A posteriori error estimates based on potential and flux reconstruction for DGMs:

algebraic error & easy incorporation of hanging nodes

Vít Dolejší, Alexandre Ern, Ivana Šebestová, and Martin Vohralík

Department of Numerical Mathematics
Charles University in Prague

Workshop Dresden-Prague on Numerical Analysis 2012

Outline

1. Setting
   - Continuous setting
   - Discrete setting
2. Upper bound
   - Flux reconstructions
   - A guaranteed a posteriori estimate
3. Lower bound
   - Stopping criteria and efficiency

Outline

Poisson equation

Poisson equation
Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a polygonal domain and $f \in L^2(\Omega)$ the source term.

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

Weak formulation
Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega) \tag{1}$$

- u is termed the *potential*
- $-\nabla u$ is termed the *flux*

Poisson equation

Poisson equation
Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a polygonal domain and $f \in L^2(\Omega)$ the source term.

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

Weak formulation
Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega) \tag{1}$$

- u is termed the *potential*
- $-\nabla u$ is termed the *flux*

Poisson equation

Poisson equation

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a polygonal domain and $f \in L^2(\Omega)$ the source term.

$$\begin{aligned} -\Delta u &= f \qquad && \text{in } \Omega, \\ u &= 0 \qquad && \text{on } \partial\Omega \end{aligned}$$

Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega) \tag{1}$$

- u is termed the *potential*
- $-\nabla u$ is termed the *flux*

Outline

1. Setting
   - Continuous setting
   - Discrete setting
2. Upper bound
   - Flux reconstructions
   - A guaranteed a posteriori estimate
3. Lower bound
   - Stopping criteria and efficiency

Meshes with hanging nodes

- $\mathcal{T}_h$ ($h > 0$): a family of partitions of Ω into a finite number of closed simplices, hanging nodes allowed
- ∂K: the boundary of element K
- $\mathcal{F}_K$: the set of the faces of element K
- $\mathcal{F}_h^{\mathrm{I}} = \{\Gamma; \Gamma = \partial K \cap \partial K', K, K' \in \mathcal{T}_h\}$
- $\mathcal{F}_h^{\mathrm{B}} = \{\Gamma; \Gamma \subset \partial\Omega, \exists K \in \mathcal{T}_h : \Gamma \in \mathcal{F}_K\}$
- $\mathcal{F}_h := \mathcal{F}_h^{\mathrm{I}} \cup \mathcal{F}_h^{\mathrm{B}}$
- $h_K = \mathrm{diam}(K)$ for $K \in \mathcal{T}_h$
  ρ_K: the diameter of the largest d-dimensional ball inscribed into K
- For $\Gamma \in \mathcal{F}_h^{\mathrm{I}}$: let K_Γ^{L} and K_Γ^{R} be such that $\Gamma \subset \overline{K_\Gamma^{\mathrm{L}}} \cap \overline{K_\Gamma^{\mathrm{R}}}$
- $\mathbf{n}_\Gamma$: a unit normal vector to Γ

Triangulation assumptions

shape regularity: $\exists C_s > 0 : \dfrac{h_K}{\rho_K} \leq C_s \; \forall\, K \in \mathcal{T}_h,$

local quasi-uniformity: $\exists C_H > 0 : \; h_K \leq C_H h_{K'} \; \forall K, K' \in \mathcal{T}_h$ sharing a face

Notation

- $H^{\boldsymbol{s}}(\Omega, \mathcal{T}_h) = \{v; v|_K \in H^{s_K}(K) \quad \forall K \in \mathcal{T}_h\}$,
  $\boldsymbol{s} := \{s_K\}_{K \in \mathcal{T}_h}, \ s_K \geq 1$
- $\nabla_h v$: the broken gradient for $v \in H^1(\Omega, \mathcal{T}_h)$
- $[v]_\Gamma$: the jump of v over Γ, $\Gamma \in \mathcal{F}_h^{\mathrm{I}}$
- $\langle v \rangle_\Gamma$: the average of v on Γ, $\Gamma \in \mathcal{F}_h^{\mathrm{I}}$
- $\langle v \rangle_\Gamma := [v]_\Gamma :=$ the trace of v on Γ, $\Gamma \in \mathcal{F}_h^{\mathrm{B}}$
- $P^{p_K}(K)$ is the space of polynomial functions on K of degree at most p_K
- $N_K := \dim(P^{p_K}(K))$
- $S_h^{\boldsymbol{p}} = \{v; v \in L^2(\Omega), v|_K \in P^{p_K}(K) \quad \forall K \in \mathcal{T}_h\}$,
  $\boldsymbol{p} := \{p_K\}_{K \in \mathcal{T}_h}, \ p_K \geq 1$
- $N := \dim(S_h^{\boldsymbol{p}})$

Discontinuous Galerkin formulation

For $u_h, v_h \in S_h^p$, we define:

$$a(u_h, v_h) := \sum_{K \in \mathcal{T}_h} \int_K \nabla u_h \cdot \nabla v_h \,\mathrm{d}\mathbf{x} - \sum_{\Gamma \in \mathcal{F}_h} \int_\Gamma \langle \nabla u_h \cdot \mathbf{n} \rangle [v_h] \,\mathrm{d}S$$
$$+ \theta \sum_{\Gamma \in \mathcal{F}_h} \int_\Gamma \langle \nabla v_h \cdot \mathbf{n} \rangle [u_h] \,\mathrm{d}S + \sum_{\Gamma \in \mathcal{F}_h} \int_\Gamma \sigma [u_h][v_h] \,\mathrm{d}S,$$
$$\ell(v_h) := \int_\Omega f v_h \,\mathrm{d}\mathbf{x},$$

- σ: a penalty parameter
- $\theta = -1$, $\theta = 1$, and $\theta = 0$ corresponds to the symmetric, nonsymmetric, and incomplete variants of the DGM, resp.

Exact Discrete problem

Exact discrete problem

Find $u_h \in S_h^p$ such that

$$a(u_h, v_h) = \ell(v_h) \quad \forall v_h \in S_h^p.$$

Matrix formulation

- $\{\varphi_j\}_{j=1..N}$: a basis of S_h^p
- $\mathbb{A} = \{\mathbb{A}_{ij}\}_{i,j=1..N} := \{a(\varphi_j, \varphi_i)\}_{i,j=1..N}$
- $U_h := \{u_h^j\}_{j=1..N}$
- $F := \{\ell(\varphi_i)\}_{i=1..N}$

Find $U_h \in \mathbb{R}^N$ such that

$$\mathbb{A} U_h = F. \tag{2}$$

Exact Discrete problem

Exact discrete problem

Find $u_h \in S_h^p$ such that

$$a(u_h, v_h) = \ell(v_h) \quad \forall v_h \in S_h^p.$$

Matrix formulation

- $\{\varphi_j\}_{j=1..N}$: a basis of S_h^p
- $\mathbb{A} = \{\mathbb{A}_{ij}\}_{i,j=1..N} := \{a(\varphi_j, \varphi_i)\}_{i,j=1..N}$
- $U_h := \{u_h^j\}_{j=1..N}$
- $F := \{\ell(\varphi_i)\}_{i=1..N}$

Find $U_h \in \mathbb{R}^N$ such that

$$\mathbb{A} U_h = F. \tag{2}$$

Exact Discrete problem

Exact discrete problem

Find $u_h \in S_h^p$ such that

$$a(u_h, v_h) = \ell(v_h) \quad \forall v_h \in S_h^p.$$

Matrix formulation

- $\{\varphi_j\}_{j=1..N}$: a basis of S_h^p
- $\mathbb{A} = \{\mathbb{A}_{ij}\}_{i,j=1..N} := \{a(\varphi_j, \varphi_i)\}_{i,j=1..N}$
- $U_h := \{u_h^j\}_{j=1..N}$
- $F := \{\ell(\varphi_i)\}_{i=1..N}$

Find $U_h \in \mathbb{R}^N$ such that

$$\mathbb{A} U_h = F. \tag{2}$$

Inexact discrete problem I

The algebraic system (2) is possibly not solved exactly.

Let an i-th step of a linear algebraic solver be given.

Algebraic residual vector

- R^i: the residual algebraic vector associated with U_h^i
- $R_K^i := \{R_{K,j}^i\}_{j=1}^{N_K}$: the subvector of the residual R^i associated with the element K, $K \in \mathcal{T}_h$

Definition (Local residual function)

$r_K^i \in P^{p_K}(K)$, $(r_K^i, \varphi_{K,j})_K = R_{K,j}^i$ for $j = 1 \dots N_K$, $K \in \mathcal{T}_h$

Definition (Residual function)

$r_h^i \in S_h^p$, $r_h^i|_K := r_K^i$ for all $K \in \mathcal{T}_h$

Inexact discrete problem I

The algebraic system (2) is possibly not solved exactly.
Let an i-th step of a linear algebraic solver be given.

Algebraic residual vector

- R^i: the residual algebraic vector associated with U_h^i
- $R_K^i := \{R_{K,j}^i\}_{j=1}^{N_K}$: the subvector of the residual R^i associated with the element K, $K \in \mathcal{T}_h$

Definition (Local residual function)

$r_K^i \in P^{p_K}(K)$, $(r_K^i, \varphi_{K,j})_K = R_{K,j}^i$ for $j = 1 \ldots N_K$, $K \in \mathcal{T}_h$

Definition (Residual function)

$r_h^i \in S_h^p$, $r_h^i|_K := r_K^i$ for all $K \in \mathcal{T}_h$

Inexact discrete problem I

The algebraic system (2) is possibly not solved exactly.
Let an i-th step of a linear algebraic solver be given.

Algebraic residual vector

- R^i: the residual algebraic vector associated with U_h^i
- $R_K^i := \{R_{K,j}^i\}_{j=1}^{N_K}$: the subvector of the residual R^i associated with the element K, $K \in \mathcal{T}_h$

Definition (Local residual function)

$r_K^i \in P^{p_K}(K)$, $(r_K^i, \varphi_{K,j})_K = R_{K,j}^i$ for $j = 1 \ldots N_K$, $K \in \mathcal{T}_h$

Definition (Residual function)

$r_h^i \in S_h^p$, $r_h^i|_K := r_K^i$ for all $K \in \mathcal{T}_h$

Inexact discrete problem I

The algebraic system (2) is possibly not solved exactly.
Let an i-th step of a linear algebraic solver be given.

Algebraic residual vector

- R^i: the residual algebraic vector associated with U_h^i
- $R_K^i := \{R_{K,j}^i\}_{j=1}^{N_K}$: the subvector of the residual R^i associated with the element K, $K \in \mathcal{T}_h$

Definition (Local residual function)

$r_K^i \in P^{p_K}(K)$, $(r_K^i, \varphi_{K,j})_K = R_{K,j}^i$ for $j = 1 \ldots N_K$, $K \in \mathcal{T}_h$

Definition (Residual function)

$r_h^i \in S_h^p$, $r_h^i|_K := r_K^i$ for all $K \in \mathcal{T}_h$

Inexact discrete problem I

The algebraic system (2) is possibly not solved exactly.
Let an i-th step of a linear algebraic solver be given.

Algebraic residual vector

- R^i: the residual algebraic vector associated with U_h^i
- $R_K^i := \{R_{K,j}^i\}_{j=1}^{N_K}$: the subvector of the residual R^i associated with the element K, $K \in \mathcal{T}_h$

Definition (Local residual function)

$r_K^i \in P^{p_K}(K)$, $(r_K^i, \varphi_{K,j})_K = R_{K,j}^i$ for $j = 1 \ldots N_K$, $K \in \mathcal{T}_h$

Definition (Residual function)

$r_h^i \in S_h^{\boldsymbol{p}}$, $r_h^i|_K := r_K^i$ for all $K \in \mathcal{T}_h$

Inexact discrete problem II

Matrix formulation

Find $U_h^i \in \mathbb{R}^N$ such that

$$\mathbb{A} U_h^i = F - R^i.$$

Inexact discrete problem

Find $u_h^i \in S_h^p$ such that

$$a(u_h^i, v_h) = \ell(v_h) - (r_h^i, v_h) \quad \forall v_h \in S_h^p. \tag{3}$$

Error components

Total error in a computational approximation: $e_h^i := u - u_h^i$

- discretization error: $u - u_h$
- algebraic error: $u_h - u_h^i$

Inexact discrete problem II

Matrix formulation

Find $U_h^i \in \mathbb{R}^N$ such that

$$\mathbb{A} U_h^i = F - R^i.$$

Inexact discrete problem

Find $u_h^i \in S_h^p$ such that

$$a(u_h^i, v_h) = \ell(v_h) - (r_h^i, v_h) \quad \forall v_h \in S_h^p. \tag{3}$$

Error components

Total error in a computational approximation: $e_h^i := u - u_h^i$

- discretization error: $u - u_h$
- algebraic error: $u_h - u_h^i$

Inexact discrete problem II

Matrix formulation

Find $U_h^i \in \mathbb{R}^N$ such that

$$\mathbb{A} U_h^i = F - R^i.$$

Inexact discrete problem

Find $u_h^i \in S_h^{\boldsymbol{p}}$ such that

$$a(u_h^i, v_h) = \ell(v_h) - (r_h^i, v_h) \quad \forall v_h \in S_h^{\boldsymbol{p}}. \tag{3}$$

Error components

Total error in a computational approximation: $e_h^i := u - u_h^i$

- discretization error: $u - u_h$
- algebraic error: $u_h - u_h^i$

Inexact discrete problem II

Matrix formulation

Find $U_h^i \in \mathbb{R}^N$ such that

$$\mathbb{A} U_h^i = F - R^i.$$

Inexact discrete problem

Find $u_h^i \in S_h^{\boldsymbol{p}}$ such that

$$a(u_h^i, v_h) = \ell(v_h) - (r_h^i, v_h) \quad \forall v_h \in S_h^{\boldsymbol{p}}. \tag{3}$$

Error components

Total error in a computational approximation: $e_h^i := u - u_h^i$

- discretization error: $u - u_h$
- algebraic error: $u_h - u_h^i$

Outline

Flux reconstruction components

Total flux reconstruction: $\mathbf{t}_h^i = \mathbf{d}_h^i + \mathbf{a}_h^i$

- $\mathbf{d}_h^i$: discretization flux reconstruction
- $\mathbf{a}_h^i$: algebraic error flux reconstruction

Construction in Raviart–Thomas–Nédélec (RTN) spaces:

- $\mathbf{RTN}_{q_K}(K) := [P^{q_K}(K)]^d + \mathbf{x}P^{q_K}(K)$ for $K \in \mathcal{T}_h$
- $\mathbf{RTN}_{\mathbf{q}}(\mathcal{T}_h) := \{v_h; v_h \in [L^2(\Omega)]^d, v_h|_K \in \mathbf{RTN}_{q_K}(K) \, \forall\, K \in \mathcal{T}_h\}$, $\mathbf{q} := \{q_K\}_{K\in\mathcal{T}_h}$, $q_K \geq 0$

Flux reconstruction components

Total flux reconstruction: $\mathbf{t}_h^i = \mathbf{d}_h^i + \mathbf{a}_h^i$

- $\mathbf{d}_h^i$: discretization flux reconstruction
- $\mathbf{a}_h^i$: algebraic error flux reconstruction

Construction in Raviart–Thomas–Nédélec (RTN) spaces:

- $\mathbf{RTN}_{q_K}(K) := [P^{q_K}(K)]^d + \mathbf{x}P^{q_K}(K)$ for $K \in \mathcal{T}_h$
- $\mathbf{RTN}_{\mathbf{q}}(\mathcal{T}_h) := \{v_h;\, v_h \in [L^2(\Omega)]^d,\, v_h|_K \in \mathbf{RTN}_{q_K}(K)\, \forall\, K \in \mathcal{T}_h\}$, $\mathbf{q} := \{q_K\}_{K\in\mathcal{T}_h}$, $q_K \geq 0$

Discretization flux reconstruction

Definition (Discretization flux reconstruction)

Let u_h^i solve (3). For all $K \in \mathcal{T}_h$, $\Gamma \in \mathcal{F}_K$ and $q_h \in P^{p_K}(\Gamma)$,

$$(\mathbf{d}_h^i \cdot \mathbf{n}, q_h)_\Gamma := (-\langle \nabla u_h^i \cdot \mathbf{n} \rangle + \sigma [u_h^i], q_h)_\Gamma \tag{4}$$

and for all $\mathbf{r}_h \in [P^{p_K - 1}(K)]^d$,

$$(\mathbf{d}_h^i, \mathbf{r}_h)_K := (-\nabla u_h^i, \mathbf{r}_h)_K + \theta \sum_{\Gamma \in \mathcal{F}_K} w_\Gamma (\mathbf{r}_h \cdot \mathbf{n}, [u_h^i])_\Gamma \tag{5}$$

where $w_\Gamma := \frac{1}{2}$ for interior faces, $w_\Gamma := 1$ for boundary faces.

Algebraic error flux reconstruction

Definition (Algebraic error flux reconstruction)

Let perform additional ν steps of the algebraic iterative solver. This gives

$$\mathbb{A} U_h^{i+\nu} = F - R^{i+\nu}$$

and

$$a(u_h^{i+\nu}, v_h) = \ell(v_h) - (r_h^{i+\nu}, v_h) \quad \forall v_h \in S_h^p.$$

Let $\mathbf{d}_h^i$ and $\mathbf{d}_h^{i+\nu}$ be given by (4), (5)
(i is replaced by $i+\nu$ in the latter case). Then, we define

$$\mathbf{a}_h^i := \mathbf{d}_h^{i+\nu} - \mathbf{d}_h^i. \tag{6}$$

Outline

Guaranteed a posteriori error estimate

Theorem (Estimate distinguishing individual error components)

Let $u \in H_0^1(\Omega)$ be the weak solution given by (1). Let an i-th algebraic solver step be given and let $u_h^i \in S_h^{\boldsymbol{p}}$, $\boldsymbol{p} = \{p_K\}_{K \in \mathcal{T}_h}$, $p_K \geq 0$, be the DGM output given by (3). Consider $\nu > 0$ additional algebraic steps. Then

$$\|\nabla_h(u - u_h^i)\| \leq 2^{1/2}\eta_{\mathrm{disc}}^i + \eta_{\mathrm{alg}}^i + \eta_{\mathrm{rem}}^i. \tag{7}$$

Estimators

- *$H_0^1(\Omega)$ nonconformity estimator:*

$$\eta_{\mathrm{PNC},K}^{i} := \|\nabla(u_h^i - \mathcal{I}_{\mathrm{Av}}(u_h^i))\|_K$$

- *residual estimator:*

$$\eta_{\mathrm{R},K}^{i} := C_{P,K} h_K \|f - \nabla \cdot \mathbf{t}_h^i - r^{i+\nu}\|_K$$

- **H**(*div*, Ω) *nonconformity estimator:*

$$\eta_{\mathrm{FNC},K}^{i} := \sum_{\Gamma \in \mathcal{F}_K} w_\Gamma h_K^{1/2} C_{\Gamma,K} \|[\mathbf{t}_h^i \cdot \mathbf{n}]\|_\Gamma$$

- *algebraic reminder estimator:*

$$\eta_{\mathrm{rem},K}^{i} := C_{F,\Omega} h_\Omega \|r^{i+\nu}\|_K$$

Estimators due to error components

- *discretization estimator:*

$$\eta^i_{\mathrm{disc},K} := \eta^i_{\mathrm{PNC},K} + \eta^i_{\mathrm{R},K} + \sum_{\Gamma \in \mathcal{F}_K} w_\Gamma h_K^{1/2} C_{\Gamma,K} \| [\mathbf{d}^i_h \cdot \mathbf{n}] \|_\Gamma + \| \nabla u^i_h + \mathbf{d}^i_h \|_K$$

- *algebraic estimator:*

$$\eta^i_{\mathrm{alg},K} := \| \mathbf{a}^i_h \|_K + \sum_{\Gamma \in \mathcal{F}_K} w_\Gamma h_K^{1/2} C_{\Gamma,K} \| [\mathbf{a}^i_h \cdot \mathbf{n}] \|_\Gamma$$

- $\eta^i_\cdot := \left\{ \sum_{K \in \mathcal{T}_h} (\eta^i_{\cdot,K})^2 \right\}^{1/2}$

$C_{P,K}$, $C_{\Gamma,K}$, and $C_{F,\Omega}$ come from the Poincaré, trace, and Friedrichs inequalities, resp.

Outline

Local stopping criteria

- $\gamma_{\mathrm{alg}}, \gamma_{rem} > 0$: user-specified weights
- stop the algebraic solver as soon as

$$\eta^{i}_{\mathrm{rem},K} \leq \gamma_{rem}(\eta^{i}_{\mathrm{disc},K} + \eta^{i}_{\mathrm{alg},K}) \quad \forall K \in \mathcal{T}_h \tag{8}$$

$$\eta^{i}_{\mathrm{alg},K} \leq \gamma_{alg}\eta^{i}_{\mathrm{disc},K} \quad \forall K \in \mathcal{T}_h \tag{9}$$

Local efficiency of the estimate

Theorem (*Local efficiency of the estimate*)

Let $u \in H_0^1(\Omega)$ be the weak solution given by (1). Let an i-th algebraic solver step be given and let u_h^i be its output given by (3). Let $\mathbf{t}_h^i = \mathbf{d}_h^i + \mathbf{a}_h^i$, with the discretization flux $\mathbf{d}_h^i$ given by (4), (5) and the algebraic flux $\mathbf{a}_h^i$ given by (6). Let f be a piecewise polynomial of degree $\boldsymbol{p}$. Let ν be chosen and the algebraic solver stopped as soon as the local stopping criteria (8) and (9) hold. Then, for a generic constant C in particular independent of i and ν,

$$(\eta_{\mathrm{disc},K}^i)^2 + (\eta_{\mathrm{alg},K}^i)^2 + (\eta_{\mathrm{rem},K}^i)^2 \leq C\|\nabla_h e_h^i\|_{\mathcal{T}_K}^2 + \sum_{\Gamma \in \widetilde{\mathcal{F}}_K} h_\Gamma^{-1} \|[u_h^i]\|_\Gamma^2,$$

where $\mathcal{T}_K$ denotes the set of the element K with its neighbors and $\widetilde{\mathcal{F}}_K$ denotes the set of faces that share at least a vertex with K.