Convergence results in balanced norms for singularly perturbed problems

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What is to be balanced?

Consider the reaction-diffusion equation

$$-\varepsilon^2 \Delta u + cu = f, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega.$$

It is known, that the solution consists of a regular part S and layer terms of the form

$$E = \exp(-\gamma x/\varepsilon).$$

The standard energy norm now yields

$$|||v|||_\varepsilon := \varepsilon\|\nabla v\|_0 + \|v\|_0 \Rightarrow |||S|||_\varepsilon \leq C, \quad |||E|||_\varepsilon \leq C\varepsilon^{1/2}.$$

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The approach of Lin and Stynes

Testing the PDE with $v - \varepsilon\Delta v$ yields

$$\varepsilon^3 (\Delta u, \Delta v) + (\varepsilon^2 + \varepsilon c)(\nabla u, \nabla v) + c(u, v) = (f, v - \varepsilon\Delta v).$$

This weak formulation is coercive w.r.t.

$$|||v|||_{LS} := \varepsilon^{3/2}\|\Delta v\|_0 + \varepsilon^{1/2}\|\nabla v\|_0 + \|v\|_0.$$

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Drawback: $v \in H^2(\Omega) \to C^1$-finite elements or mixed formulation

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Overview

Reaction-Diffusion Problems

Convection-Diffusion Problems

Uniform convergence on a standard mesh

Lemma

Assume $u_0 \in H^1(\Omega)$. Then

$$\varepsilon^3 |u|_2^2 + \varepsilon |u|_1^2 \le C \left(\varepsilon |u_0|_1^2 + \|u_0\|_{0,\partial\Omega}^2 \right),$$

moreover,

$$\|u - u_0\|_0^2 \le C \left(\varepsilon^2 |u_0|_1^2 + \varepsilon \|u_0\|_{0,\partial\Omega}^2 \right).$$

Theorem

If $|u_0|_1$ and $\|u_0\|_{0,\partial\Omega}$ are bounded, the error of the finite element method with linear or bilinear finite elements on a shape-regular mesh satisfies the uniform in ε estimate

$$\|u - u_h\|_\varepsilon \le C h^{1/2}.$$

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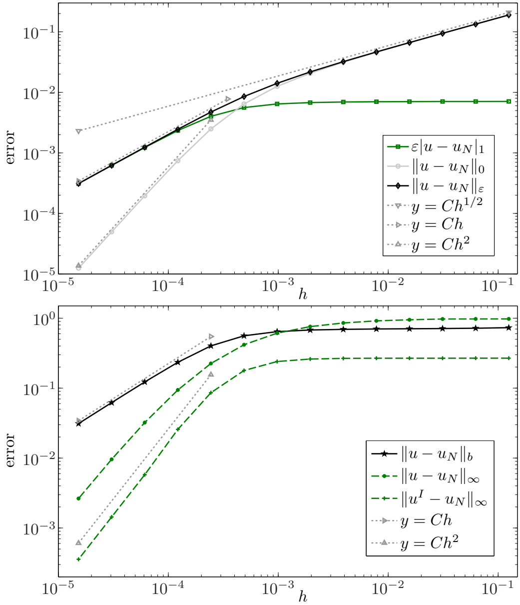
Reaction-Diffusion Problems



Errors of the Galerkin FEM with bilinear elements on uniform meshes, $\varepsilon = 10^{-4}$.

Another balanced norm

Define

$$|||v|||_{RD} := \varepsilon^{1/2}\|\nabla v\|_0 + \|v\|_0.$$

Apparently, this is a balanced norm as

$$|||S|||_{RD} \le C, \quad |||E|||_{RD} \le C.$$

But we do not have coercivity in this norm!

Theorem

For the nodal bilinear interpolant u^I on a standard Shishkin mesh holds

$$\left|\left|\left| u - u^I \right|\right|\right|_{RD} \le CN^{-1} \ln N.$$

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Analysis

Assume a standard Shishkin mesh and the Galerkin FEM:
Find $u^N \in V^N = \{v^N \in H_0^1(\Omega) : v^N|_\tau \in \mathcal{Q}_1(\tau)\}$ such that

$$a_{Gal}(u^N, v^N) := \varepsilon^2(\nabla u^N, \nabla v^N) + c(u^N, v^N) = (f, v^N), \forall v^N \in V^N.$$

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Sketch of proof

$$u - u^N = u - \pi u + \pi u - u^N \qquad (\triangle)$$

Galerkin orthogonality yields for $\pi u \in V^N$:

$$\varepsilon^2 |\pi u - u^N|_1^2 + c\,\|\pi u - u^N\|_0^2 = a_{Gal}(\pi u - u^N, \pi u - u^N)$$
$$= \varepsilon^2 \Big(\nabla(\pi u - u), \nabla(\pi u - u^N)\Big) + c\,\Big(\pi u - u, \pi u - u^N\Big).$$

With π as the L_2-projection onto V^N follows $(\pi u - u, v^N) = 0$, hence

$$|\pi u - u^N|_1 \le |\pi u - u|_1 \quad \overset{(\triangle)}{\Longrightarrow} \quad |u - u^N|_1 \le 2|u - \pi u|_1$$

Now to handle $|u - \pi u|_1$ use inverse estimates, L_∞-stability of L_2-projection and estimates of $u - u^I$.

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numerical verification

N	$\varepsilon = 10^{-2}$ error	$\varepsilon = 10^{-2}$ rate	$\varepsilon = 10^{-3}$ error	$\varepsilon = 10^{-3}$ rate	$\varepsilon = 10^{-k}, k \in \{4, \dots, 8\}$ error	$\varepsilon = 10^{-k}, k \in \{4, \dots, 8\}$ rate
8	1.86e-1		1.88e-1		1.88e-1	
		0.98		0.97		0.97
16	1.25e-1		1.27e-1		1.27e-1	
		0.98		0.98		0.98
32	7.89e-2		8.00e-2		8.01e-2	
		0.99		0.99		0.99
64	4.76e-2		4.82e-2		4.83e-2	
		1.00		1.00		1.00
128	2.78e-2		2.82e-2		2.82e-2	
		1.00		1.00		1.00
256	1.59e-2		1.61e-2		1.61e-2	
		1.00		1.00		1.00
512	8.93e-3		9.07e-3		9.08e-3	
		1.00		1.00		1.00
1024	4.96e-3		5.04e-3		5.05e-3	

Error $\|u - u^N\|_{RD}$ of the Galerkin FEM with bilinear elements on a sequence of Shishkin-meshes, convergence rates as powers of $(N^{-1} \ln N)$.

interior penalty

$$B_\alpha^\pm(w,v) := \varepsilon^{3/2+\alpha} \sum_{T\in\Omega^N} (\Delta w, \Delta v)_T + \varepsilon^{1/2+\alpha}(c+\varepsilon^{1/2-\alpha})(\nabla w, \nabla v) + c(w,v)$$

$$+ \sum_{e\in\mathcal{E}^N} \left(-\varepsilon^{1/2+\alpha}\, c\left(\left[\!\left[\frac{\partial v}{\partial n} \right]\!\right], w \right)_e \pm \varepsilon^{1/2+\alpha}\, c\left(\left[\!\left[\frac{\partial w}{\partial n} \right]\!\right], v \right)_e + \left(\sigma_e \left[\!\left[\frac{\partial w}{\partial n} \right]\!\right], \left[\!\left[\frac{\partial v}{\partial n} \right]\!\right] \right)_e \right)$$

$$L_\alpha(v) := (f,v) - \varepsilon^{1/2+\alpha} \sum_{T\in\Omega^N} (f, \Delta v)_T$$

Theorem

Fix $\alpha \in [1/4, 1/2]$ and choose σ_e adequately. Then

$$|\!|\!| u - u^h |\!|\!|_\alpha \le C N^{-1} \ln N$$

for biquadratic elements on a Shishkin mesh with

$$|\!|\!| v |\!|\!|_\alpha^2 = B_\alpha^+(v,v).$$

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Again, it's unbalanced

Consider the convection-diffusion problem

$$-\varepsilon\Delta u - bu_x + cu = f \qquad \text{in } \Omega = (0,1)^2,$$
$$u = 0 \qquad \text{on } \partial\Omega.$$

Here the solution u has a regular component S and two different layer components of the forms $E_1 = \exp(-x/\varepsilon)$ and $E_2 = \exp(-y/\sqrt{\varepsilon})$. With the standard energy norm

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A balanced norm

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$$|||v|||_{CD} := \varepsilon^{1/2}\|v_x\|_0 + \varepsilon^{1/4}\|v_y\|_0 + \|v\|_0.$$

Again, we do not have coercivity. We try to use a similar analysis.
Problem: We will need a different projection that is still L_∞-stable.

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A version of SDFEM

Define the SDFEM on a standard Shishkin mesh and with piecewise bilinear elements by:
Find $u^N \in V^N$ such that for all $v^N \in V^N$

$$a_{SD}(u^N, v^N) = f_{SD}(v^N),$$

where

$$\begin{aligned} a_{SD}(v, w) &:= \varepsilon(\nabla v, \nabla w) + (cv - bv_x, w) + \\ &\qquad \sum_\tau (\varepsilon \Delta v + bv_x - cv, \delta_\tau b w_x)_\tau, \\ f_{SD}(v) &:= (f, v) - \sum_\tau (f, \delta_\tau b v_x)_\tau. \end{aligned}$$

A version of SDFEM

The difference to the standard SDFEM-formulation is the choice of δ_τ as a stabilisation function on τ given by

$$\delta_{\tau_{ij}} := \min\left\{\frac{h_i}{2\varepsilon}, \frac{1}{\|b\|_{\infty,\tau_{ij}}}\right\} h_i \frac{(x_i - x)(x - x_{i-1})}{h_i^2}.$$

Analysis

Let a projection operator $\pi : C(\Omega) \to V^N$ be given by

$$a_{proj}(\pi u - u, v^N) = 0 \quad \text{for all } v^N \in V^N$$

where

$$a_{proj}(v, w) = \varepsilon(v_x, w_x) + (cv - bv_x, w) + \sum_\tau (\varepsilon v_{xx} + bv_x - cv, \delta_\tau b w_x)_\tau.$$

Then follows from $a_{SD}(\pi u - u^N, \pi u - u^N)$ with coercivity and Galerkin orthogonality

$$\|(\pi u - u^N)_y\|_0 \left|\left|\left|\pi u - u^N\right|\right|\right|_{SD} \le C\|(u - \pi u)_y\|_0 \left|\left|\left|\pi u - u^N\right|\right|\right|_{SD} + C\varepsilon^{-1/2}\left|\sum_\tau(\varepsilon(\pi u - u)_{yy}, \delta_\tau b(\pi u - u^N)_x)_\tau\right|.$$

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Preliminary Results

Lemma

The projection π is L_∞-stable, i.e.

$$\|\pi v\|_\infty \le C\|v\|_\infty \quad \text{for } v \in C(\Omega)$$

and

$$\|(u - \pi u)_y\|_0 \le C\varepsilon^{-1/4}N^{-1}(\ln N)^{3/2}.$$

It follows finally (after a lengthy analysis)

$$\|(u - u^N)_y\|_0 \le C\varepsilon^{-1/4}N^{-1}(\ln N)^{3/2}.$$

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Lemma

The projection π is L_∞-stable, i.e.

$$\|\pi v\|_\infty \leq C\|v\|_\infty \quad \text{for } v \in C(\Omega)$$

and

$$\|(u - \pi u)_y\|_0 \leq C\varepsilon^{-1/4} N^{-1} (\ln N)^{3/2}.$$

It follows finally (after a lengthy analysis)

$$\|(u - u^N)_y\|_0 \leq C\varepsilon^{-1/4} N^{-1} (\ln N)^{3/2}.$$

Result

Theorem

For the SDFEM solution u^N holds in the standard energy norm

$$\left|\left|\left| u - u^N \right|\right|\right|_{\varepsilon} \le CN^{-1}(\ln N)^{3/2},$$

and in the balanced norm

$$\left|\left|\left| u - u^N \right|\right|\right|_{CD} \le CN^{-1}(\ln N)^{3/2}.$$