b and c, respectively. Let $\tilde{E}_N \in \tilde{V}_N^E := \operatorname{span}\{\psi_i,\ i = 0, \ldots, \ell - 1\}$ s.t. $\tilde{E}_N(0) = -S_N(0)$ and

$$a(\tilde{E}_N, v) = 0, \quad \text{for all } v \in \tilde{V}_N^E \cap H_0^1(\Omega).$$

Then $\tilde{u}_N := S_N + \tilde{E}_N$ satisfies

$$\|u - \tilde{u}_N\|_\varepsilon \leq CN^{-1}.$$

In the case $x_1 := \frac{1}{N} \geq \frac{\varepsilon}{\beta} \ln N$ the approximation $\tilde{u}_N$ can be calculated in a simple **post-processing**:

$$\tilde{u}_N(x) = S_N(x) - S_N(0)\psi_0(x).$$

Work reduction(2):

Assume $\|S\|_2 \leq C$ and the bounds on E from the solution decomposition. Suppose $\varepsilon \leq C_2 h_N$ and let $\bar{b}$ and $\bar{c}$ be $\mathcal{O}(N^{-1})$ piecewise constant approximations of b and c, respectively. Let $\tilde{E}_N \in \tilde{V}_N^E := \mathrm{span}\{\psi_i,\ i = 0, \ldots, \ell - 1\}$ s.t. $\tilde{E}_N(0) = -S_N(0)$ and

$$a(\tilde{E}_N, v) = 0, \quad \text{for all } v \in \tilde{V}_N^E \cap H_0^1(\Omega).$$

Then $\tilde{u}_N := S_N + \tilde{E}_N$ satisfies

$$\|u - \tilde{u}_N\|_\varepsilon \leq CN^{-1}.$$

In the case $x_1 := \frac{1}{N} \geq \frac{\varepsilon}{\beta} \ln N$ the approximation $\tilde{u}_N$ can be calculated in a simple **post-processing**:

$$\tilde{u}_N(x) = S_N(x) - S_N(0)\psi_0(x).$$

Test problem 1

$$\mathcal{L}u := -\varepsilon u'' - u' + \frac{1}{2}u = f \quad \text{in } \Omega = (0,1),$$
$$u(0) = u(1) = 0,$$

where $0 < \varepsilon \ll 1$ and f such that $u = S + E$ with

$$S(x) = 1 - 2x + x^3,$$
$$E(x) = \frac{1}{1 - \mathrm{e}^{-2\sigma}} \left(\mathrm{e}^{-x/(2\varepsilon)+\sigma(x-2)} - \mathrm{e}^{-\left(1/(2\varepsilon)+\sigma\right)x} \right),$$

and $\sigma = \sqrt{b^2 + 4\varepsilon c}/(2\varepsilon)$.

N	$\lvert S - S_N \rvert_1$		$\lVert S - S_N \rVert_0$		$\lVert S - S_N \rVert_\infty$	
	error	rate	error	rate	error	rate
32	3.12e-2		4.08e-4		7.25e-4	
64	1.56e-2	1.00	1.02e-4	2.00	1.82e-4	2.00
128	7.81e-3	1.00	2.55e-5	2.00	4.57e-5	2.00
256	3.91e-3	1.00	6.38e-6	2.00	1.14e-5	2.00
512	1.95e-3	1.00	1.59e-6	2.00	2.86e-6	2.00
1024	9.77e-4	1.00	3.99e-7	2.00	7.15e-7	2.00
2048	4.88e-4	1.00	9.97e-8	2.00	1.79e-7	2.00
4096	2.44e-4	1.00	2.49e-8	2.00	4.47e-8	2.00
8192	1.22e-4	1.00	6.23e-9	2.00	1.12e-8	2.00
16384	6.10e-5	1.00	1.56e-9	2.00	2.79e-9	2.00
32768	3.05e-5	1.00	3.89e-10	2.00	6.98e-10	2.00

errors and convergence rates for $S - S_N$, $\varepsilon = 10^{-2k}$, $k = 2, \dots, 6$

Test problem 2

$$\mathcal{L}u := -\varepsilon u'' - (1+2x)u' + (1+\sqrt{x})u = 0 \quad \text{in } \Omega = (0,1),$$
$$u(0) = 1, \quad u(1) = 0,$$

where $0 < \varepsilon \ll 1$.

N	$\varepsilon = 10^{-8}$				$\varepsilon \in \{10^{-6}, 10^{-8}, 10^{-10}, 10^{-12}\}$	
	$\sqrt{\varepsilon}\|G(u_N^R) - \tilde{u}'_N\|_0$		$\|u_N^R - \tilde{u}_N\|_0$		$\sqrt{\varepsilon}\|G(u_N^R) - \tilde{u}'_N\|_0$	$\|u_N^R - \tilde{u}_N\|_0$
	error	rate	error	rate	max error	max error
32	1.47e-2		1.53e-6		1.47e-2	1.53e-5
		0.96		0.98		
64	7.59e-3		7.72e-7		7.59e-3	7.72e-6
		0.98		0.99		
128	3.85e-3		3.88e-7		3.85e-3	3.88e-6
		0.99		1.00		
256	1.94e-3		1.95e-7		1.94e-3	1.95e-6
		0.99		1.00		
512	9.73e-4		9.75e-8		9.73e-4	9.73e-7
		1.00		1.00		
1024	4.87e-4		4.88e-8		4.87e-4	4.86e-7
		1.00		1.00		
2048	2.44e-4		2.44e-8		2.44e-4	2.42e-7
		1.00		1.00		
4096	1.22e-4		1.22e-8		1.22e-4	1.20e-7
		1.00		1.00		
8192	6.10e-5		6.10e-9		6.10e-5	5.93e-8
		1.00		1.00		
16384	3.05e-5		3.05e-9		3.05e-5	2.88e-8
		1.00		1.00		
32768	1.53e-5		1.53e-9		1.53e-5	1.35e-8

estimated errors and convergence rates for variable coefficients