Numerical simulation of fluid-structure interaction motivated by the self-oscilation of human vocal folds

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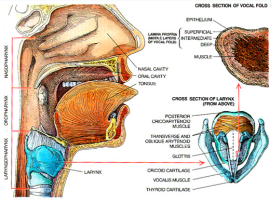
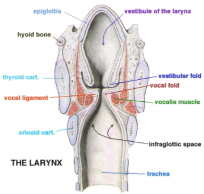
Goal and motivation

- Modelling of fluid-structure interaction
- Importance of this problem in several domains of human activity
  - Development of airplanes and turbines
  - Some problems of civil engineering
  - Car industry
  - Medicine, etc.
- Interested in the problem of the simulation of the airflow through the human vocal folds

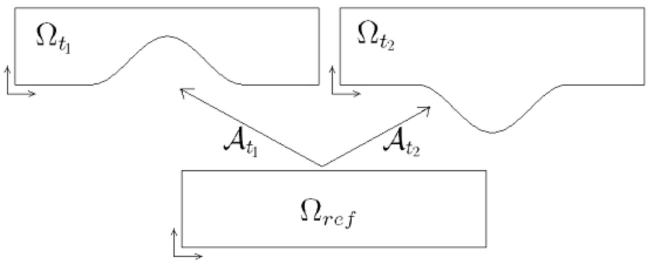




ALE mapping

Regular one-to-one ALE mapping:

$$\begin{array}{rcl} \mathcal{A}_t : \bar{\Omega}_{ref} & \longrightarrow & \bar{\Omega}_t \\ \mathbf{X} \subset \bar{\Omega}_{ref} & \longmapsto & \mathbf{x} = \mathbf{x}(\mathbf{X}, t) = \mathcal{A}_t(\mathbf{X}) \subset \bar{\Omega}_t \end{array}$$



Domain velocity

Domain velocity

$$\tilde{\mathbf{z}} : \bar{\Omega}_{ref} \times (0, T) \longrightarrow \boldsymbol{R}^2$$
$$\tilde{\mathbf{z}}(\mathbf{X}, t) = \frac{\partial}{\partial t}\mathbf{x}(\mathbf{X}, t) = \frac{\partial}{\partial t}\mathcal{A}_t(\mathbf{X})$$
$$\mathbf{z}(\mathbf{x}, t) = \tilde{\mathbf{z}}(\mathcal{A}_t^{-1}(\mathbf{x}), t), \ \ t \in (0, T), \ \ \mathbf{x} \in \bar{\Omega}_t$$

ALE derivative

- ALE derivative of a function $f = f(\mathbf{x}, t),\ \mathbf{x} \in \Omega_t,\ \ t \in (0, T)$:

$$\frac{D^{\mathcal{A}}}{Dt} f(\mathbf{x}, t) = \frac{\partial \tilde{f}}{\partial t}(\mathbf{X}, t), \quad \mathbf{X} = \mathcal{A}_t^{-1}(\mathbf{x})$$
$$\tilde{f}(\mathbf{X}, t) = f(\mathcal{A}_t(\mathbf{X}), t),\ \mathbf{X} \in \Omega_{ref},\ \ t \in (0, T)$$

- Form of the time derivative of the function f

$$\frac{\partial f}{\partial t} = \frac{D^{\mathcal{A}}}{Dt} f + f \operatorname{div}(\mathbf{z}) - \operatorname{div}(f\mathbf{z})$$

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Continuous problem in time dependent domains

- Bounded domain $\Omega \subset \mathbb{R}^2$, $\partial\Omega_t = \Gamma_I \cup \Gamma_O \cup \Gamma_{W_t}, t \in [0, T]$
- Navier-Stokes equations in the conservative ALE form

$$\frac{D^{\mathcal{A}}\mathbf{w}}{Dt} + \sum_{s=1}^{2} \frac{\partial \mathbf{g}_s(\mathbf{w})}{\partial x_s} + \mathbf{w}\,\mathrm{div}\,\mathbf{z} = \sum_{s=1}^{2} \frac{\partial \mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w})}{\partial x_s}, \quad (1)$$

where $\mathbf{g}_s(\mathbf{w}) = \mathbf{f}_s(\mathbf{w}) - z_s\mathbf{w}, \; s = 1, 2$

- Termodynamical relations

$$\begin{aligned} p &= (\gamma - 1)(E - \rho\,|\mathbf{v}|^2/2) \\ \theta &= \left(\frac{E}{\rho} - \frac{1}{2}|\mathbf{v}|^2\right)/c_v \end{aligned}$$

- State vector $\mathbf{w}$: $\mathbf{w} = (\rho, \rho v_1, \rho v_2, E)^T \in \boldsymbol{R}^4$
- Inviscid flux of the quantity $\mathbf{w}$ in the direction x_s :

$$\mathbf{f}_s(\mathbf{w}) = (\rho v_s, \rho v_1 v_s + \delta_{1s} p, \rho v_2 v_s + \delta_{2s} p, (E+p) v_s)^T, \;\; s = 1, 2$$

- Viscous flux of the quantity $\mathbf{w}$ in the direction x_s :

$$\begin{aligned} \mathbf{R}_s(\mathbf{w}, \nabla\mathbf{w}) &= (0, \tau_{s1}, \tau_{s2}, \tau_{s1} v_1 + \tau_{s2} v_2 + k\frac{\partial\theta}{\partial x_s})^T, \;\; s = 1, 2 \\ \tau_{ij} &= \lambda\delta_{ij}\mathrm{div}\mathbf{v} + 2\mu d_{ij}(\mathbf{w}), \;\; i, j = 1, 2 \\ d_{ij}(\mathbf{w}) &= \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \;\; i, j = 1, 2 \end{aligned}$$

- Initial condition

$$\mathbf{w}(\mathbf{x},0)=\mathbf{w}^0(\mathbf{x}), \quad \mathbf{x}\in\Omega$$

- Boundary conditions

$$\begin{aligned}
\text{Inlet } \Gamma_I : \quad & \rho|_{\Gamma_I\times(0,T)}=\rho_D, \\
& \mathbf{v}|_{\Gamma_I\times(0,T)}=\mathbf{v}_D=(v_{D1},v_{D1})^T, \\
& \sum_{j=1}^{2}\left(\sum_{i=1}^{2}\tau_{ij}n_i\right)v_j+k\frac{\partial\theta}{\partial\mathbf{n}}=0 \quad \text{on } \Gamma_I\times(0,T); \\
\text{Wall } \Gamma_{W_t} : \quad & \mathbf{v}_{\Gamma_{W_t}}=\mathbf{z}, \quad \frac{\partial\theta}{\partial\mathbf{n}}=0; \\
\text{Outlet } \Gamma_O : \quad & \sum_{i=1}^{2}\tau_{ij}n_i=0, \quad \frac{\partial\theta}{\partial\mathbf{n}}=0 \; j=1,2;
\end{aligned}$$

Space semidiscretization

Discontinuous Galerkin finite element method (DGFEM)

- Partition $\mathcal{T}_{ht}$ of $\bar{\Omega}_{ht}$ (polygonal approximation of the domain $\bar{\Omega}_t$) consisting of triangles K_i, $i \in I$, $\Gamma_{ij} = \partial K_i \cup \partial K_j$
- Space of the approximate solution - discontinuous piecewise polynomial functions:

$$\mathbf{S}_{ht} = [S_{ht}]^4,$$
$$S_{ht} = \{\varphi_h;\ \varphi_h|_K \in P^r(K) \quad \forall K \in \mathcal{T}_{ht}\}^4,$$

  - $r \geq 1$ - integer and $P^r(K)$ denotes the space of all polynomials on K of degree $\leq r$
  - $\varphi_h \in \mathbf{S}_{ht}$ - in general discontinuous on interfaces of the elements

Derivation of the discrete problem

- multiply system (1) by a test function $\varphi_h \in \mathbf{S}_{ht}$
- integrate over $K \in T_{ht}$
- use Green's theorem
- sum over all $K \in T_{ht}$
- introduce the concept of the numerical flux

$$\sum_{K_i \in T_{ht}} \int_{K_i} \frac{D^{\mathcal{A}}\mathbf{w}(t)}{Dt} \cdot \varphi_h d\mathbf{x} + b_h\left(\mathbf{w}, \varphi_h\right)$$
$$+J_h\left(\mathbf{w}, \varphi_h\right) + a_h\left(\mathbf{w}, \varphi_h\right) + d_h\left(\mathbf{w}, \varphi_h\right) = l_h\left(\mathbf{w}, \varphi_h\right)$$

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Approximate solution

The approximate solution is defined as $\mathbf{w}_h \in \mathbf{S}_{ht}$ such that

$$\sum_{K \in T_{ht}} \int_K \frac{D^{\mathcal{A}} \mathbf{w}_h(t)}{Dt} \cdot \varphi_h d\mathbf{x} + b_h \left(\mathbf{w}_h(t), \varphi_h \right) + a_h \left(\mathbf{w}_h(t), \varphi_h \right) + J_h \left(\mathbf{w}_h(t), \varphi_h \right) + d_h \left(\mathbf{w}_h(t), \varphi_h \right) = l_h \left(\mathbf{w}_h(t), \varphi_h \right)$$

holds for all $\varphi_h \in \mathbf{S}_{ht}$, all $t \in (0, T)$ and $\mathbf{w}_h(0) = \mathbf{w}_h^0$ (=approximation of the initial state $\mathbf{w}^0$)

Time discretization - semi-implicit scheme

- Partition $0 = t_0 < t_1 < t_2 < \dots$ of the interval $(0, T)$
- Time step $\tau_k = t_{k+1} - t_k$
- ALE derivative - the first order backward difference

$$\frac{D^{\mathcal{A}}\mathbf{w}_h}{Dt}(\mathbf{x}, t_{k+1}) \approx \frac{\mathbf{w}_h^{k+1}(\mathbf{x}) - \hat{\mathbf{w}}_h^k(\mathbf{x})}{\tau_k}, \quad \mathbf{x} \in \Omega_{ht_{k+1}}$$

$$\hat{\mathbf{w}}_h^j(\mathbf{x}) = \mathbf{w}^j\left(\mathcal{A}_{t_j}\left(\mathcal{A}_{t_{k+1}}^{-1}\right)(\mathbf{x})\right), \quad \mathbf{x} \in \Omega_{ht_{k+1}}$$

- Remaining terms treated with the aid of a linearization and extrapolation

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Semi-implicit discrete problem

Problem is linear with respect to $\mathbf{w}_h^{k+1}$

$$
\begin{aligned}
&\mathbf{w}_h^{k+1} \in \mathbf{S}_{ht_{k+1}}, \\
&\left(\frac{\mathbf{w}_h^{k+1} - \hat{\mathbf{w}}_h^k}{\tau_k}, \varphi_h \right) + \hat{b}_h \left(\hat{\mathbf{w}}_h^k, \mathbf{w}_h^{k+1}, \varphi_h \right) \\
&\quad + \hat{a}_h \left(\hat{\mathbf{w}}_h^k, \mathbf{w}_h^{k+1}, \varphi_h \right) + J_h \left(\mathbf{w}_h^{k+1}, \varphi_h \right) + d_h \left(\mathbf{w}_h^{k+1}, \varphi_h \right) \\
&\quad = l_h \left(\mathbf{w}_B, \varphi_h \right) \\
&\quad \forall \varphi_h \in \mathbf{S}_{ht_{k+1}}, \;\; k = 0, 1, \ldots
\end{aligned}
$$

Remarks

- Discret problem is equivalent on each time level to a linear algebraic system, which is solved by GMRES with a block diagonal preconditioning.
- Scheme for the solution of inviscid flow:
  - $\mu = \lambda = \kappa = 0$
  - Boundary conditions in the form $\hat{b}_h$ (the determination of the state $\mathbf{w}_h^{k+1}|_{\Gamma_{ij}}$ for $\Gamma_{ij} \subset \partial\Omega_{ht_{k+1}}$) realized by linearized local initial-boundary value Riemann problem

Continuous problem for elasticity

- Bounded domain $\Omega^b \subset \boldsymbol{R}^2$, $\partial\Omega^b = \Gamma_D^b \cup \Gamma_W$, $t \subset [0, T]$
- Dynamic equations for the isotropic elastic body

$$\rho^b \frac{\partial^2 u_i}{\partial t^2} + C\rho^b \frac{\partial u_i}{\partial t} + \sum_{j=1}^{2} \frac{\partial \tau_{ij}^b}{\partial x_j} = f_i \text{ in } \Omega_b, \; t \subset [0, T], \; i = 1, 2, \quad (2)$$

where the term $C\rho^b \frac{\partial u_i}{\partial t}$, $i = 1, 2$ with $C \geq 0$ performs damping

- Initial condition

$$\mathbf{u}(0, \cdot) = \mathbf{u}^0, \quad \frac{\partial \mathbf{u}}{\partial t}(0, \cdot) = \mathbf{r}^0 \text{ in } \Omega_b$$

- Boundary condition

$$\begin{aligned} \mathbf{u} &= 0 \text{ in } (0, T) \times \Gamma_D^b \\ \sum_{j=1}^{2} \tau_{ij}^b n_j &= T_i^{\mathbf{n}} \text{ in } (0, T) \times \Gamma_W^b \end{aligned}$$

Space discretization

Finite element method (FEM)

- Partition T_{ht} of $\bar{\Omega}_h^b$ (polygonal approximation of the domain $\bar{\Omega}^b$) consisting of triangles $K_i,\ i \in I,\ \Gamma_{ij} = \partial K_i \cup \partial K_j$
- Space of the approximate solution - piecewise polynomial functions:

$$\mathbf{X}_h = [X_h]^2,$$
$$X_h = \{y_h;\ y_h|_K \in P^r(K) \quad \forall K \in T_h\}^2,$$

- $r \geq 1$ - integer and $P^r(K)$ denotes the space of all polynomials on K of degree $\leq r$

Derivation of the discrete problem

- multiply system (2) by a test function $\mathbf{y}_h \in \mathbf{X}_h$
- integrate over Ω_h^b
- use Green's theorem

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\rho^b \mathbf{u}(t), \mathbf{y}_h\right)_{0,\Omega_h^b} + \frac{\mathrm{d}}{\mathrm{d}t}\left(C\rho^b \mathbf{u}(t), \mathbf{y}_h\right)_{0,\Omega_h^b} + a\left(\mathbf{u}, \mathbf{y}_h; t\right)$$
$$= \left(\mathbf{f}(t), \mathbf{y}_h\right)_{0,\Omega_h^b} + \left(\mathbf{T}^{\mathbf{n}}(t), \mathbf{y}_h\right)_{0,\Gamma_{Wh}}$$

- application of the generalized Hook's law for an isotropic material

$$a\left(\mathbf{u}, \mathbf{y}_h; t\right) = \int_{\Omega_h^b} \sum_{i,j=2}^{2} \left(\tilde{\lambda}\,\mathrm{div}\mathbf{u}(t)\delta_{ij} + 2\tilde{\mu} e_{ij}(\mathbf{u}(t))\right) \frac{\partial y_i}{\partial x_j} dx,$$

where $\tilde{\lambda}$ and $\tilde{\mu}$ are the Lame koeficients and $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, $i, j = 1, 2$ is the tenzor of small deformation

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Approximate solution

The approximate solution is defined as $\mathbf{u}_h \in \mathbf{V}_h$ such that there exist $\mathbf{u}'_h$ and $\mathbf{u}''_h$ and holds

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\rho^b \mathbf{u}_h(t), \mathbf{y}_h\right)_{0,\Omega_h^b} + \frac{\mathrm{d}}{\mathrm{d}t}\left(C\rho^b \mathbf{u}_h(t), \mathbf{y}_h\right)_{0,\Omega_h^b} + a\left(\mathbf{u}_h, \mathbf{y}_h; t\right)$$
$$= (\mathbf{f}(t), \mathbf{y}_h)_{0,\Omega_h^b} + (\mathbf{T^n}_h(t), \mathbf{y}_h)_{0,\Gamma_{Wh}}$$

holds for all $\mathbf{y}_h \in \mathbf{V}_h = \left\{\mathbf{y}_h \in \mathbf{X}_h \middle|\ \mathbf{y}_h|_{\Gamma^b_{Dh}} = 0\right\}$, almost all $t \in (0, T)$ and

$$\begin{aligned} \mathbf{u}_h(\mathbf{x}, 0) &= \mathbf{u}_h^0(\mathbf{x}),\ \mathbf{x} \in \Omega_h^b, \\ \mathbf{u}'_h(\mathbf{x}, 0) &= \mathbf{u}_r^0(\mathbf{x}),\ \mathbf{x} \in \Omega_h^b, \end{aligned}$$

(= approximation of the initial state $\mathbf{u}^0$ a $\mathbf{u}_r^0$)

Time discretization

- There is the system of second order ordinary differential equations arising from the space discretization of the elasticity problem.
- Suitable solution of the system can be obtain by **Newmark method** $\Rightarrow$ linear algebraic system.

Computation of the ALE mapping

- Motion of the computational mesh in the domain occupied by the fluid solved as a special stationary problem of linear elasticity

$$\sum_{j=1}^{2} \frac{\partial \tau_{ij}^{a}}{\partial x_j} = 0 \text{ in } \Omega_f \ \ i = 1, 2$$

- Use of finite element method (FEM) - linear elements
- Resulting linear system solved by the method of conjugated gradients

Transmission conditions

The interaction between the flow and the structure takes place on their common boundary $\tilde{\Gamma}_{W_t}$ at the time t :

$$\tilde{\Gamma}_{W_t} = \left\{ \mathbf{x} \in \boldsymbol{R}^2;\ \mathbf{x} = \mathbf{X} + \mathbf{u}(\mathbf{X}, t),\ \mathbf{X} \in \Gamma_W^b \right\}.$$

Transmission condition (fluid $\to$ structure):

$$\sum_{j=1}^{2} \tau_{ij}^{b}(\mathbf{X}) n_j(\mathbf{X}) = -\sum_{j=1}^{2} \tau_{ij}^{f}(\mathbf{x}) n_j(\mathbf{X}), \quad i = 1, 2$$
$$\text{where } \mathbf{x} = \mathbf{X} + \mathbf{u}(\mathbf{X}, t)$$

Transmission condition (structure $\to$ fluid):

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{z}_D(\mathbf{x}, t) = \frac{\partial \mathbf{u}(\mathbf{X}, \mathbf{t})}{\partial t}$$

Weak coupling

Computational scheme:

1. Compute the approximate solution of the flow problem on the time level t_m.
2. Compute the stress tensor of the fluid τ_{ij}^f and the aerodynamical force acting on the structure and transform it to the interface Γ_{Wh}^b.
3. Solve the elasticity problem, compute the deformation $\mathbf{u}_{h,m}$ at time t_m and approximate the domain $\Omega_{ht_{m+1}}$.
4. Determine the ALE mapping $\mathcal{A}_{t_{m+1}h}$ and approximate the domain velocity $\mathbf{z}_{h,m+1}$.
5. Set $m := m + 1$, go to 1).

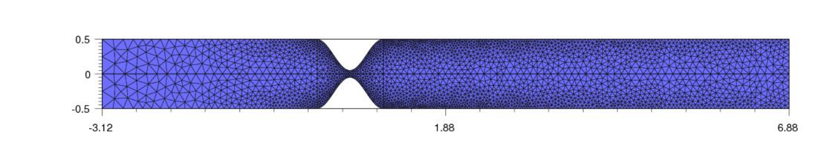
Strong coupling

Computational scheme:

1. Assume that the approximate solution $\mathbf{w}_h^m$ of the flow problem and the deformation $\mathbf{u}_{h,m}$ of the structure are known on the time level t_m.
2. Set $\mathbf{u}^0_{h,m+1} := \mathbf{u}_{h,m}$, $k := 1$ and apply the iterative process:
   1. Compute the stress tensor of the fluid τ^f_{ij} and the aerodynamical force acting on the structure and transform it to the interface Γ^b_{Wh}.
   2. Solve the elasticity problem, compute the approximation of the deformation $\mathbf{u}^k_{h,m+1}$ and construct the approximation $\Omega^k_{ht_{m+1}}$ of the flow domain at time t_{m+1}.
   3. Determine the approximations of ALE mapping $\mathcal{A}^k_{t_{m+1}h}$ and the domain velocity $\mathbf{z}^k_{h,m+1}$.
   4. Solve the flow problem in $\Omega^k_{ht_{m+1}}$ and obtain the approximate solution $\mathbf{w}^k_{h,m+1}$.
   5. If the variation of the displacement $\mathbf{u}^k_{h,m+1}$ and $\mathbf{u}^{k-1}_{h,m+1}$ is larger than the prescribed tolerance, go to a) and $k := k+1$. Else $\Omega_{ht_{m+1}} := \Omega^k_{ht_m}$, $\mathbf{w}_h^{m+1} := \mathbf{w}^k_{h,m+1}$, $\mathbf{u}_h^{m+1} := \mathbf{u}^k_{h,m}$, $m := m+1$ and goto 2).

Computational geometry

Geometry of the channel inspired by the shape of vocal folds and a part of supraglottal space



Flow parameters

Input parameters and boundary conditions for the airflow

- inlet flow velocity 4 m/s
- viscosity $15 \cdot 10^{-6}\ kgm^{-1}s^{-1}$
- density 1.225 kg/m^3
- outlet pressure 97611 Pa
- $\mathrm{Re} = 5227$
- $\mathrm{k} = 2.428 \cdot 10^{-2}\ kgms^{-3}K^{-1}$
- $c_v = 721.428\ m^2s^{-2}K^{-1}$

Parameters of the solid body and for the computation of the ALE mapping

Parameters of the solid body

- density 1040 kgm^{-3}
- damping coefficient $C = 0.1$
- Young modulus $E^b = 25000\ Pa$
- Poisson ration $\sigma^b = 0.4$

Parameters for the computation of the ALE mapping

- $E = 10000.0\ Pa$
- $\sigma = 0.49$

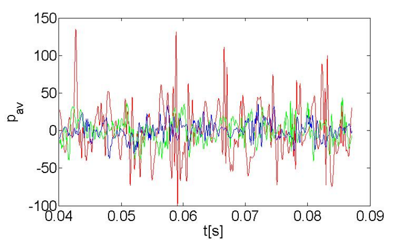
Comparison of the influence of the density of the computational mesh

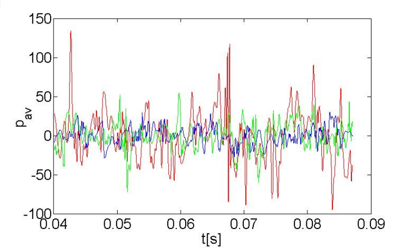
Testing meshes:

- red . . . 5398/1998 elements in flow/structure part of the channel
- green . . . 10130/2806 elements in flow/structure part of the channel
- blue . . . 20484/4076 elements in flow/structure part of the channel

Observed quantity: $p_{av} = \int_{\Gamma_O} \left(p(x,t) - \frac{1}{T} \int_0^T p(x,t) \mathrm{d}t \right) / \int_{\Gamma_O} \mathrm{d}S$

Weak x strong coupling

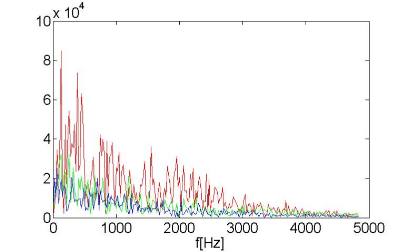


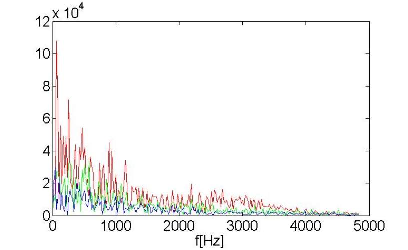


Used dimensionless time step $\tau = 0.001$.

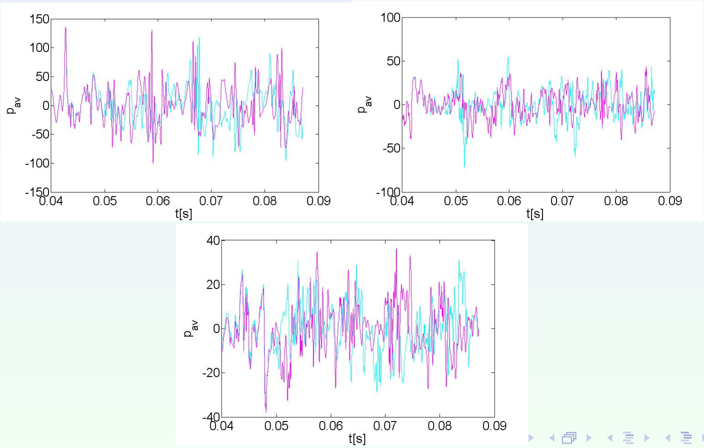
Comparison of the influence of the density of the computational mesh using the Fourier analysis

Weak x strong coupling



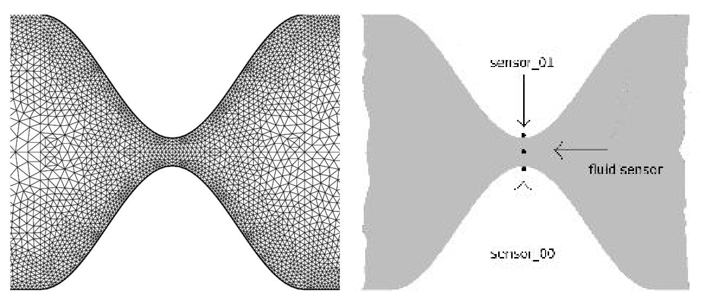


Comparison of weak and strong coupling procedure on the different meshes (5398, 10130, 20484 el. in the fluid part)



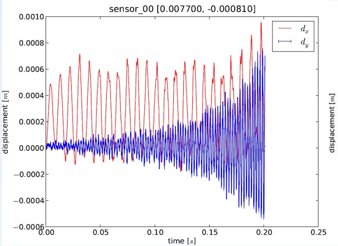
Comparison of the behaviour of the fluid and the structure in the narrowest part of the channel

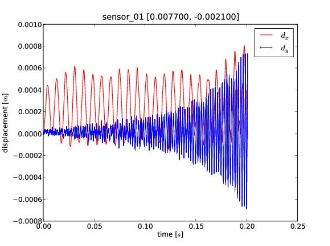
Detail of the computational mesh and placement of sensors

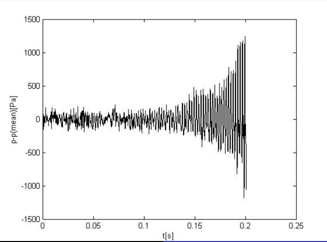


Used computational mesh: 10130/2806 elements in flow/structure part of the channel.

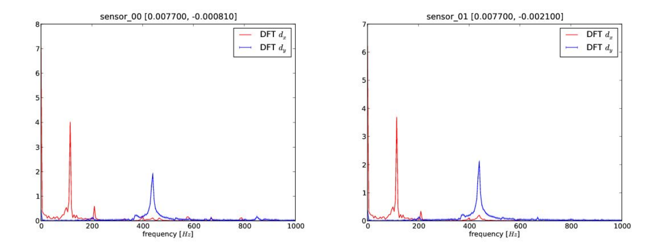
Displacement of the structure and the pressure of the fluid in the narrowest part of the channel

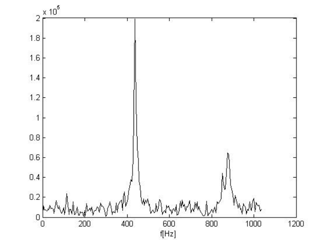






Displacement of the structure and the pressure of the fluid in the narrowest part of the channel - Fourier analysis





Isolines of velocity and pressure

What has been done

- Fluid-structure interaction
  - Weak coupling
  - Strong coupling
- Study of several computational geometries and input parameters
- Preliminary tests of the influence of the mesh size on results - observation of the convergence tendency

Further work

Suitable tests of the influence of the mesh size on results

Thank you for your attention!

Details of the space discretization of the flow problem

Convective terms

Approximation of fluxes

$$\int_{\Gamma_{ij}} \sum_{s=1}^{2} \mathbf{g}_s(\mathbf{w}) \left(n_{ij}\right)_s \cdot \varphi dS \approx \int_{\Gamma_{ij}} \mathbf{H}_g(\mathbf{w}_h(t)|_{\Gamma_{ij}}, \mathbf{w}_h(t)|_{\Gamma_{ji}}, \mathbf{n}_{ij}) \cdot \varphi dS$$

$$b_h\left(\mathbf{w}, \varphi\right) = -\sum_{K_i \in T_{ht}} \int_{K_i} \sum_{s=1}^{2} \mathbf{g}_s(\mathbf{w}(t)) \cdot \frac{\partial \varphi}{\partial x_s} d\mathbf{x}$$
$$+ \sum_{K_i \in T_{ht}} \sum_{j \in S_t(i)} \int_{\Gamma_{ij}} \mathbf{H}_g(\mathbf{w}_h(t)|_{\Gamma_{ij}}, \mathbf{w}_h(t)|_{\Gamma_{ji}}, \mathbf{n}_{ij}) \cdot \varphi dS$$

Diffusion form (IIPG)

$$
\begin{aligned}
a_h(\mathbf{w}, \varphi) = & \sum_{i \in I} \int_{K_i} \sum_{s=1}^{2} \mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w}) \cdot \frac{\partial \varphi}{\partial x_s} dx \\
& - \sum_{i \in I} \sum_{j \in s(i), j < i} \int_{\Gamma_{ij}} \sum_{s=1}^{2} \langle \mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w}) \rangle (n_{ij})_s \cdot [\varphi] \, dS \\
& - \sum_{i \in I} \sum_{j \in \gamma_D(i)} \int_{\Gamma_{ij}} \sum_{s=1}^{2} \mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w}) (n_{ij})_s \cdot \varphi dS
\end{aligned}
$$

Interior and boundary penalty jump terms

$$J_h(\mathbf{w},\varphi)=\sum_{i\in I}\sum_{j\in s(i),j<i}\int_{\Gamma_{ij}}\sigma\,[\mathbf{w}]\cdot[\varphi]\,dS+\sum_{i\in I}\sum_{j\in\gamma_D(i)}\int_{\Gamma_{ij}}\sigma\mathbf{w}\cdot\varphi dS,$$

where $\sigma|_{\Gamma_{ij}}=\frac{C_W\mu}{d(\Gamma_{ij})},\;\; C_W>0$

Form d_h

$$d_h\left(\mathbf{w},\varphi_h\right)=\sum_{K_i\in T_{ht}}\int_{K_i}\mathrm{div}\mathbf{z}\left(\mathbf{w}\cdot\varphi\right)d\mathbf{x}$$

Right-hand side form

$$l_h(\mathbf{w}, \varphi) = \sum_{i \in I} \sum_{j \in \gamma_D(i)} \int_{\Gamma_{ij}} \sigma \mathbf{w}_B \cdot \varphi dS,$$

where $\mathbf{w}_B$ is defined on the basis of the Dirichlet boundary conditions and extrapolation:

$$\mathbf{w}_B = \left(\rho_{ij}, \rho_{ij} z_1, \rho_{ij} z_2, c_v \rho_{ij} \theta_{ij} + \frac{1}{2} \rho_{ij} \left|\mathbf{z}\right|^2\right) \quad \text{on } \Gamma_{W_t}$$

$$\mathbf{w}_B = \left(\rho_D, \rho_D v_{D1}, \rho_D v_{D2}, c_v \rho_D \theta_D + \frac{1}{2} \rho_D \left|\mathbf{v}_D\right|^2\right) \quad \text{on } \Gamma_I$$

$$\mathbf{w}_B = \mathbf{w}_{ij} \quad \text{on } \Gamma_O$$

Details of the time discretization of the flow problem

Semi-implicit linearized scheme

Linearization of the term $b_h(\mathbf{w}, \varphi_h)$:

$$b_h(\mathbf{w}, \varphi_h) = -\sum_{K_i \in T_{ht}} \int_{K_i} \sum_{s=1}^{2} \mathbf{g}_s(\mathbf{w}(t)) \cdot \frac{\partial \varphi_h}{\partial x_s} d\mathbf{x}$$

$$+ \sum_{K_i \in T_{ht}} \sum_{j \in S_t(i)} \int_{\Gamma_{ij}} \mathbf{H}_g(\mathbf{w}_h(t)|_{\Gamma_{ij}}, \mathbf{w}_h(t)|_{\Gamma_{ji}}, \mathbf{n}_{ij}) \cdot \varphi dS$$

Based on relation:

$$\mathbf{g}_s(\mathbf{w}_h^{k+1}) = \left(\mathbb{A}_s(\mathbf{w}_h^{k+1}) - z_s^{k+1}\mathbb{I}\right) \mathbf{w}_h^{k+1}$$

$$\approx \left(\mathbb{A}_s(\hat{\mathbf{w}}_h^{k}) - z_s^{k+1}\mathbb{I}\right) \mathbf{w}_h^{k+1},$$

where $\mathbb{A}_s(w) = \frac{D\mathbf{f}_s(w)}{D\mathbf{w}} = \left(\frac{\partial f_{si}(\mathbf{w})}{\partial w_j}\right)_{i,j=1}^{4}$

Linearization of the first term of $b_h(.,.)$:

$$\sum_{K \in T_{ht_{k+1}}} \int_K \sum_{s=1}^{2} \mathbf{g}_s(\mathbf{w}_h^{k+1}) \cdot \frac{\partial \varphi_h}{\partial x_s} d\mathbf{x}$$

$$\approx \sum_{K \in T_{ht_{k+1}}} \int_K \sum_{s=1}^{2} \left(\mathbb{A}_s(\hat{\mathbf{w}}_h^k) - z_s^{k+1}\mathbb{I} \right) \mathbf{w}_h^{k+1} \cdot \frac{\partial \varphi_h}{\partial x_s} d\mathbf{x},$$

The second term of $b_h(.,.)$ is linearized with the aid of the Vijayasundaram numerical flux:

$$\mathbf{H}_g(\mathbf{w}_h^{k+1}|_{\Gamma_{ij}}, \mathbf{w}_h^{k+1}|_{\Gamma_{ji}}, \mathbf{n}_{ij})$$

$$\approx \mathbb{P}^+ \left(\left\langle \hat{\mathbf{w}}_h^k \right\rangle_{\Gamma_{ij}}, \mathbf{n}_{ij} \right) \mathbf{w}_h^{k+1}|_{\Gamma_{ij}} + \mathbb{P}^- \left(\left\langle \hat{\mathbf{w}}_h^k \right\rangle_{\Gamma_{ij}}, \mathbf{n}_{ij} \right) \mathbf{w}_h^{k+1}|_{\Gamma_{ji}}$$

where $\mathbb{P}^+$ and $\mathbb{P}^-$ are positive and negative part of the matrix $\sum_{s=1}^{2} \left(\mathbb{A}_s(\hat{\mathbf{w}}_h^k(\mathbf{x})) - z_s^{k+1}\mathbb{I} \right) n_s$

$$
\begin{aligned}
&\hat{b}_h\left(\hat{\mathbf{w}}_h^k, \mathbf{w}_h^{k+1}, \varphi_h\right) \\
&= -\sum_{K \in T_{ht_{k+1}}} \int_K \sum_{s=1}^{2} \left(\mathbb{A}_s(\hat{\mathbf{w}}_h^k) - z_s^{k+1}\mathbb{I}\right) \mathbf{w}_h^{k+1} \cdot \frac{\partial \varphi_h}{\partial x_s} d\mathbf{x} \\
&\quad + \sum_{K_i \in T_{ht_{k+1}}} \sum_{j \in S_{t_{k+1}}(i)} \int_{\Gamma_{ij}} \left[\mathbb{P}^+ \left(\left\langle \hat{\mathbf{w}}_h^k \right\rangle_{\Gamma_{ij}}, \mathbf{n}_{ij}\right) \mathbf{w}_h^{k+1}|_{\Gamma_{ij}} \right. \\
&\quad \left. + \mathbb{P}^- \left(\left\langle \hat{\mathbf{w}}_h^k \right\rangle_{\Gamma_{ij}}, \mathbf{n}_{ij}\right) \mathbf{w}_h^{k+1}|_{\Gamma_{ji}}\right] \cdot \varphi_h dS
\end{aligned}
$$

Linearization of the form a_h

Based on the fact that $\mathbf{R}_s\left(\mathbf{w},\nabla\mathbf{w}\right)$ is nonlinear in $\mathbf{w}$ but linear in $\nabla\mathbf{w}$ and the following formula: $\mathbf{R}_s\left(\mathbf{w},\nabla\mathbf{w}\right)=\sum_{j=1}^{2}\mathbb{K}_{ij}\left(\mathbf{w}\right)\frac{\partial\mathbf{w}}{\partial x_j}$

$$
\begin{aligned}
& a_h\left(\mathbf{w}^{\mathbf{k+1}},\varphi\right)\approx\hat{a}_h\left(\hat{\mathbf{w}}^k,\mathbf{w}^{k+1},\varphi\right) \\
& :=\sum_{i\in I}\int_{K_i}\sum_{s=1}^{2}\sum_{t=1}^{2}\mathbb{K}_{st}\left(\hat{\mathbf{w}}_h^k\right)\frac{\partial\mathbf{w}_h^{k+1}}{\partial x_t}\cdot\frac{\partial\varphi_h}{\partial x_s}dx \\
& -\sum_{i\in I}\sum_{j\in s(i),j<i}\int_{\Gamma_{ij}}\sum_{s=1}^{2}\langle\sum_{t=1}^{2}\mathbb{K}_{st}\left(\hat{\mathbf{w}}_h^k\right)\frac{\partial\mathbf{w}_h^{k+1}}{\partial x_t}\rangle(n_{ij})_s\cdot[\varphi]\,dS \\
& -\sum_{i\in I}\sum_{j\in\gamma_D(i)}\int_{\Gamma_{ij}}\sum_{s=1}^{2}\sum_{t=1}^{2}\mathbb{K}_{st}\left(\hat{\mathbf{w}}_h^k\right)\frac{\partial\mathbf{w}_h^{k+1}}{\partial x_t}(n_{ij})_s\cdot\varphi dS,
\end{aligned}
$$

Semi-implicit discrete problem

Problem is linear with respect to $\mathbf{w}_h^{k+1}$

$$\begin{aligned}
&\mathbf{w}_h^{k+1} \in \mathbf{S}_{ht_{k+1}}, \\
&\left(\frac{\mathbf{w}_h^{k+1} - \hat{\mathbf{w}}_h^k}{\tau_k}, \varphi_h\right) + \hat{b}_h\left(\hat{\mathbf{w}}_h^k, \mathbf{w}_h^{k+1}, \varphi_h\right) \\
&\quad + \hat{a}_h\left(\hat{\mathbf{w}}_h^k, \mathbf{w}_h^{k+1}, \varphi_h\right) + J_h\left(\mathbf{w}_h^{k+1}, \varphi_h\right) + d_h\left(\mathbf{w}_h^{k+1}, \varphi_h\right) \\
&\quad = l_h\left(\mathbf{w}_B, \varphi_h\right) \\
&\quad \forall \varphi_h \in \mathbf{S}_{ht_{k+1}}, \;\; k = 0, 1, \ldots
\end{aligned}$$

Details of the computation of the ALE mapping

Computation of the ALE mapping

- Bounded domain $\Omega^f \subset \boldsymbol{R}^2$, $\partial\Omega^f = \Gamma^f_{D_1} \cup \Gamma^f_{D_2}$
- Stationary equations of linear elasticity for the isotropic elastic body

$$\sum_{j=1}^{2} \frac{\partial \tau^a_{ij}}{\partial x_j} = 0 \text{ in } \Omega_f \;\; i = 1, 2, \tag{3}$$

- Boundary condition

$$\begin{aligned} \mathbf{u} &= \mathbf{u}^0 \text{ in } \Gamma^f_{D_1} \\ \mathbf{u} &= 0 \text{ in } \Gamma^f_{D_2} \end{aligned}$$

Space discretization

Finite element method (FEM)

- Partition T_{ht} of $\bar{\Omega}_h^f$ (polygonal approximation of the domain $\bar{\Omega}^f$) consisting of triangles $K_i,\ i \in I,\ \Gamma_{ij} = \partial K_i \cup \partial K_j$
- Space of the approximate solution - discontinuous piecewise polynomial functions:

$$\mathbf{X}_h = [X_h]^2,$$
$$X_h = \{y_h;\ y_h|_K \in P^r(K) \quad \forall K \in T_h\}^2,$$

- $r \geq 0$ - integer and $P^r(K)$ denotes the space of all polynomials on K of degree $\leq r$

Derivation of the discrete problem

- multiply system (2) by a test function $\mathbf{y}_h \in \mathbf{X}_h$
- integrate over Ω_h^f
- use Green's theorem
- application of the generalized Hook's law for an isotropic material

$$\int_{\Omega_h^f} \sum_{i,j=2}^{2} \left(\bar{\lambda}\mathrm{div}\mathbf{u}(t)\delta_{ij} + 2\bar{\mu}e_{ij}(\mathbf{u}(t))\right) \frac{\partial y_i}{\partial x_j} d\mathbf{x} = 0,$$

where $\bar{\lambda}$ and $\bar{\mu}$ are the Lame koeficients and $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \; i,j = 1,2$ is the tenzor of small deformation

- Resulting linear system solved by the method of conjugated gradients