TECHNISCHE UNIVERSITÄT DRESDEN

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A New Successive Linear Programming Framework for Constrained Equations

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Contents

Problem

$$F(z) = 0 \qquad \text{s.t.} \qquad z \in \Omega$$

with

- $F : \mathbb{R}^n \to \mathbb{R}^m$ (at least) locally Lipschitz continuous,
- $\Omega \subseteq \mathbb{R}^n$ is nonempty, closed and convex,
- nonempty solution set Z,
- let $G : \mathbb{R}^n \to \mathbb{R}^{m \times n}$ be a given map

A New Successive Linear Programming Algorithm

For given z^k determine a solution $(z(z^k), \gamma(z^k))$ of the problem

$$\begin{aligned} \min_{z,\gamma} \quad & \gamma \\ & \|F(z^k) + G(z^k)(z - z^k)\| \leq \gamma \|F(z^k)\|^2, \\ & \|z - z^k\| \leq \gamma \|F(z^k)\|, \\ & z \in \Omega, \ \gamma \geq 0. \end{aligned} \tag{1}$$

Set $z^{k+1} := z(z^k)$.

For any $z^k \in \mathbb{R}^n$ the subproblem has a solution. Thus, the algorithm is well-defined for any $z^0 \in \mathbb{R}^n$.

If Ω is polyhedral and $\|\cdot\| = \|\cdot\|_\infty$ then (1) is a linear program.

Local Convergence Properties

Assumptions

Let $z^* \in Z$ be an arbitrary but fixed solution.
Moreover, let $\delta > 0$ denote the radius of the ball

$$\mathcal{B}_\delta(z^*) := \{w \in \mathbb{R}^n \mid \|w - z^*\| \leq \delta\}.$$

Assumption 1 (Error Bound):

There is $\ell > 0$ so that

$$\text{dist}[s, Z] \leq \ell \|F(s)\|$$

for all $s \in \mathcal{B}_\delta(z^*) \cap \Omega$.

Assumption 2:

There is $\Gamma > 0$ so that for any $s \in \mathcal{B}_\delta(z^*) \cap \Omega$ there exists $s^\diamond \in \Omega$ such that

$$\begin{aligned} \|s^\diamond - s\| &\leq \Gamma\|F(s)\|, \\ \|F(s) + G(s)(s^\diamond - s)\| &\leq \Gamma\|F(s)\|^2. \end{aligned}$$

Assumption 3:

There is $\hat{\alpha} > 0$ so that for all $s \in \mathcal{B}_\delta(z^*) \cap \Omega$ and all $\alpha \in [0, \delta]$

$$w \in \mathcal{L}(s, \alpha) \quad \text{implies} \quad \|F(w)\| \leq \hat{\alpha}\alpha^2$$

where

$$\mathcal{L}(s, \alpha) := \left\{ w \in \Omega \mid \|w - s\| \leq \alpha, \ \|F(s) + G(s)(w - s)\| \leq \alpha^2 \right\}.$$

Convergence Result

Theorem 1

Let Assumptions 1 – 3 be satisfied. Then, there is $\epsilon > 0$ so that $z^0 \in \mathcal{B}_\epsilon(z^*) \cap \Omega$ implies that the sequence $\{z^k\}$ generated by the SLP Algorithm converges Q-quadratically to some $\hat{z} \in Z$.

Discussion of Assumptions

Relations to Classical Smoothness Conditions

Proposition 2

Let F be $C^{1,1}$ (differentiable with locally Lipschitz continuous Jacobian) and $G(s) := F'(s)$. Then,

- Assumption 2 is satisfied, if Assumption 1 (Error Bound) holds,
- Assumption 3 is satisfied.

Instead of assuming that F' is locally Lipschitz, Proposition 1 remains valid if a smoothness condition in Kanzow/Yamashita/Fukushima 2004 is used, i.e., if there is $\kappa > 0$ so that

$$\|F(s) + G(s)(w - s) - F(w)\| \leq \kappa \|w - s\|^2$$

holds for all $w, s \in \mathcal{B}_{2\delta}(z^*) \cap \Omega$.

Relations to Classical Smoothness Conditions (Continuued)

Proposition 3

Let F be strongly semismooth at z^* and $G(s) \in \partial_B F(s)$. Then,

- Assumption 2 is satisfied, if z^* is a locally isolated solution,
- Assumption 3 is satisfied, if $m \geq n$ and $\partial_B F(z^*)$ has rank n.

Together with appropriate regularity conditions, Assumptions 2 and 3 are implied by state-of-the-art smoothness conditions (Qi and Qi/Sun).

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Assumptions 2 and 3 for Reformulated KKT Systems

Let us consider the KKT system

$$\begin{aligned} \Psi(x,u) := H(x) + \sum_{i=1}^{m_g} u_i \nabla g_i(x) &= 0, \\ u \geq 0,\ g(x) \leq 0,\ u^\top g(x) &= 0, \end{aligned}$$

where $H : \mathbb{R}^{n_0} \to \mathbb{R}^{n_0}$ and $g : \mathbb{R}^{n_0} \to \mathbb{R}^{m_g}$ are assumed to be sufficiently smooth.

Reformulation:

$$F(z) := F(x,u,y) := \begin{pmatrix} \Psi(x,u) \\ g(x)+y \\ \min\{u_1, y_1\} \\ \vdots \\ \min\{u_{m_g}, y_{m_g}\} \end{pmatrix} = 0 \quad \text{s.t.} \quad z \in \Omega \qquad (2)$$

with

$$\Omega := \mathbb{R}^{n_0} \times \mathbb{R}_+^{m_g} \times \mathbb{R}_+^{m_g} \subseteq \mathbb{R}^n.$$

Theorem 4

If Assumption 1 holds and if there is a neighborhood $\mathcal{U}(z^*)$ of $z^* = (x^*, u^*, v^*)$ so that $z = (x, u, v) \in Z \cap \mathcal{U}(z^*)$ implies $x = x^*$ then Assumption 2 is satisfied.

Theorem 5

Assumption 3 is satisfied for problem (2).

Corollary 6

Suppose that at z^* the following second-order condition holds:

$$d^\top \Psi_x(x^*, u^*)d \neq 0 \tag{3}$$

for all $d \in \mathbb{R}^{n_0}$ with

$$\nabla g_i(x)^\top d \leq 0 \, \forall i \in \mathcal{I}_0(x^*) \quad \text{and} \quad \nabla g_i(x)^\top d = 0 \, \forall i \in \mathcal{I}_0(x^*) \text{ with } u_i^* > 0.$$

Then, the x-part of the solution $z^* = (x^*, u^*, v^*)$ is locally unique and Assumption 1 is satisfied ($\rightarrow$ Fernández/Solodov 2009).

Hence, the SLP algorithm converges locally quadratically to a solution of (2).

Outlook

- Globalization
- Applications to GNEPS
- Inexact versions of the SLP algorithm