TECHNISCHE UNIVERSITÄT DRESDEN

Optimal Control Subject to a Singularly Perturbed Convection-Diffusion Equation

Christian Reibiger, Hans-Görg Roos

Consider

$$\min_{y,u} J(y,q) := \min_{y,q} \frac{1}{2}\|y - y_d\|_0^2 + \frac{\lambda}{2}\|q\|_0^2,$$

$$Ly := -\varepsilon y'' + by' + cy = f + q \text{ in } (0,1),$$
$$y(0) = y(1) = 0.$$

Assume

$$0 < \varepsilon \ll 1,$$
$$\lambda > 0,$$
$$|b(x)| \geq \beta > 0,$$
$$c > 0,$$
$$b, c, f, y_d \text{ sufficiently smooth.}$$

Using the adjoint state p (cf. Tröltzsch 2009)

$$\begin{aligned} \lambda q + p &= 0, \\ L^* p = -\varepsilon p'' - bp' + (c - b')p &= y - y_0, \\ p(0) = p(1) &= 0 \end{aligned}$$

gives equivalent problem

$$\begin{aligned} -\varepsilon y'' + by' + cy \qquad + \frac{1}{\lambda} p &= f, \qquad y(0) = y(1) = 0, \\ -\varepsilon p'' - bp' + (c - b')p - y &= -y_0, \quad p(0) = p(1) = 0. \end{aligned}$$

This leads to a system

$$
\begin{aligned}
-\varepsilon u_1'' + a_1 u_1' + b_{11} u_1 + b_{12} u_2 &= f_1, \quad u_1(0) = u_1(1) = 0,\\
-\varepsilon u_2'' - a_2 u_2' + b_{22} u_2 - b_{21} u_1 &= f_2, \quad u_2(0) = u_2(1) = 0,
\end{aligned}
$$

assuming

$$
\begin{aligned}
a_1,\ a_2 &\geq \alpha > 0,\\
b_{11},\ b_{22} &\geq 0,\\
b_{12} b_{21} &> 0, \quad b_{12},\ b_{21} \geq \beta > 0
\end{aligned}
$$

Theorem

If the assumptions

$$2b_{11}b_{21} - (a_1 b_{21} + \varepsilon b_{21}')' \geq 0$$
$$2b_{22}b_{12} + (a_2 b_{12} - \varepsilon b_{12}')' \geq 0$$

hold the system has an unique weak solution $u \in H_0^1(0,1)$.

Theorem

Assume

$$b_{11}b_{22} + b_{12}b_{21} - a_2 b_{12}\left(\frac{b_{11}}{b_{12}}\right)' \geq 0 \text{ or}$$

$$b_{11}b_{22} + b_{12}b_{21} + a_1 b_{21}\left(\frac{b_{22}}{b_{21}}\right)' \geq 0.$$

Then the reduced problem ($\varepsilon = 0$) has an unique solution u with

$$|u_1|_{k+1} + |u_2|_{k+1} < C(\|f_1\|_k + \|f_2\|_k),$$

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$$|u_1|_{k+1} + |u_2|_{k+1} < C(\|f_1\|_k + \|f_2\|_k),$$

- Constant coefficients fulfill the prerequisites of the last two theorems.

Decompose u in

$$u_1 = \tilde{S}_1 + E_{10} + E_{11} + R_{1\,n}, \quad u_2 = \tilde{S}_2 + E_{20} + E_{21} + R_{2\,n}$$

$$\tilde{S}_1 = \sum_{k=0}^{n} \varepsilon^k u_{1,k}, \quad E_{10} = \sum_{k=0}^{n} \varepsilon^k v_k(\xi), \quad E_{11} = \sum_{k=0}^{n} \varepsilon^k w_k(\eta),$$

$$\tilde{S}_2 = \sum_{k=0}^{n} \varepsilon^k u_{2,k}, \quad E_{20} = \sum_{k=0}^{n} \varepsilon^k r_k(\xi), \quad E_{21} = \sum_{k=0}^{n} \varepsilon^k s_k(\eta)$$

with the local variables $\xi := x/\varepsilon$, $\eta := (1-x)/\varepsilon$.

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$$L\left(\tilde{S}_1,\ \tilde{S}_2\right) = f + \mathcal{O}(\varepsilon^{n+1}), \qquad \tilde{S}_1(0) = 0,\ \tilde{S}_2(1) = 0,$$

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$$L\left(\tilde{E_{10}},\ \tilde{E}_{20}\right) = \mathcal{O}(\varepsilon^{n+1}), \qquad E_{20}(0) = -\tilde{S}_2(0),$$

$$L\left(\tilde{E_{11}},\ \tilde{E}_{21}\right) = \mathcal{O}(\varepsilon^{n+1}), \qquad E_{11}(1) = -\tilde{S}_1(1).$$

This leads to (cf. H.-G. Roos, M.Stynes, L.Tobiska 2008)

$$
\begin{aligned}
a_1 u_{1,0}' + b_{11} u_{1,0} + b_{12} u_{2,0} &= f_1 \quad u_{1,0}(0) = 0,\\
-a_2 u_{2,0}' + b_{22} u_{2,0} - b_{21} u_{1,0} &= f_2 \quad u_{2,0}(1) = 0,
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$k \geq 0$:

$$-v_k'' + \tilde{a}_{1,0} v_k' = g_{k1}\left(v_1,\, v_1',\, \dots,\, v_{k-1},\, v_{k-1}',\, r_1,\, \dots,\, r_{k-1}\right) \qquad \lim_{\xi\to\infty} v_k(\xi) = 0,$$

$$-r_k'' - \tilde{a}_{2,0} r_k' = g_{k2}\left(r_1,\, r_1',\, \dots,\, r_{k-1},\, r_{k-1}',\, v_1,\, \dots,\, v_{k-1}\right) \qquad \lim_{\xi\to\infty} r_k(\xi) = 0, \quad r_k(0) = -u_{2,k}(0),$$

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$$-w_k'' - \hat{a}_{1,0} w_k' = h_{k1}\left(w_1,\, w_1',\, \dots,\, w_{k-1},\, w_{k-1}',\, s_1,\, \dots,\, s_{k-1}\right) \qquad \lim_{\eta\to\infty} w_k(\eta) = 0, \quad w_k(0) = -u_{1,k}(1),$$

$$-s_k'' + \hat{a}_{2,0} s_k' = h_{k2}\left(s_1,\, s_1',\, \dots,\, s_{k-1},\, s_{k-j}',\, w_1,\, \dots,\, w_{k-1}\right) \qquad \lim_{\eta\to\infty} s_k(\eta) = 0,$$

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$k \geq 0$:

$$-v_k'' + \tilde{a}_{1,0} v_k' = g_{k1}\left(v_1, v_1', \ldots, v_{k-1}, v_{k-1}', r_1, \ldots, r_{k-1}\right) \qquad \lim_{\xi\to\infty} v_k(\xi) = 0,$$

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$$-s_k'' + \hat{a}_{2,0} s_k' = h_{k2}\left(s_1, s_1', \ldots, s_{k-1}, s_{k-j}', w_1, \ldots, w_{k-1}\right) \qquad \lim_{\eta\to\infty} s_k(\eta) = 0,$$

$k \geq 1$:

$$a_1 u_{1,k}' + b_{11} u_{1,k} + b_{12} u_{2,k} = u_{1,k-1}'' \quad u_{1,k}(0) = -v_k(0),$$
$$-a_2 u_{2,k}' + b_{22} u_{2,k} - b_{21} u_{1,k} = u_{2,k-1}'' \quad u_{2,k}(1) = -s_k(1).$$

Corollary

The terms of boundary layer correction have the form

$$v_i(\xi) \in \mathbb{P}_{i-1}(\xi)\, e^{-a_2(0)\xi}, \quad w_i(\eta) \in \mathbb{P}_i(\eta)\, e^{-a_1(1)\eta},$$
$$r_i(\xi) \in \mathbb{P}_i(\xi)\, e^{-a_2(0)\xi}, \qquad s_i(\eta) \in \mathbb{P}_{i-1}(\eta)\, e^{-a_1(1)\eta},$$

where $\mathbb{P}_n(x)$ *denotes the set of polynomials in the unknown* x *of degree less or equal to* n*.*

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where $\mathbb{P}_n(x)$ *denotes the set of polynomials in the unknown* x *of degree less or equal to* n.
In particular

$$v_0(\xi) = 0, \qquad w_0(\eta) = -u_{1,0}(1)\, e^{-a_1(1)\eta},$$
$$r_0(\xi) = -u_{2,0}(0)\, e^{-a_2(0)\xi}, \quad s_0(\eta) = 0$$

holds. Therefore E_{10} *and* E_{21} *are only weak layers.*

Corollary

The remainders $R_{i\,n},\ i \in \{1,2\}$ *satisfy*

$$
\begin{aligned}
R_{i\,n}(0) &\in \mathcal{O}(\varepsilon^{n+1}),\\
R_{i\,n}(1) &\in \mathcal{O}(\varepsilon^{n+1}),\\
L(R_{1\,n}, R_{2\,n}) &\in \mathcal{O}(\varepsilon^{n+1/2}).
\end{aligned}
$$

Corollary

The remainders $R_{i\,n}$, $i \in \{1,2\}$ satisfy

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$$L(R_{1\,n}, R_{2\,n}) \in \mathcal{O}(\varepsilon^{n+1/2}).$$

This provides

$$|R_{i\,n}|_1 \leq \varepsilon^{-1} \|f_{i\,n}\| \leq \varepsilon^n C,$$
$$|R_{i\,n}|_2 \leq \varepsilon^{-1} \left(\|f_{i\,n}\| + \|R_{i\,n}\|_1\right) \leq \varepsilon^{n-1} C.$$

Theorem

For sufficient smooth coefficients a, b and inhomogeneity f we can decompose the solution in

$$u_1 = S_1 + E_{10} + E_{11},$$
$$u_2 = S_2 + E_{20} + E_{21},$$

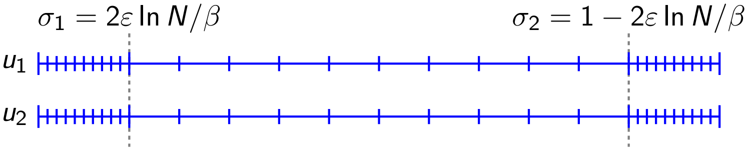
with

$$\left\| S_1^{(k)} \right\|, \left\| S_2^{(k)} \right\| \leq C,$$
$$\left| E_{10}^{(k)} \right| \leq C\varepsilon^{1-k} \mathcal{E}_l(x), \quad \left| E_{11}^{(k)} \right| \leq C\varepsilon^{-k} \mathcal{E}_r(x),$$
$$\left| E_{20}^{(k)} \right| \leq C\varepsilon^{-k} \mathcal{E}_l(x), \quad \left| E_{21}^{(k)} \right| \leq C\varepsilon^{1-k} \mathcal{E}_r(x)$$

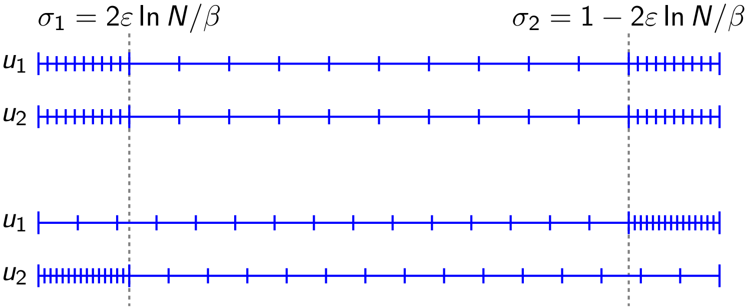
for $k \leq 2$. Where the generic constant C is independent of ε and

$$\mathcal{E}_l(x) := e^{-\alpha \frac{x}{\varepsilon}}, \quad \mathcal{E}_r(x) := e^{-\alpha \frac{1-x}{\varepsilon}}.$$

Used Mesh


$\sigma_1 = 2\varepsilon \ln N/\beta$
$\sigma_2 = 1 - 2\varepsilon \ln N/\beta$
u_1
u_2

Used Mesh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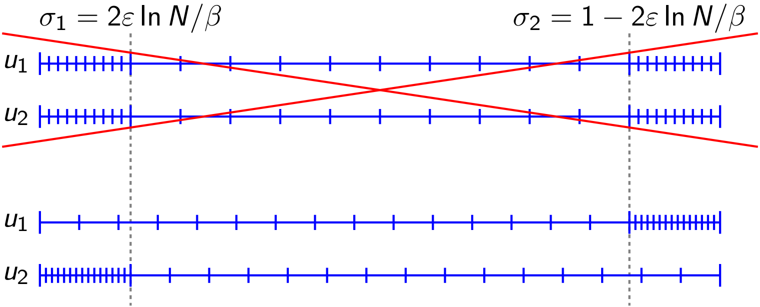
u_1
u_2
u_1
u_2

Used Mesh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u_1
u_2
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u_2

Theorem

Provided the former estimates of the asymptotic expansion hold we have for the nodal interpolant u^I to the solution u

$$\|u - u^I\|_\varepsilon \leq CN^{-1} \ln N,$$

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Proof.

Estimate as in the standard case

$$\|S_1 - S_1^I\|_\varepsilon + \|E_{11} - E_{11}^I\|_\varepsilon \leq CN^{-1} \ln N.$$

By standard interpolation results we can estimate

$$\|E_{10} - E_{10}^I\|_0 \leq \left\{ \begin{matrix} \tilde{C}h|E_{10}|_1 \leq C\varepsilon^{1/2}N^{-1}, & \varepsilon < N^{-1} \\ \tilde{C}h^2|E_{10}|_2 \leq C\varepsilon^{-1/2}N^{-2}, & \varepsilon \geq N^{-1} \end{matrix} \right\} \leq CN^{-3/2},$$

$$|E_{10} - E_{10}^I|_1 \leq \tilde{C}h|E_{10}|_2 \leq C\varepsilon^{-1/2}N^{-1}.$$

□

Theorem

If the former estimates of the asymptotic expansion hold we have for the solution u^N of the discretised problem

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- Use a symmetric version of the differential equation system. Thus we have the coerzitivity of the associated bilinear form a and the Galerkin orthogonality of our method.

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- Use a symmetric version of the differential equation system. Thus we have the coerzitivity of the associated bilinear form a and the Galerkin orthogonality of our method.
- Using $\chi := u^I - u^N$ and $\psi := u^I - u$ this provides

$$\gamma \|\chi\|_\varepsilon^2 \leq a(\chi, \chi) = a(\psi, \chi).$$

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- Using $\chi := u^I - u^N$ and $\psi := u^I - u$ this provides
$$\gamma \|\chi\|_\varepsilon^2 \leq a(\chi, \chi) = a(\psi, \chi).$$
- Finally one can show (cf. T. Linss 2010)
$$a(\psi, \chi) \leq C\|\psi\|_\varepsilon \|\chi\|_\varepsilon.$$

□

We solve the following problem numerically

$$-\varepsilon u_1'' + \sqrt{2}\, u_1' + u_2 = 2, \qquad u_1(0) = u_1(1) = 0,$$
$$-\varepsilon u_2'' - \sqrt{2}\, u_2' - u_1 = 1, \qquad u_2(0) = u_2(1) = 0.$$

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- We know the exact solution.

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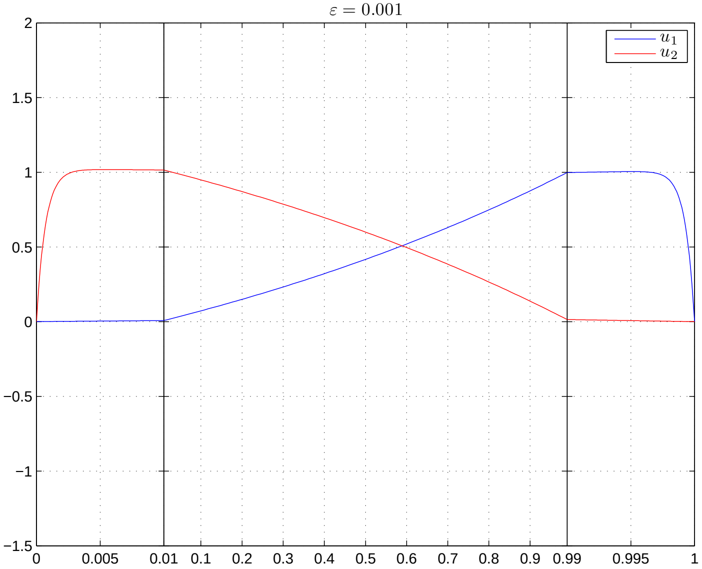
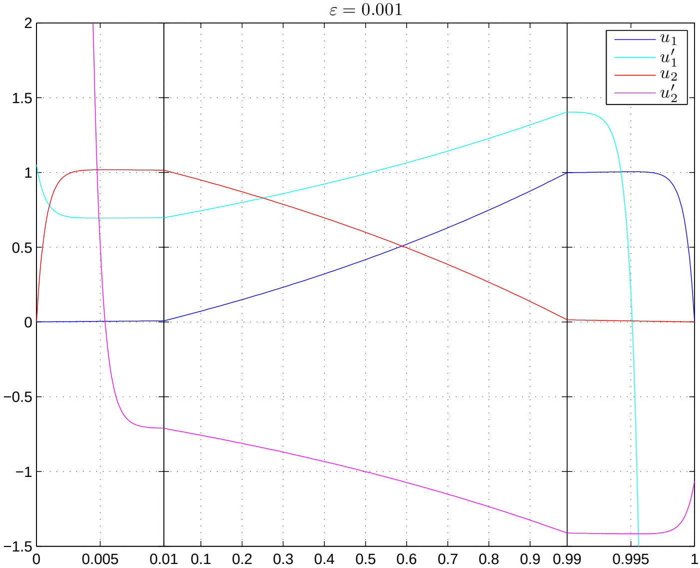
\end{aligned}
$$

- We know the exact solution.
- All assumptions are met.

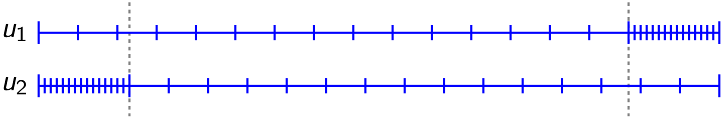

$\varepsilon = 0.001$
u_1
u_2
2
1.5
1
0.5
0
−0.5
−1
−1.5
0
0.005
0.01
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
0.99
0.995
1

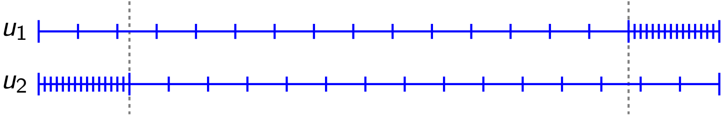
The Problem

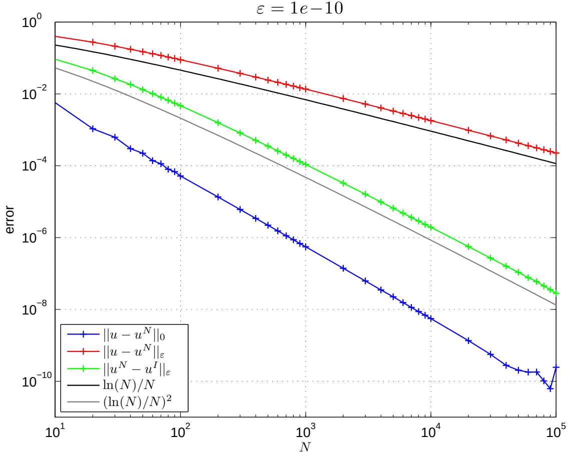


$\varepsilon = 0.001$
u_1
u_1'
u_2
u_2'
2
1.5
1
0.5
0
−0.5
−1
−1.5
0
0.005
0.01
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
0.99
0.995
1

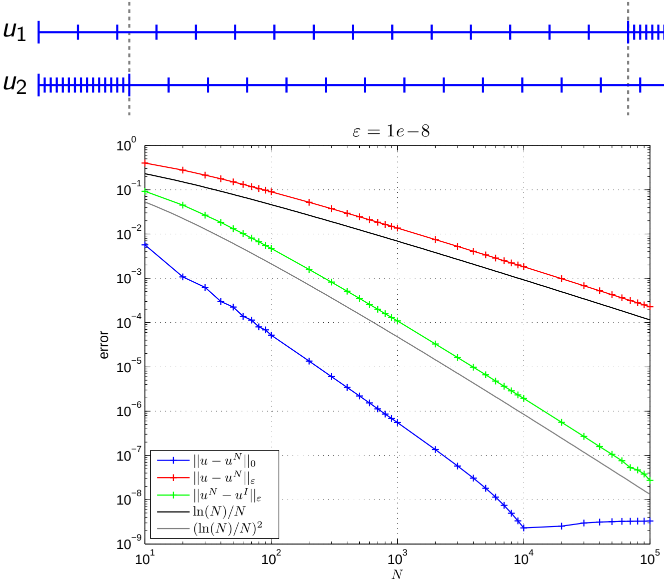
Results



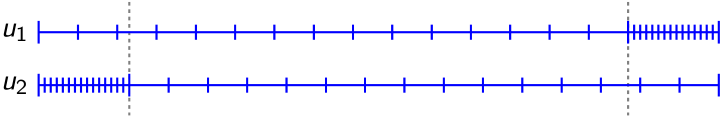

u_1
u_2

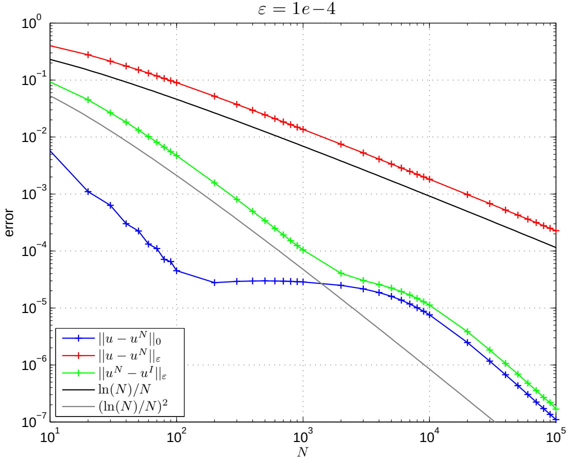

$\varepsilon = 1e{-}10$
error
N
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{1}
10^{2}
10^{3}
10^{4}
10^{5}
$||u - u^N||_0$
$||u - u^N||_\varepsilon$
$||u^N - u^I||_\varepsilon$
$\ln(N)/N$
$(\ln(N)/N)^2$

Results

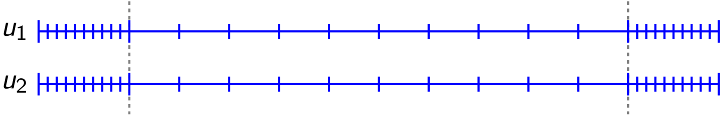

u_1
u_2
$\varepsilon = 1e{-}8$
error
N
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^1
10^2
10^3
10^4
10^5
$||u - u^N||_0$
$||u - u^N||_\varepsilon$
$||u^N - u^I||_\varepsilon$
$\ln(N)/N$
$(\ln(N)/N)^2$

Results

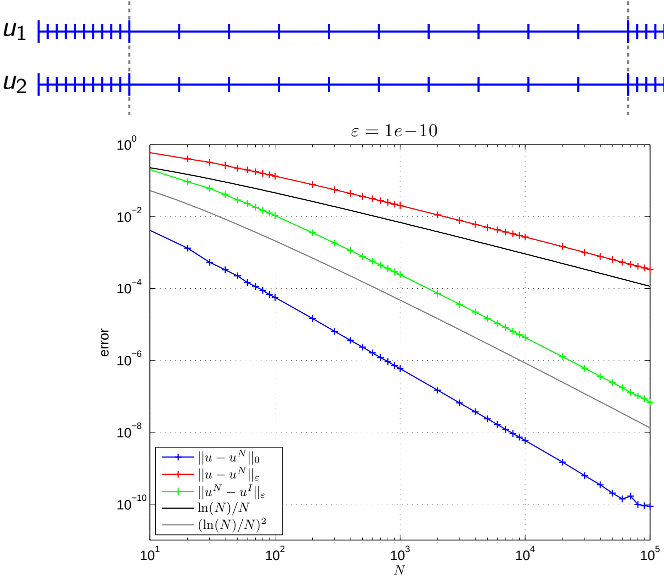

u_1
u_2


$\varepsilon = 1e-4$
error
N
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^1
10^2
10^3
10^4
10^5
$\|u - u^N\|_0$
$\|u - u^N\|_\varepsilon$
$\|u^N - u^I\|_\varepsilon$
$\ln(N)/N$
$(\ln(N)/N)^2$

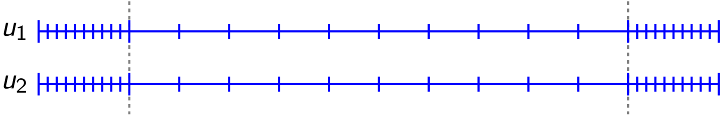
Results

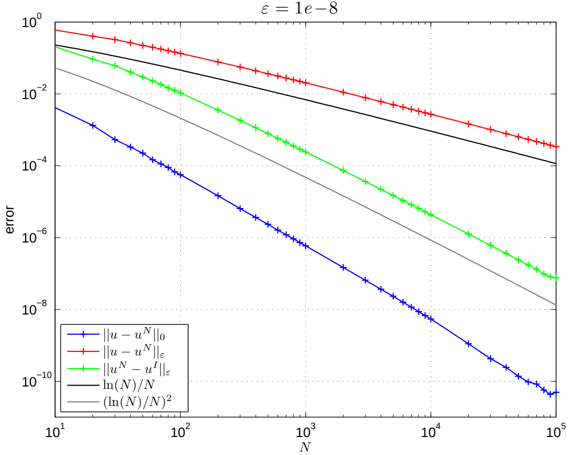


Results


u_1
u_2
$\varepsilon = 1e{-}10$
error
N
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^1
10^2
10^3
10^4
10^5
$\|u - u^N\|_0$
$\|u - u^N\|_\varepsilon$
$\|u^N - u^I\|_\varepsilon$
$\ln(N)/N$
$(\ln(N)/N)^2$

Results


u_1
u_2


$\varepsilon = 1e{-}8$
error
N
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^1
10^2
10^3
10^4
10^5
$||u - u^N||_0$
$||u - u^N||_\varepsilon$
$||u^N - u^I||_\varepsilon$
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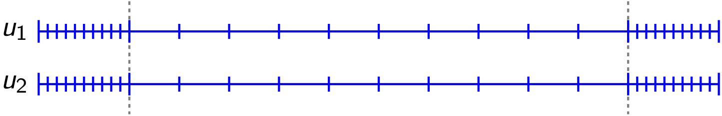
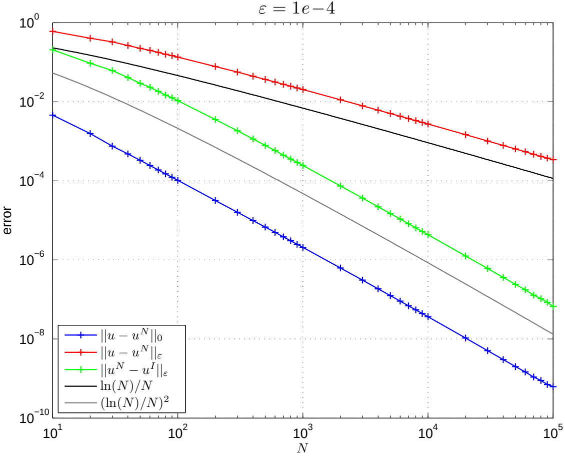
Results


u_1
u_2


$\varepsilon = 1e{-}4$
error
N
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^1
10^2
10^3
10^4
10^5
$\|u - u^N\|_0$
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Consider

$$\min_{y,u} J(y,q) := \min_{y,q} \frac{1}{2}\|y - y_d\|_0^2 + \frac{\lambda}{2}\|q\|_0^2,$$

$$Ly := -\varepsilon y'' + by' + cy = f + q \text{ in } (0,1),$$
$$y(0) = y(1) = 0,$$
$$q \in Q_{\text{ad}} := \{q \in L_2 \mid q_a \leq q \leq q_b\}$$

Assume

$$|f^{(k)}| \leq C\left(1 + \varepsilon^{-k}\mathcal{E}_l(x) + \varepsilon^{-k-\frac{1}{2}}\mathcal{E}_r(x)\right),$$
$$|q_a^{(k)}|, |q_b^{(k)}| \leq C\left(1 + \varepsilon^{-k}\mathcal{E}_l(x) + \varepsilon^{-k-\frac{1}{2}}\mathcal{E}_r(x)\right) \text{ or } \infty,$$
$$|y_d^{(k)}| \leq C\left(1 + \varepsilon^{-k-\frac{1}{2}}\mathcal{E}_l(x) + \varepsilon^{-k}\mathcal{E}_r(x)\right)$$

for $k \in \{0,1\}$.

Using the adjoint state p (cf. Tröltzsch 2009)

$$q = -\lambda^{-1}\Pi(p) := -\lambda^{-1}\max\Big(-\lambda q_b, \min(p, -\lambda q_a)\Big)$$

$$L^* p = -\varepsilon p'' - bp' + (c - b')p = y - y_0,$$
$$p(0) = p(1) = 0$$

gives equivalent problem

$$\begin{aligned} -\varepsilon y'' + by' + cy \qquad & + \frac{1}{\lambda}\Pi(p) = f, \qquad & y(0) = y(1) = 0,\\ -\varepsilon p'' - bp' + (c - b')p - y \qquad & = -y_0, \qquad & p(0) = p(1) = 0. \end{aligned}$$

Theorem

The following estimates hold

$$|p^{(k)}| \leq C\left(1+\varepsilon^{-k}\mathcal{E}_l(x)+\varepsilon^{1-k}\mathcal{E}_r(x)\right),$$
$$|y^{(k)}| \leq C\left(1+\varepsilon^{1-k}\mathcal{E}_l(x)+\varepsilon^{-k}\mathcal{E}_r(x)\right)$$

for $k \in \{0,1,2\}$ *and*

$$|q^{(k)}| \leq C\left(1+\varepsilon^{-k}\mathcal{E}_l(x)+\varepsilon^{1-k}\mathcal{E}_r(x)\right)$$

for $k \in \{0,1\}$.

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4. Analysis shows bounds of the form $y = S_y + E_y$, $\|S_y\|_1 \leq C$, $\left|E_y^k(x)\right| \leq \varepsilon^{-k}\mathcal{E}_r(x)$, $k \in \{0, 1\}$
5. Analysis shows bounds for $p = S_p + E_p$, $\|S_p\|_1 \leq C$, $\left|E_p^k(x)\right| \leq \varepsilon^{-k}\mathcal{E}_l(x)$, $k \in \{0, 1\}$

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7. Analysis shows bounds for y and y'
8. Analysis shows bounds for p and p'
9. We know $q = -\lambda^{-1}\Pi(p)$ therfore we have bounds for q'
10. We can deduce the remaining bounds for y'' and p''

Theorem

If the former estimates hold and we have

$$\left(L^N\right)^* = \left(L^*\right)^N$$

we can show for the solution u^N of the discretised problem

$$\left\|q - q^N\right\|_\varepsilon \leq CN^{-1} \ln N.$$

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1. $\left(L^N\right)^* = \left(L^*\right)^N$ implies we should use the same mesh for the discretisation of q, p and y
2. But the boundary layer structure differs for p and y
3. $\left(L^N\right)^* = \left(L^*\right)^N$ gives that the discrete problem is an optimisation problem
4. The other assumptions are quite reasonable

Other Way of Proof - Use the System

We consider the weak formultaion of

$$\begin{aligned} -\varepsilon y'' + by' + cy \qquad\qquad + \frac{1}{\lambda}\Pi(p) &= f, \qquad y(0) = y(1) = 0, \\ -\varepsilon p'' - bp' + (c - b')p - y \qquad &= -y_0, \quad p(0) = p(1) = 0. \end{aligned}$$

Assuming $c - b'/2 \geq \lambda^{-1/2}$ we can conclude

$$a\left(\begin{pmatrix} y_1 \\ p_1 \end{pmatrix}, \begin{pmatrix} y_1 - y_2 \\ p_1 - p_2 \end{pmatrix}\right) - a\left(\begin{pmatrix} y_2 \\ p_2 \end{pmatrix}, \begin{pmatrix} y_1 - y_2 \\ p_1 - p_2 \end{pmatrix}\right) \geq C\left(\|y_1 - y_2\|_\varepsilon^2 + \|p_1 - p_2\|_\varepsilon^2\right),$$

the monotony of the operator. We have continuity of the operator. Thus we have an unique solution of the system and its discrete counterparts (cf. E. Zeidler, 1990).

Assuming $q_a \leq 0 \leq q_b$ we get from the monotony of a by $y_2 = 0$ and $p_2 = 0$

$$a\left(\begin{pmatrix} y_1 \\ p_1 \end{pmatrix}, \begin{pmatrix} y_1 \\ p_1 \end{pmatrix}\right) \geq C\left(\|y_1\|_\varepsilon^2 + \|p_1\|_\varepsilon^2\right).$$

Thus we can use the standard analysis for linear operators to attain

$$\left\|y^I - y^N\right\|_\varepsilon, \left\|p^I - p^N\right\|_\varepsilon \leq CN^{-1} \ln N.$$

and we can derive first order error estimates for the numerical method for arbitrary discretisations with sufficient interpolation properties. But we have the prerequisits

$$c - b'/2 \geq \lambda^{-1/2} \quad \text{and} \quad q_a \leq 0 \leq q_b.$$

Thank you for your attention.

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